Analysis of Algorithms

T. M. Murali

August 22, 2018
What is Algorithm Analysis?

- Measure resource requirements: how does the amount of time and space an algorithm uses scale with increasing input size?
- How do we put this notion on a concrete footing?
- What does it mean for one function to grow faster or slower than another?
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- What does it mean for one function to grow faster or slower than another?

Goal

Develop algorithms that provably run quickly and use low amounts of space.
Worst-case Running Time

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- Bound the largest possible running time the algorithm over all inputs of size $n$, as a function of $n$. 
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- Input size = number of elements in the input.
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- Bound the largest possible running time the algorithm over all inputs of size $n$, as a function of $n$.
- Why worst-case? Why not average-case or on random inputs?
- Input size = number of elements in the input. Values in the input do not matter, except for specific algorithms.
- Assume all elementary operations take unit time: assignment, arithmetic on a fixed-size number, comparisons, array lookup, following a pointer, etc.
Polynomial Time

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- An algorithm has a polynomial running time if there exist constants $c > 0$ and $d > 0$ such that on every input of size $n$, the running time of the algorithm is bounded by $cn^d$ steps.
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- An algorithm has a \emph{polynomial} running time if there exist constants \( c > 0 \) and \( d > 0 \) such that on every input of size \( n \), the running time of the algorithm is bounded by \( cn^d \) steps.

Definition

An algorithm is \emph{efficient} if it has a polynomial running time.
Comparing Mathematical Functions

- Assume all (mathematical) functions take only positive arguments and values.
- Different algorithms for the same problem may have different (worst-case) running times.
- Example of sorting:
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- Example of sorting: bubble sort, insertion sort, quick sort, merge sort, etc.
- Bubble sort and insertion sort take roughly $n^2$ comparisons while quick sort (only on average) and merge sort take roughly $n \log_2 n$ comparisons.
  - “Roughly” hides potentially large constants, e.g., running time of merge sort may in reality be $100n \log_2 n$. 

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How can make statements such as the following, in order to compare the running times of different algorithms?

▶ $100n \log_2 n \leq n^2$
▶ $10000n \leq n^2$
▶ $5n^2 - 4n \geq 1000n \log_2 n$
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10000n vs. O(n^2)
Upper Bound

Definition

Asymptotic upper bound: A function $f(n)$ is $O(g(n))$ if for all $n$, $f(n) \leq c \cdot g(n)$.

Graph showing $10000n$ is $O(n^2)$.
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- $10000n$ is $O(n^2)$, $c = 1$, $n_0 = 10000$. 

![Graph showing $10000n$ and $n^2$ as functions of $n$.]
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$100n \log_2 n$ and $n^2$

$100n \log_2 n$ is $O(n^2)$, $c = 100$, $n_0 = 1$
Lower Bound

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Asymptotic lower bound: A function \( f(n) \) is \( \Omega(g(n)) \) if for all \( n \), we have \( f(n) \geq c \cdot g(n) \).

\[ n \log_2 n / 10 \text{ is } \Omega(n), \quad c = 1, \quad n_0 = 1024 \]
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Meaning of “Lower Bound” in Different Contexts

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- Mathematical functions: $n$ is a lower bound for $n \log n/10$, i.e., $n \log n/10 = \Omega(n)$. 

Algorithms: The lower bound on the running time of bubble sort is $\Omega(n^2)$.

Problems: The problem of sorting $n$ numbers has a lower bound of $\Omega(n \log n)$. For any comparison-based sorting algorithm, there is at least one input for which that algorithm will take $\Omega(n \log n)$ steps.
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- Problems: The problem of sorting $n$ numbers has a lower bound of $\Omega(n \log n)$. For any comparison-based sorting algorithm, there is at least one input for which that algorithm will take $\Omega(n \log n)$ steps.
Tight Bound

**Definition**

Asymptotic tight bound: A function $f(n)$ is $\Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$. 

[Table example: Insert table here]
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- In all these definitions, $c$ and $n_0$ are constants independent of $n$.
- Abuse of notation: say $g(n) = O(f(n))$, $g(n) = \Omega(f(n))$, $g(n) = \Theta(f(n))$. 
Properties of Asymptotic Growth Rates

**Transitivity**
- If \( f = O(g) \) and \( g = O(h) \), then \( f = O(h) \).
- If \( f = \Omega(g) \) and \( g = \Omega(h) \), then \( f = \Omega(h) \).
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- Similar statements hold for lower and tight bounds.
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- If \( f = O(g) \), then \( f + g = \Theta(g) \).
Examples

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  - $O(n^d)$ is the definition of *polynomial time*. 
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- Is an algorithm with running time $O(n^{1.59})$ a polynomial-time algorithm?
- $O(\log_a n) = O(\log_b n)$ for any pair of constants $a, b > 1$.
- For every constant $x > 0$, $\log n = O(n^x)$.
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- For every constant $x > 0$, $\log n = O(n^x)$.
- For every constant $r > 1$ and every constant $d > 0$, $n^d = O(r^n)$. 
Different functions of $n$

- $n$
- $n \log n$
- $n^2$
- $n^3$
- $2^n$

Graph showing the growth of different functions of $n$ with $x$-axis representing $n$ and $y$-axis representing values up to 2000.
More functions of $n$

- $n$
- $\log_2 n$
- $\log_3 n$
- $n^{0.5}$
**Linear Time**

- Running time is at most a constant factor times the size of the input.
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- Sub-linear time.
- Running time is at most a constant factor times the size of the input.
- Finding the minimum, merging two sorted lists.
- Computing the median (or kth smallest) element in an unsorted list. “Median-of-medians” algorithm.
- Sub-linear time. Binary search in a sorted array of n numbers takes $O(\log n)$ time.
Any algorithm where the costliest step is sorting.
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Given a set of $n$ points in the plane, find the pair that are the closest.
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- Given a set of $n$ points in the plane, find the pair that are the closest. Surprising fact: will solve this problem in $O(n \log n)$ time later in the semester.
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Algorithm: For each subset \( S \) of \( k \) nodes, check if \( S \) is a clique. If the answer is yes, report it.
Does a graph have a *clique* of size $k$, where $k$ is a constant, i.e. there are $k$ nodes such that every pair is connected by an edge?

**Algorithm:** For each subset $S$ of $k$ nodes, check if $S$ is a clique. If the answer is yes, report it.

**Running time** is $O(k^2 \binom{n}{k}) = O(n^k)$.
What is the largest size of a clique in a graph with $n$ nodes?

AlGORITHM: For each $1 \leq i \leq n$, check if the graph has a clique of size $i$. Output largest clique found.
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What is the running time?

\[ O(n^2) \]
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What is the running time? $O(n^2 2^n)$. 