Applications of Network Flow

T. M. Murali

October 29, 31, November 5, 2018

Maximum Flow and Minimum Cut

- Two rich algorithmic problems.
- Fundamental problems in combinatorial optimization.
- Beautiful mathematical duality between flows and cuts.
- Numerous non-trivial applications:
  - Bipartite matching.
  - Network connectivity.
  - Data mining.
  - Project selection.
  - Airline scheduling.
  - Baseball elimination.
  - Image segmentation.
  - Open-pit mining.
  - Network reliability.
  - Distributed computing.
  - Egalitarian stable matching.
  - Security of statistical data.
  - Network intrusion detection.
  - Multi-camera scene reconstruction.
  - Gene function prediction.

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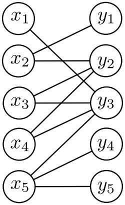
  - Gene function prediction.
- We will only sketch proofs. Read details from the textbook.

Matching in Bipartite Graphs



- *Bipartite Graph*: a graph $G(V, E)$ where
  1. $V = X \cup Y$, X and Y are disjoint and
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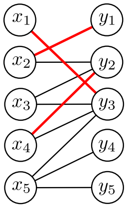

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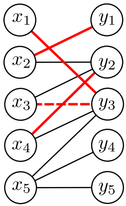

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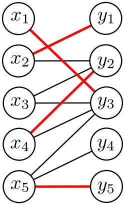

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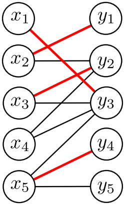

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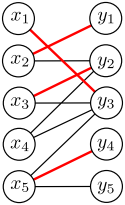

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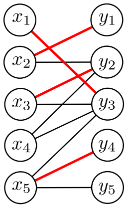

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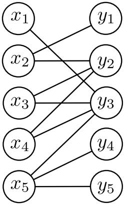
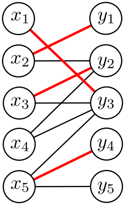
  - The graph in the figure does not have a perfect matching because both y_4 and y_5 are adjacent only to x_5.

Bipartite Graph Matching Problem



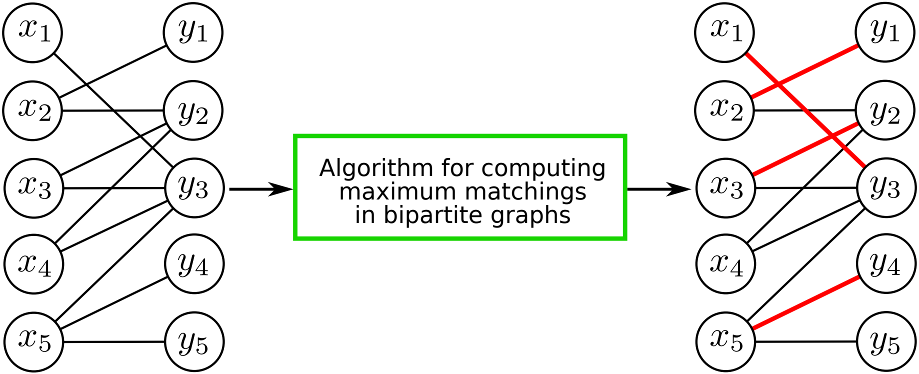


BIPARTITE MATCHING

INSTANCE: A Bipartite graph G.

SOLUTION: The matching of largest size in G.

Normal Approach for Solving a Problem



- Develop algorithm for computing maximum matchings in bipartite graphs.
- Prove that the algorithm is correct, i.e., for every possible input, it compute the size of the largest matching in the bipartite graph accurately.
- Analyze running time of the algorithm.

Alternative Approach for Solving a Problem

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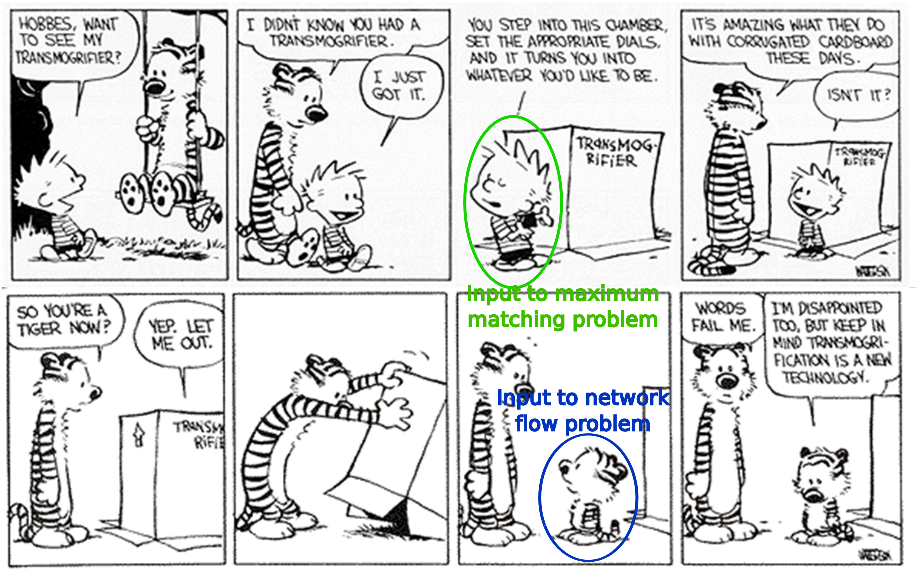

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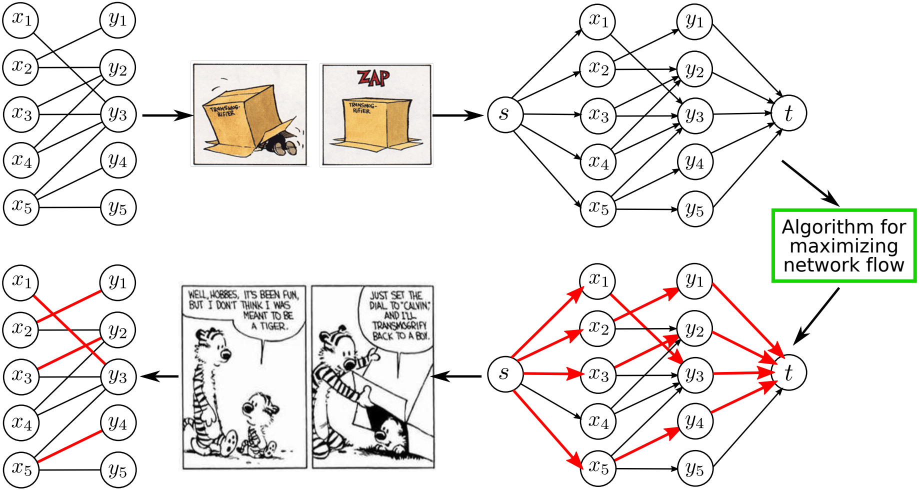
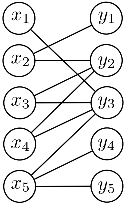
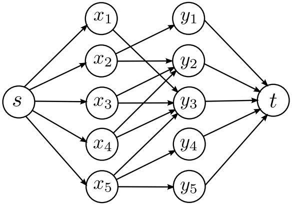


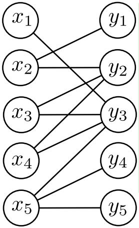
Algorithm for Bipartite Graph Matching



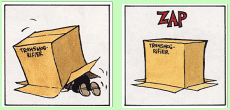


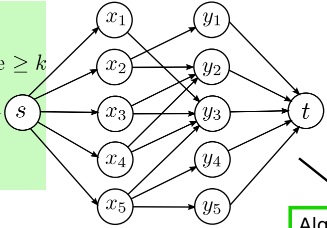
- Convert G to a flow network G': direct edges from X to Y, add nodes s and t, connect s to each node in X, connect each node in Y to t, set all edge capacities to 1.
- Compute the maximum flow in G'.
- Convert the maximum flow in G' into a matching in G.
- Claim: the value of the maximum flow in G' is the size of the maximum matching in G.
- In general, there is matching with size k in G if and only if there is a (integer-valued) flow of value k in G'.

Strategy for Proving Correctness

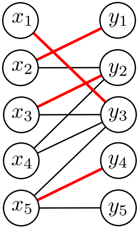


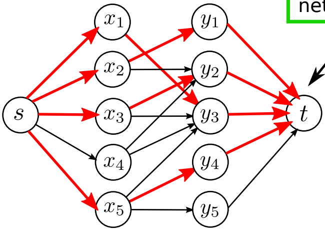
Matching $\Rightarrow$ flow:
If G has a matching with k edges,
then G' must have a flow with value $\geq k$





Algorithm for maximizing network flow





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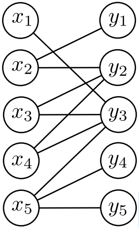

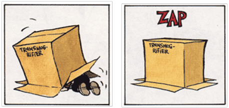

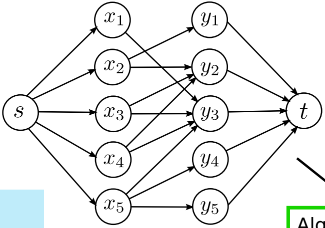

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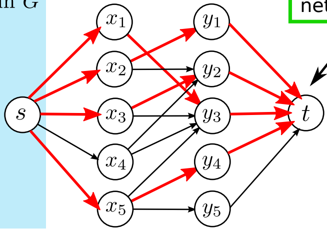
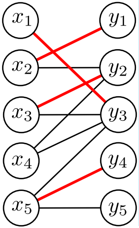


Flow $\Rightarrow$ matching:
If G' has a flow of size k,
then we can construct a matching in G with k edges



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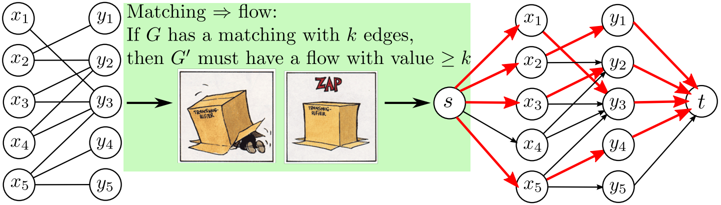

- Matching $\Rightarrow$ flow: if there is a matching with k edges in G, there is an s-t flow of value k in G'.

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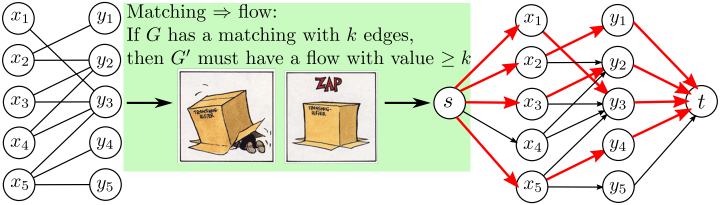

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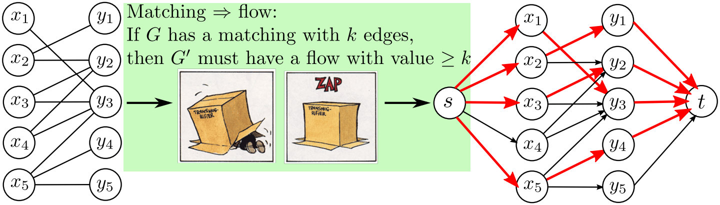

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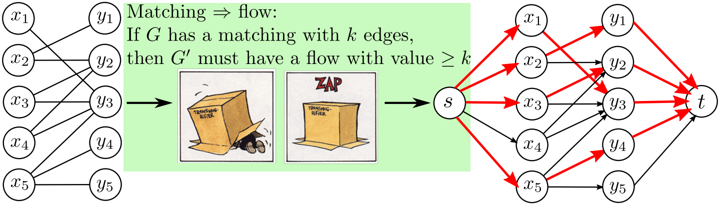

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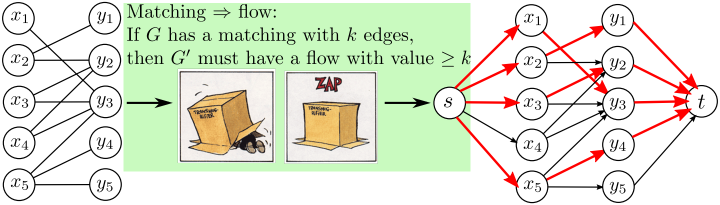

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  - Conservation constraint:

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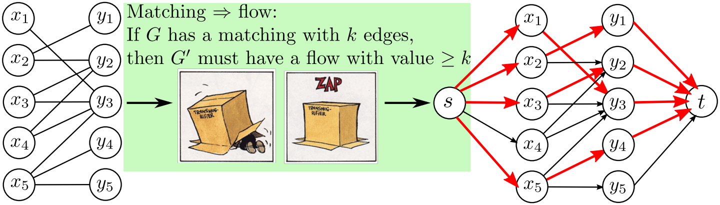

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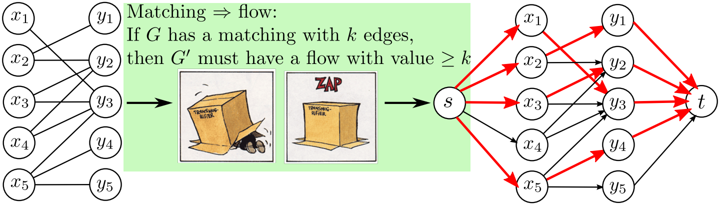

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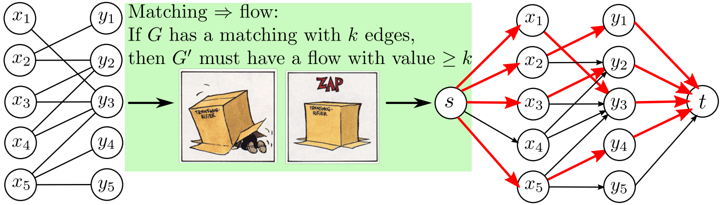

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- Why have we constructed a flow?
  - Capacity constraint: No edge receives a flow > 1 because we started with a matching.
  - Conservation constraint: Every node other than s and t has one incoming unit and one outgoing unit of flow because we started with a matching.
- What is the value of the flow? k, since exactly that many nodes out of s carry flow.

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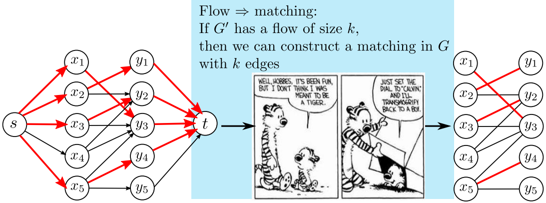

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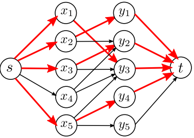

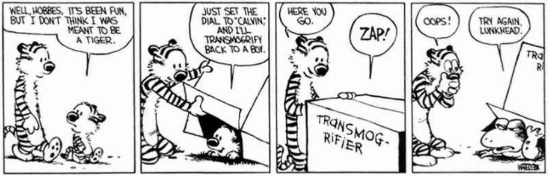

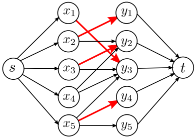

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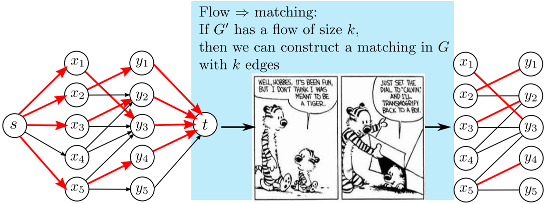

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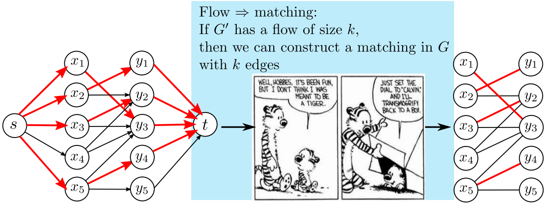

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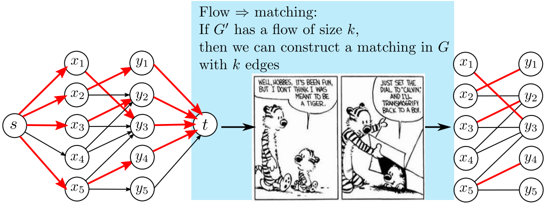

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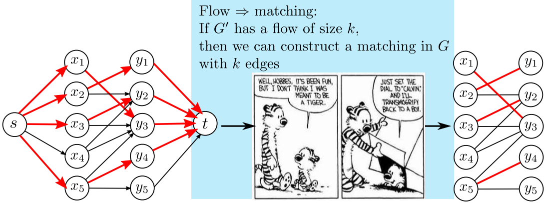

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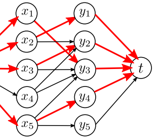

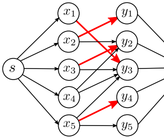

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  - Let M be the set of edges not incident on s or t with flow equal to 1.
  - Claim: M contains k edges.
  - Claim: Each node in X (respectively, Y) is the tail (respectively, head) of at most one edge in M.
- Conclusion: size of the maximum matching in G is equal to the value of the maximum flow in G'; the edges in this matching are those that carry flow from X to Y in G'.

Correctness of Bipartite Graph Matching Algorithm

- Flow $\Rightarrow$ matching: if there is a flow f' in G' with value k, there is a matching M in G with k edges.
  - There is an integer-valued flow f' of value k $\Rightarrow$ flow along any edge is 0 or 1.
  - Let M be the set of edges not incident on s or t with flow equal to 1.
  - Claim: M contains k edges.
  - Claim: Each node in X (respectively, Y) is the tail (respectively, head) of at most one edge in M.
- Conclusion: size of the maximum matching in G is equal to the value of the maximum flow in G'; the edges in this matching are those that carry flow from X to Y in G'.
- Read the book on what augmenting paths mean in this context.

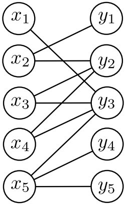
Running time of Bipartite Graph Matching Algorithm

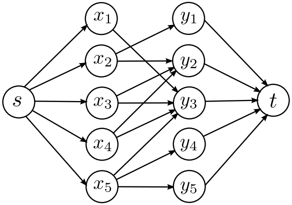
- Suppose G has m edges and n nodes in X and in Y.

Running time of Bipartite Graph Matching Algorithm

- Suppose G has m edges and n nodes in X and in Y.
- $C \leq n$.
- Ford-Fulkerson algorithm runs in $O(mn)$ time.
- Scaling algorithm takes $O(m^2)$ time ($C = 1$ for this algorithm).
- Edmonds-Karp algorithm takes $O(m^2 n)$ time.

Bipartite Graphs without Perfect Matchings





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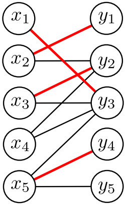

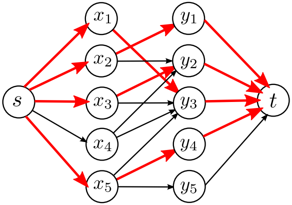

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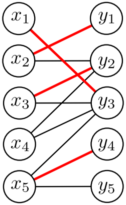
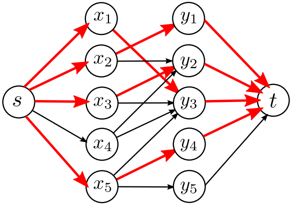
Bipartite Graphs without Perfect Matchings





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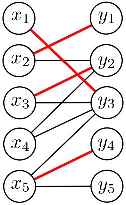
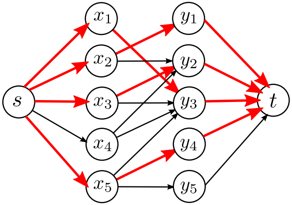
Bipartite Graphs without Perfect Matchings





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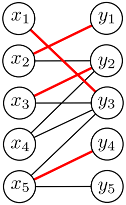
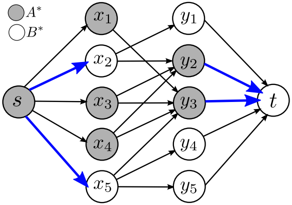


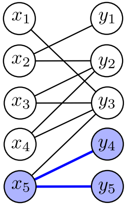


- How do we determine if a bipartite graph G has a perfect matching? Find the maximum matching and check if it is perfect.
- Suppose G has no perfect matching. Can we exhibit a short "certificate" of that fact? What can such certificates look like?
- G has no perfect matching iff there is a cut in G' with capacity less than n. Therefore, the cut is a certificate.

Bipartite Graphs without Perfect Matchings

- We would like the certificate in terms of G.

Bipartite Graphs without Perfect Matchings



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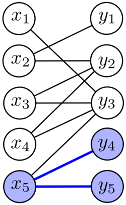
Bipartite Graphs without Perfect Matchings



- We would like the certificate in terms of G.
  - For example, two nodes in Y with one incident edge each with the same neighbour in X.
  - Generally, a subset $A \subseteq X$ with neighbours $\Gamma(A) \subseteq Y$, such that $|A| > |\Gamma(A)|$.
- Hall's Theorem: Let $G(X \cup Y, E)$ be a bipartite graph such that $|X| = |Y|$. Then G either has a perfect matching or there is a subset $A \subseteq Y$ such that $|A| > |\Gamma(A)|$. We can compute a perfect matching or such a subset in $O(mn)$ time.

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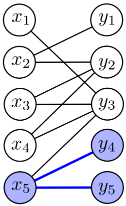

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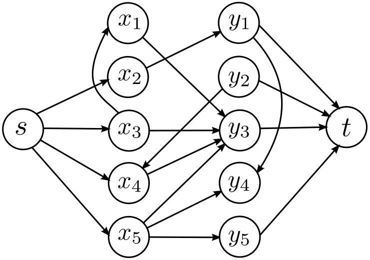
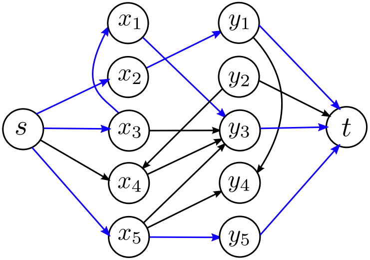
Edge-Disjoint Paths





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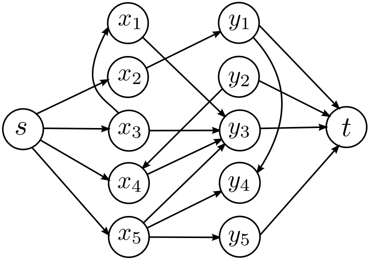

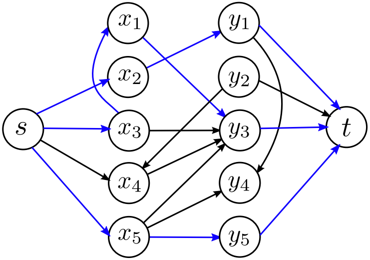

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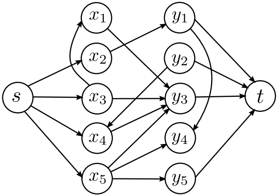
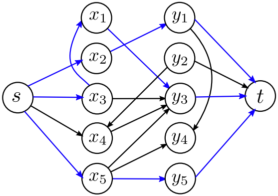
DIRECTED EDGE-DISJOINT PATHS

INSTANCE: Directed graph $G(V, E)$ with two distinguished nodes s and t.

SOLUTION: The maximum number of edge-disjoint paths between s and t.

Mapping to the Max-Flow Problem





- Convert G into a flow network:

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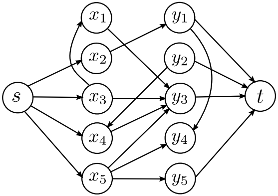

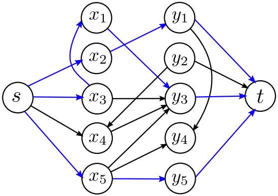

- Convert G into a flow network: s is the source, t is the sink, each edge has capacity 1.
- Claim: There are k edge-disjoint paths from s to t in a directed graph G if and only if there is a s-t flow in G with value $\geq k$.

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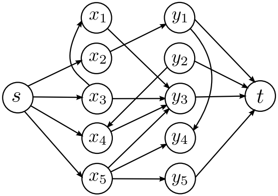

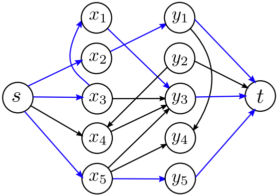

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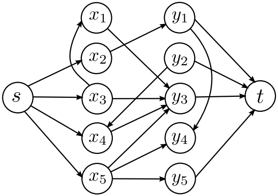

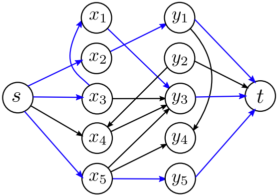

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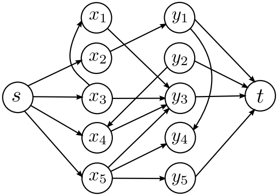

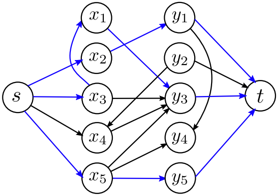

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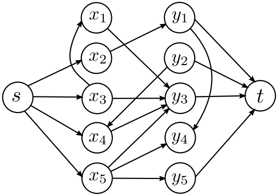

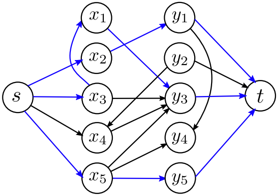

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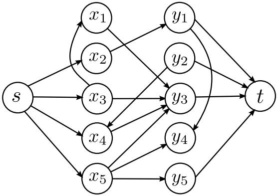

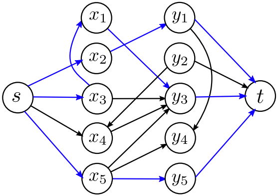

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  - Claim: if f is a 0-1 valued flow of value $\nu(f) = k$, then the set of edges with flow $f(e) = 1$ contains a set of k edge-disjoint paths.

Completing the Proof

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  Base case: $\nu = 0$. Nothing to prove.

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  Inductive step: Construct a set of k s-t paths from f. Work out on the board.
- Note: Formulating the inductive hypothesis precisely can be tricky.
- Strategy is to try to prove the inductive step first.
- During this proof, you will observe two types of "smaller" flows:
  (i) When you succeed in finding an s-t path, you get a new flow f' that is smaller, i.e., $\nu(f') < k$ carrying flow on fewer edges, i.e., $\kappa(f') < \kappa(f)$.
  (ii) When you run into a cycle, you get a new flow f' with $\nu(f') = k$ but carrying flow on fewer edges, i.e., $\kappa(f') < \kappa(f)$ edges.
- You can combine both situations in the inductive hypothesis.

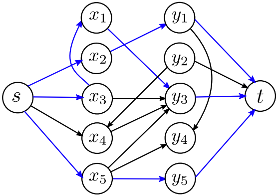
Running Time of the Edge-Disjoint Paths Algorithm

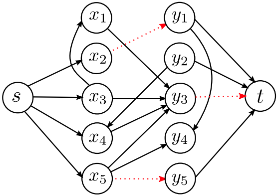
- Given a flow of value k, how quickly can we determine the k edge-disjoint paths?

Running Time of the Edge-Disjoint Paths Algorithm

- Given a flow of value k, how quickly can we determine the k edge-disjoint paths? $O(mn)$ time.
- Corollary: The Ford-Fulkerson algorithm can be used to find a maximum set of edge-disjoint s-t paths in a directed graph G in $O(mn)$ time.

Certificate for Edge-Disjoint Paths Algorithm





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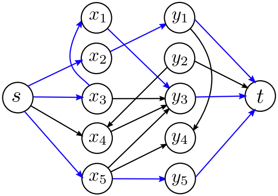
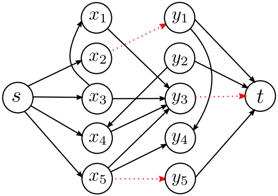
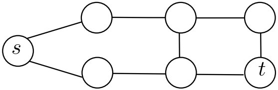




- A set $F \subseteq E$ of edge separates s and t if the graph $(V, E - F)$ contains no s-t paths.
- Menger's Theorem: In every directed graph with nodes s and t, the maximum number of edge-disjoint s-t paths is equal to the minimum number of edges whose removal disconnects s from t.

Edge-Disjoint Paths in Undirected Graphs



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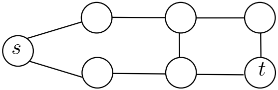

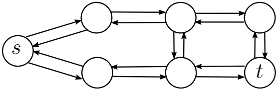

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- Replace each edge with two directed edges of capacity 1 and apply the algorithm for directed graphs.

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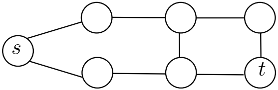

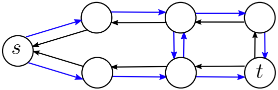

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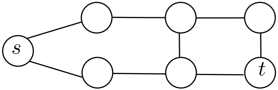

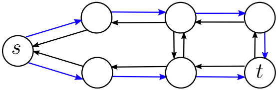

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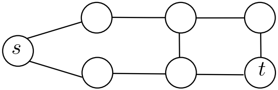

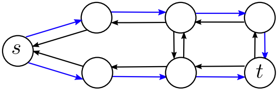

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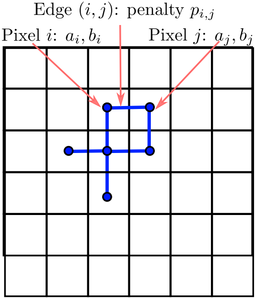
- Can obtain an integral flow where only one of the directed counterparts of (u, v) has non-zero flow.
- We can find the maximum number of edge-disjoint paths in $O(mn)$ time.
- We can prove a version of Menger's theorem for undirected graphs: in every undirected graph with nodes s and t, the maximum number of edge-disjoint s–t paths is equal to the minimum number of edges whose removal separates s from t.

Image Segmentation

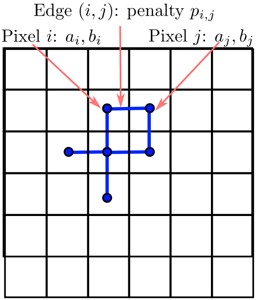
- A fundamental problem in computer vision is that of segmenting an image into coherent regions.
- A basic segmentation problem is that of partitioning an image into a foreground and a background: label each pixel in the image as belonging to the foreground or the background.
  - Note that the image on the right shows segmentation into multiple regions but we are interested in the segmentation into two regions.

Formulating the Image Segmentation Problem



- Let V be the set of pixels in an image.
- Let E be the set of pairs of neighbouring pixels.
- V and E yield an undirected graph $G(V, E)$.

Formulating the Image Segmentation Problem



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- These likelihoods are specified in the input to the problem.

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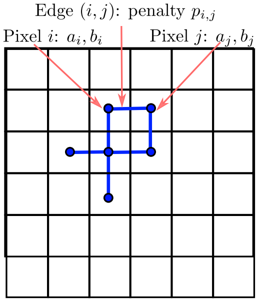

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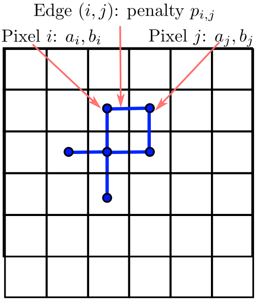

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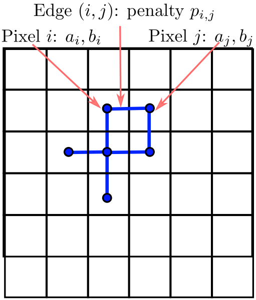
- These likelihoods are specified in the input to the problem.
- We want the foreground/background boundary to be smooth: For each pair (i, j) of pixels, there is a separation penalty $p_{ij} \geq 0$ for placing one of them in the foreground and the other in the background.

The Image Segmentation Problem



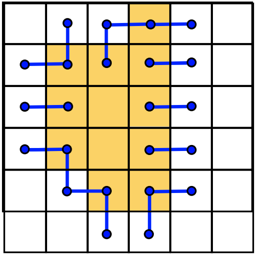


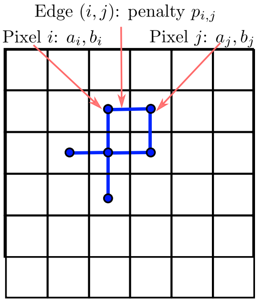
IMAGE SEGMENTATION

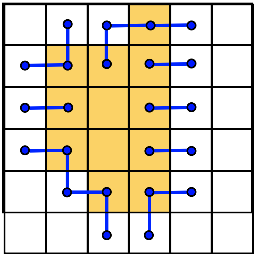
INSTANCE: Pixel graphs $G(V, E)$, likelihood functions $a, b : V \to \mathbb{R}^+$, penalty function $p : E \to \mathbb{R}^+$

SOLUTION: *Optimum labelling*: partition of the pixels into two sets A and B that maximises

$$q(A, B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

Developing an Algorithm for Image Segmentation





$$q(A, B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

- There is a similarity between cuts and labellings.
- But there are differences:
  - We are maximising an objective function rather than minimising it.
  - There is no source or sink in the segmentation problem.
  - We have values on the nodes.
  - The graph is undirected.

Maximization to Minimization

- Let $Q = \sum_i (a_i + b_i)$.

Maximization to Minimization

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- Notice that $\sum_{i \in A} a_i + \sum_{j \in B} b_j = Q - \sum_{i \in A} b_i - \sum_{j \in B} a_j$.
- Therefore, maximising

$$\begin{aligned} q(A, B) &= \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cup \{i,j\}|=1}} p_{ij} \\ &= Q - \sum_{i \in A} b_i - \sum_{j \in B} a_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1}} p_{ij} \end{aligned}$$

is identical to minimising

$$q'(A, B) = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1}} p_{ij}$$

Solving the Other Issues

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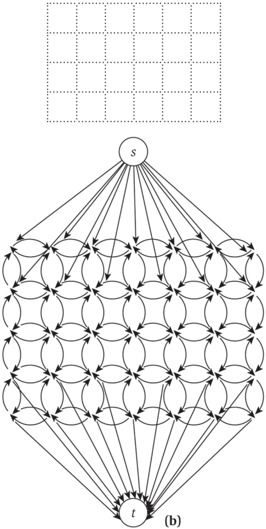
- Add a new "super-source" s to represent the foreground.
- Add a new "super-sink" t to represent the background.
- Connect s and t to every pixel and assign capacity a_i to edge (s, i) and capacity b_i to edge (i, t).
- Direct edges away from s and into t.
- Replace each edge (i, j) in E with two directed edges of capacity p_{ij}.



(b)

Cuts in the Flow Network

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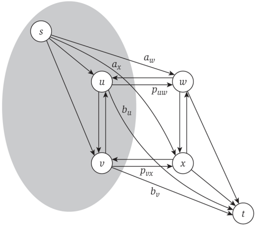

Figure 7.19 An s-t cut on a graph constructed from four pixels. Note how the three types of terms in the expression for $q'(A, B)$ are captured by the cut.

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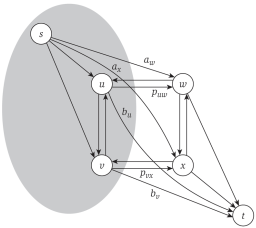
  - $(u, w), u \in A, w \in B$ contributes p_{uw}.



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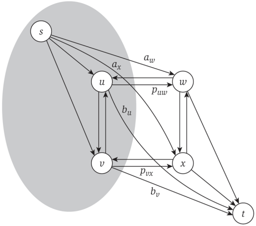

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$$c(A, B) \quad = \quad \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}|=1}} p_{ij} = q'(A, B).$$

Solving the Image Segmentation Problem

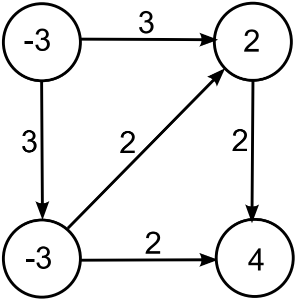
- The capacity of a s-t cut $c(A, B)$ exactly measures the quantity $q'(A, B)$.
- To maximise $q(A, B)$, we simply compute the s-t cut (A, B) of minimum capacity.
- Deleting s and t from the cut yields the desired segmentation of the image.

Extension of Max-Flow Problem

- Suppose we have a set S of multiple sources and a set T of multiple sinks.
- Each source can send flow to any sink.
- Let us not maximise flow here but formulate the problem in terms of demands and supplies.

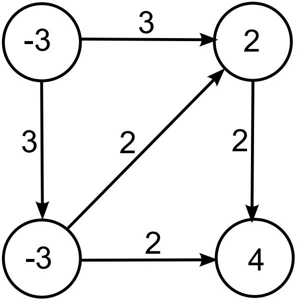
Circulation with Demands

- We are given a graph $G(V, E)$ with capacity function $c : E \rightarrow \mathbb{Z}^+$ and a demand function $d : V \rightarrow \mathbb{Z}$:



Circulation with Demands

- We are given a graph $G(V, E)$ with capacity function $c : E \rightarrow \mathbb{Z}^+$ and a demand function $d : V \rightarrow \mathbb{Z}$:
  - $d_v > 0$: node is a sink, it has a "demand" for d_v units of flow.
  - $d_v < 0$: node is a source, it has a "supply" of $-d_v$ units of flow.
  - $d_v = 0$: node simply receives and transmits flow.



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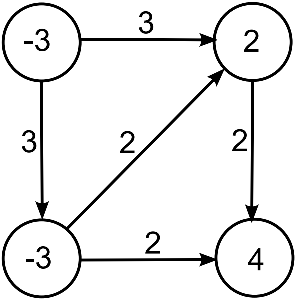

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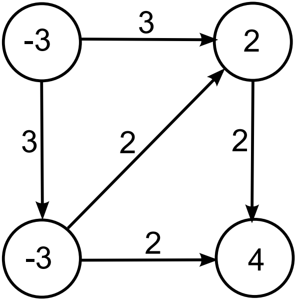

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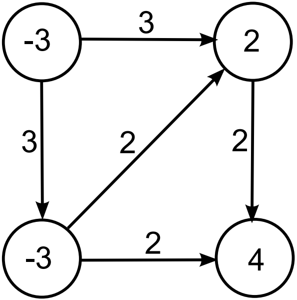

- A *circulation* with demands is a function $f : E \rightarrow \mathbb{R}^+$ that satisfies
  1. (*Capacity conditions*) For each $e \in E$, $0 \leq f(e) \leq c(e)$.
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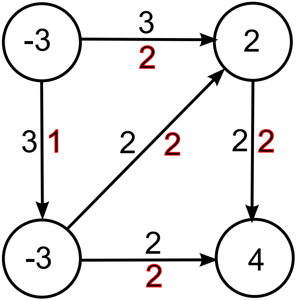

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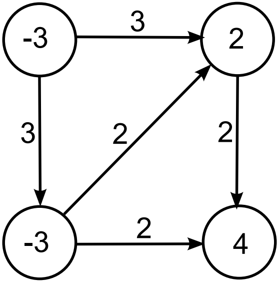
CIRCULATION WITH DEMANDS

INSTANCE: A directed graph $G(V, E)$, $c : E \to \mathbb{Z}^+$, and $d : V \to \mathbb{Z}$.

SOLUTION: Does a *feasible* circulation exist, i.e., it meets the capacity and demand conditions?

Properties of Feasible Circulations



- Claim: if there exists a feasible circulation with demands, then $\sum_v d_v = 0$.

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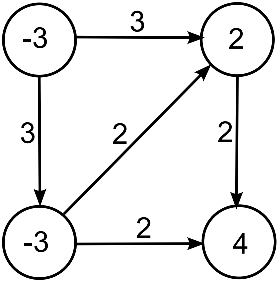


- Claim: if there exists a feasible circulation with demands, then $\sum_v d_v = 0$.
- Corollary: $\sum_{v, d_v > 0} d_v = \sum_{v, d_v < 0} -d_v$. Let D denote this common value.

Mapping Circulation to Maximum Flow

- Create a new graph $G' = G$ and
  1. create two new nodes in G': a source s^* and a sink t^*;
  2. connect s^* to each node v in S using an edge with capacity $-d_v$;
  3. connect each node v in T to t^* using an edge with capacity d_v.

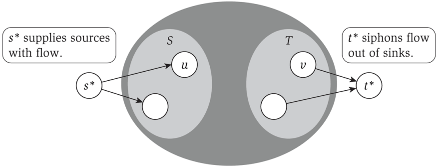
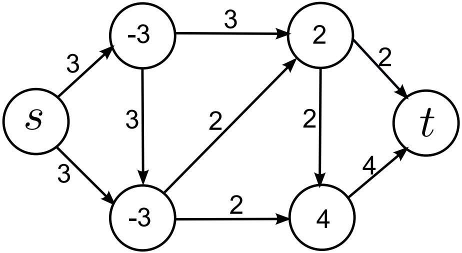
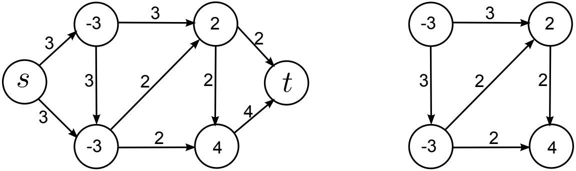


Figure 7.14 Reducing the Circulation Problem to the Maximum-Flow Problem.



Computing a Feasible Circulation



- We will look for a maximum s^*-t^* flow f in G'; $\nu(f)$

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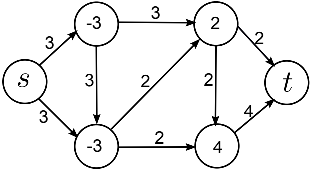
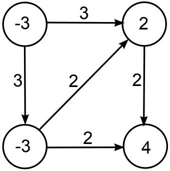
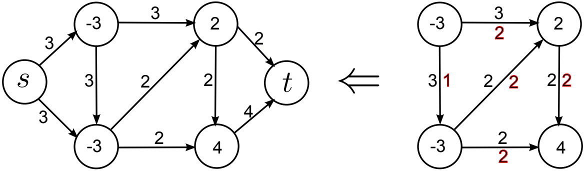




- We will look for a maximum s^*-t^* flow f in G'; $\nu(f) \leq D$.

Computing a Feasible Circulation



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- Circulation $\Rightarrow$ flow.

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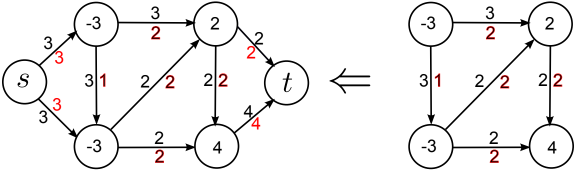
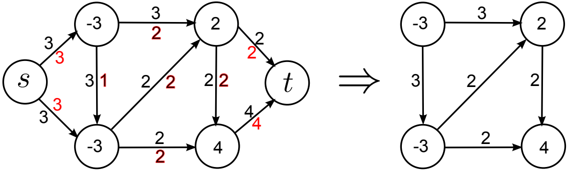


- We will look for a maximum s^*-t^* flow f in G'; $\nu(f) \leq D$.
- Circulation $\Rightarrow$ flow. If there is a feasible circulation, we send $-d_v$ units of flow along each edge (s^*, v) and d_v units of flow along each edge (v, t^*). The value of this flow is D. (Prove it yourself.)

Computing a Feasible Circulation



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- Flow $\Rightarrow$ circulation. If there is an s^*-t^* flow of value D in G',

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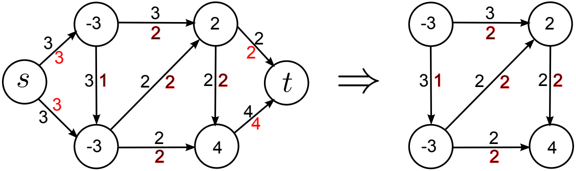

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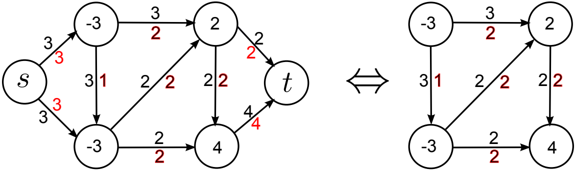
Computing a Feasible Circulation



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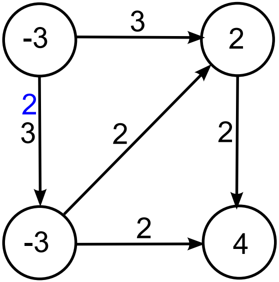
- Flow $\Rightarrow$ circulation. If there is an s^*-t^* flow of value D in G', edges incident on s^* and on t^* must be saturated with flow. Deleting these edges from G' yields a feasible circulation in G. (Prove it yourself.)
- We have proved that there is a feasible circulation with demands in G iff the maximum s^*-t^* flow in G' has value D.

Circulation with Demands and Lower Bounds



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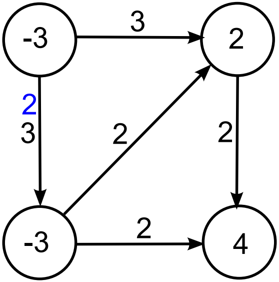
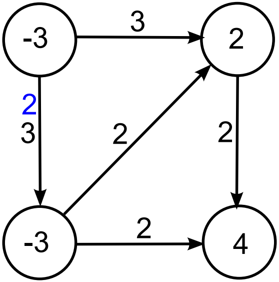


- We want to force the flow to use certain edges.
- We are given a graph $G(V, E)$ with a capacity $c(e)$ and a lower bound $0 \leq l(e) \leq c(e)$ on each edge and a demand d_v on each vertex.

Circulation with Demands and Lower Bounds



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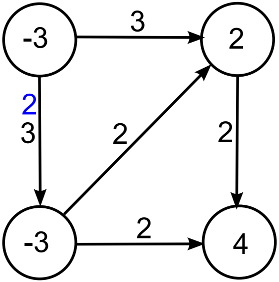

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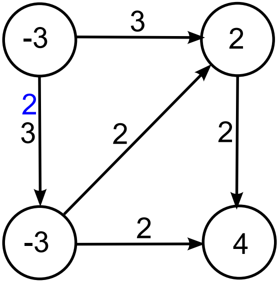
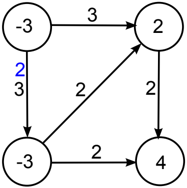
Circulation with Demands and Lower Bounds



- We want to force the flow to use certain edges.
- We are given a graph $G(V, E)$ with a capacity $c(e)$ and a lower bound $0 \leq l(e) \leq c(e)$ on each edge and a demand d_v on each vertex.
- A *circulation* with demands and lower bounds is a function $f : E \rightarrow \mathbb{R}^+$ that satisfies
  1. (*Capacity conditions*) For each $e \in E$, $l(e) \leq f(e) \leq c(e)$.
  2. (*Demand conditions*) For each node v, $f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$.
- Problem we want to solve: Is there a feasible circulation?

Algorithm for Circulation with Lower Bounds



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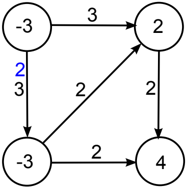

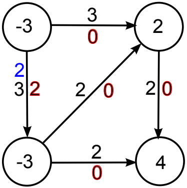

- Strategy is to reduce the problem to one with no lower bounds on edges.
- Suppose we define a circulation f_0 that satisfies lower bounds on all edges, i.e., set $f_0(e) = l(e)$ for all $e \in E$. What can go wrong?

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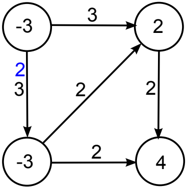

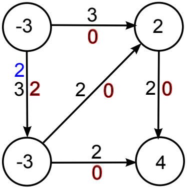

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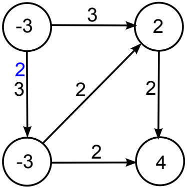

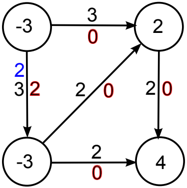

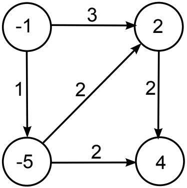

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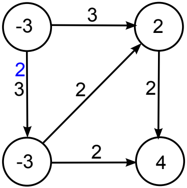
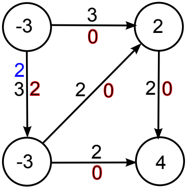
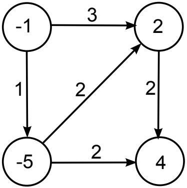
- If $L_v \neq d_v$ for every node, let us try to superimpose a circulation f_1 on top of f_0 such that $f_1^{\text{in}}(v) - f_1^{\text{out}}(v) = d_v - L_v$.

Algorithm for Circulation with Lower Bounds







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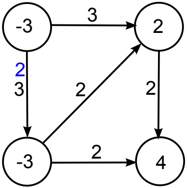
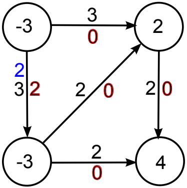
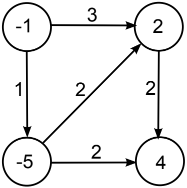
Algorithm for Circulation with Lower Bounds







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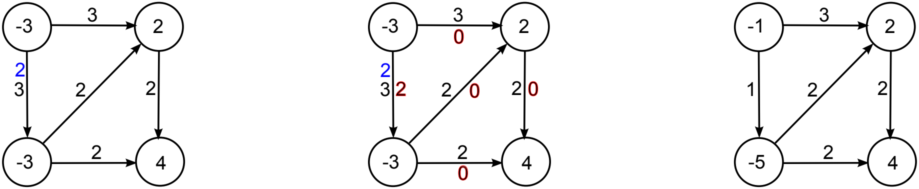
Algorithm for Circulation with Lower Bounds



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- How much capacity do we have left on each edge? $c(e) - l(e)$.
- Approach: define a new graph G' with the same nodes and edges: each edge e has lower bound 0, capacity $c(e) - l(e)$; demand of each node v is $d_v - L_v$.
- Claim: there is a feasible circulation in G iff there is a feasible circulation in G'. Read the proof in the textbook.

Airline Scheduling

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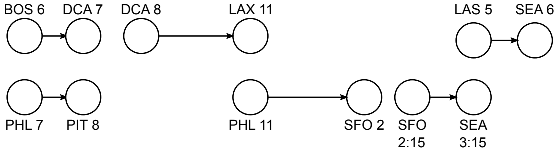
- Produce schedules for thousands of routes.
- Make these schedules efficient in terms of crew allocation, equipment usage, fuel costs, customer satisfaction, etc.
- Modelling these problems realistically is out of the scope of the course.
- We will focus on a "toy" problem that cleanly captures some of the resource allocation problems they have to deal with.

Creating Flight Schedules



- Desire to serve m specific flight segments.
- Each flight segment (or flight) specified by four parameters: origin airport, destination airport, departure time, arrival time.

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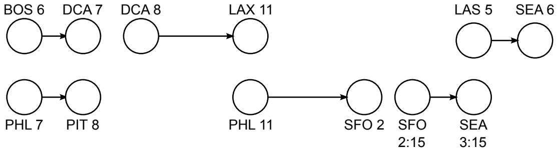
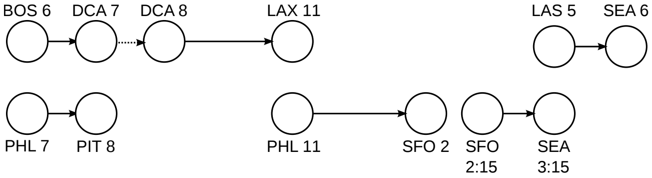


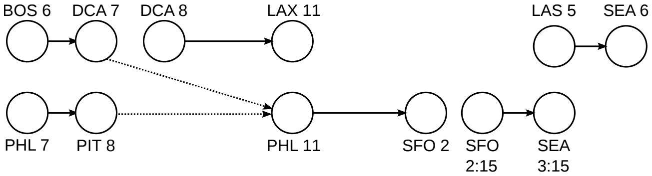
- Desire to serve m specific flight segments.
- Each flight segment (or flight) specified by four parameters: origin airport, destination airport, departure time, arrival time.
- We can use a single plane for flight i and later for flight j if
  (i) the destination of i is the same as the origin of j and there is enough time to perform maintenance on the plane between the two flights, or
  (ii) we can add a flight that takes the plane from the destination of i to the origin of j with enough time for maintenance.
- Goal is to schedule all m flights using at most k planes.

Reachability



- Flight j is *reachable* from flight i if the same plane can be used for both flights subject to the constraints described earlier.

Reachability



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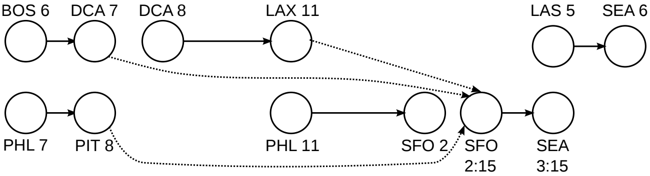

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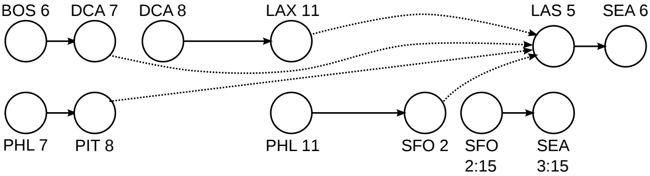

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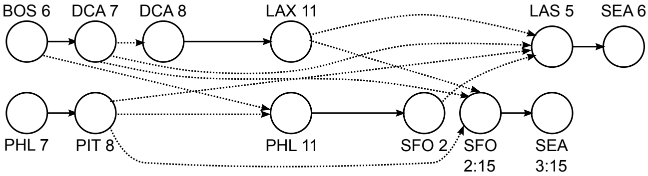

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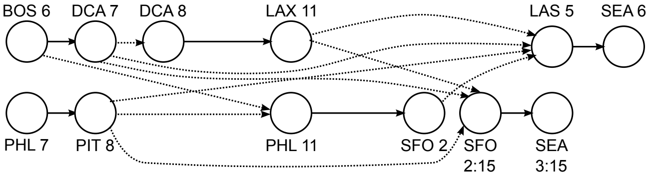
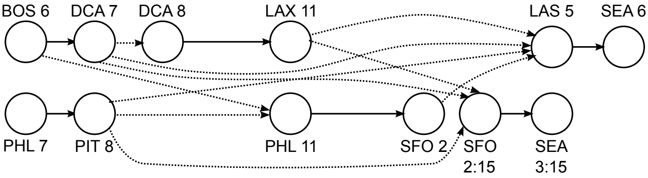


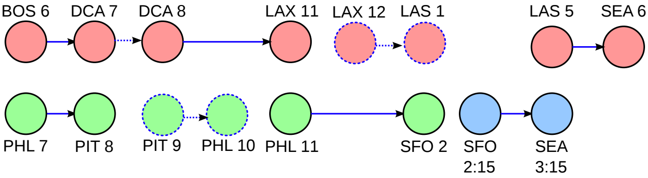
- Flight j is *reachable* from flight i if the same plane can be used for both flights subject to the constraints described earlier.
- Assume input includes pairs (i, j) of reachable flights, i.e., in each pair j is reachable from i.
  - Pairs form a DAG.
  - *Flights* are reachable from one another, not *airports*.
  - Construction of reachable pairs will take maintenance time into account.
  - Definition of reachability can be more complex; input pairs can encode this complexity.

The Airline Scheduling Problem



Airline Scheduling
INSTANCE: Set S of m flight segments (u_i, v_i), $1 \leq i \leq m$, a set R of reachable pairs of flights (i, j), $1 \leq i, j \leq m$, and an integer bound k
SOLUTION:

The Airline Scheduling Problem



The dotted circles are meant only to illustrate the new flights added.

AIRLINE SCHEDULING

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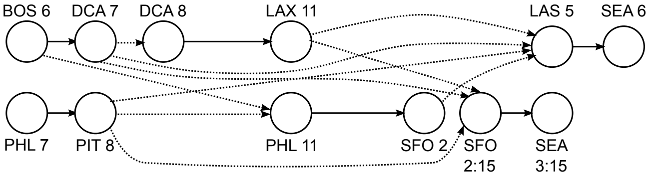
SOLUTION: *Feasible scheduling*:

(a) Set T of $n \geq 0$ new flight segments (u_j, v_j), $1 \leq j \leq n$ and

(b) A partition of $S \cup T$ into at most k sequences such that in each sequence, flight i is reachable from flight $i-1$, for all $1 < i \leq l$, where l is the length of the sequence.

The Airline Scheduling Problem



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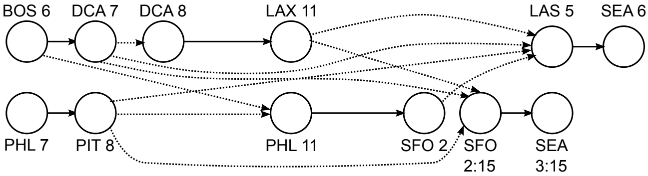

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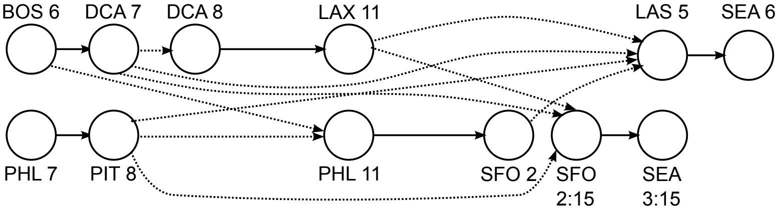
- Where are flight departure and arrival times in the input? In a flight segment, u_i specifies both origin airport and departure time; v_i specifies both arrival airport and arrival time.

Intuition Underlying Algorithm



- Nodes in the flow network are airports.
- Planes correspond to units of flow.

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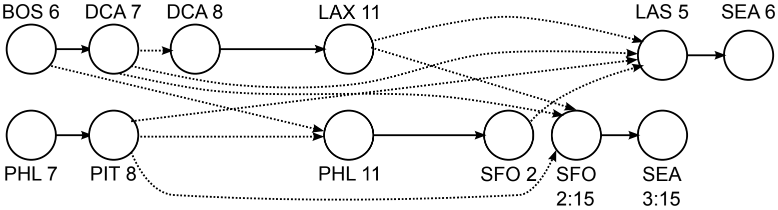

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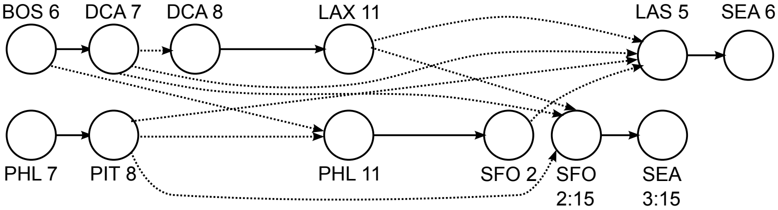

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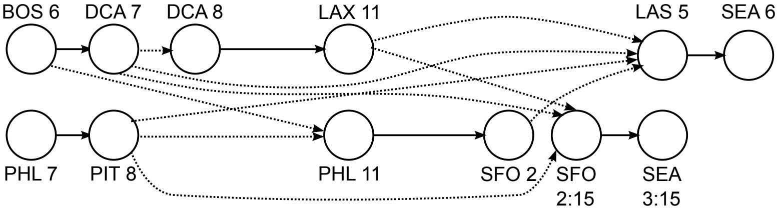

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- How do we represent reachability? If (i, j) is a reachable pair, there is an edge from v_i to u_j with lower bound of 0 and a capacity of 1.

Designing the Flow Network

Nodes:

- For each flight i, graph G has two nodes u_i and v_i.
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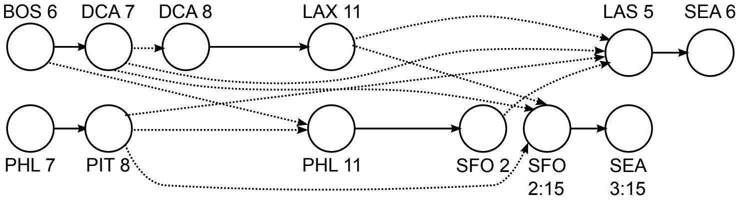
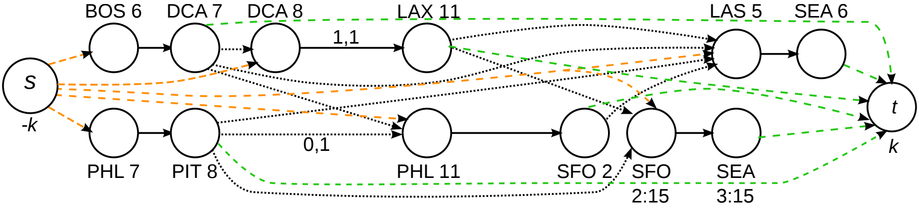
- Excess planes: G contains an edge directed from s to t with lower bound 0 and capacity k.

Demands: Node s has demand $-k$, node t has demand k, all other nodes have demand 0.

Goal: Compute whether G has a feasible circulation.

Example of Circulation Formulation



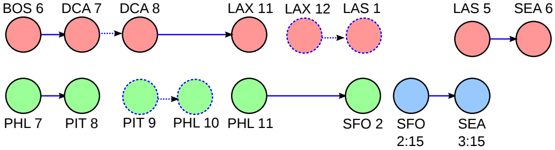


The image does not show the edge between s and t.

Proof of Correctness: Part 1

- Claim: We can schedule all flights in S using at most k planes iff G has a feasible circulation.

Proof of Correctness: Part 1



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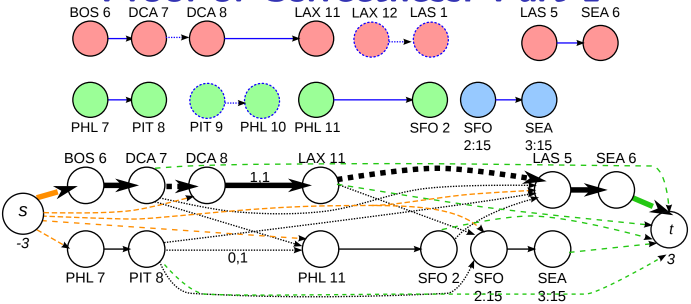

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  - Send one unit of flow along the edges of that path P_l and along the edges (s, s_l) and (t_l, t).

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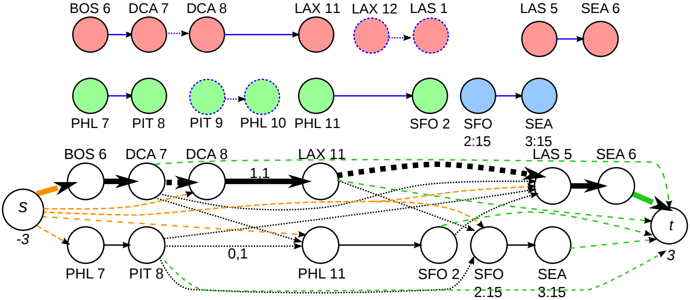

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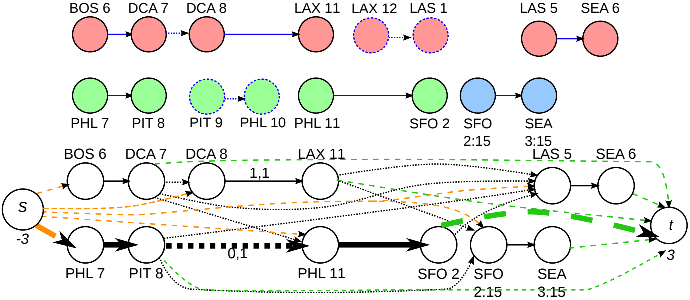

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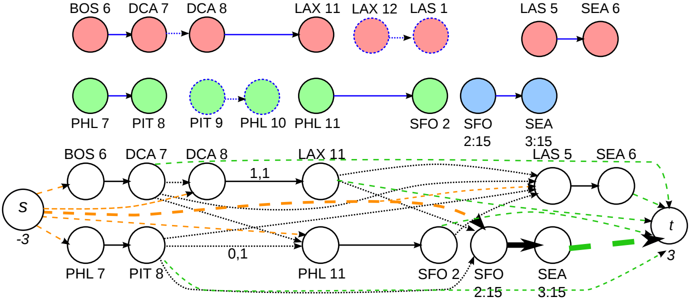

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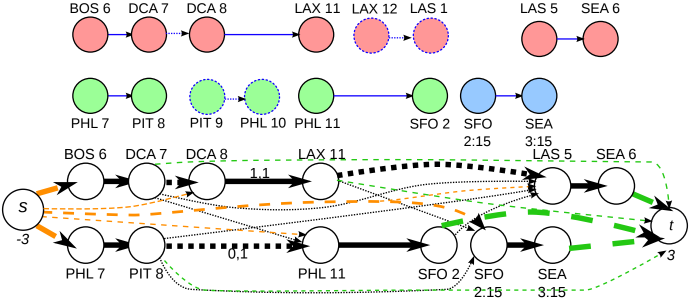

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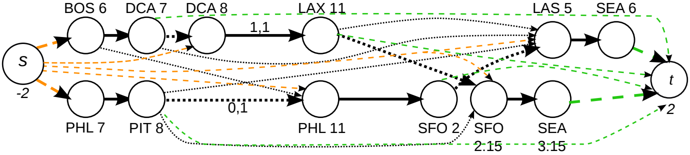
  - Send one unit of flow along the edges of that path P_l and along the edges (s, s_l) and (t_l, t).
  - To satisfy excess demands at s and t, send $k - k'$ units of flow along (s, t).
  - Why does the resulting circulation satisfy all demand, lower bound, and capacity constraints?

Proof of Correctness: Part 2

- Claim: We can schedule all flights in S using at most k planes iff G has a feasible circulation.

Proof of Correctness: Part 2



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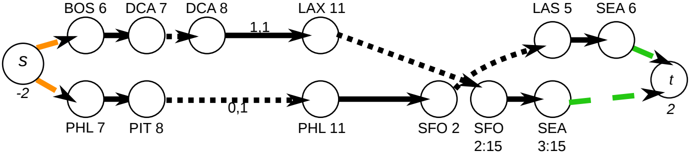

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- Feasible circulation $\Rightarrow$ feasible schedule:
  - Flow on each edge must be 0 or 1. Flow on the edges for flights must be 1.

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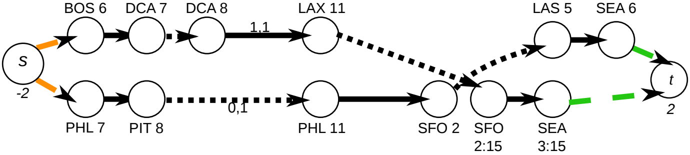

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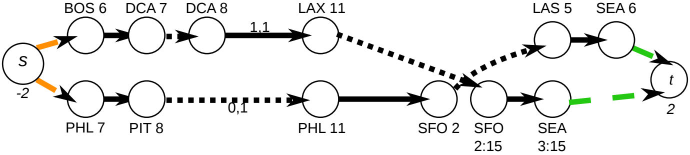

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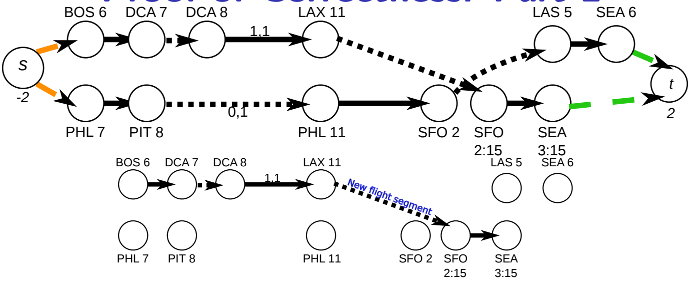

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  - Convert set of edges that carry flow into k' edge-disjoint s-t paths.

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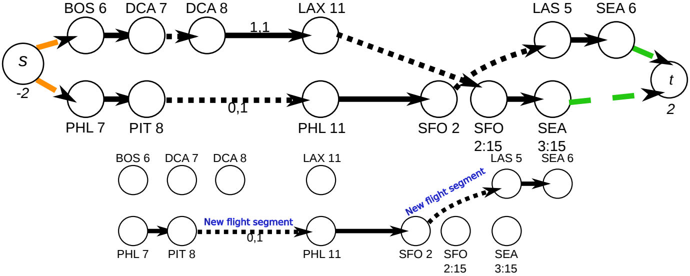

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  - Convert set of edges that carry flow into k' edge-disjoint s-t paths.
  - Each path starts at exactly one of the k' edges of the form (s, u), $u \neq t$ that carry flow. Use the proof for the edge-disjoint paths problem to compute path.

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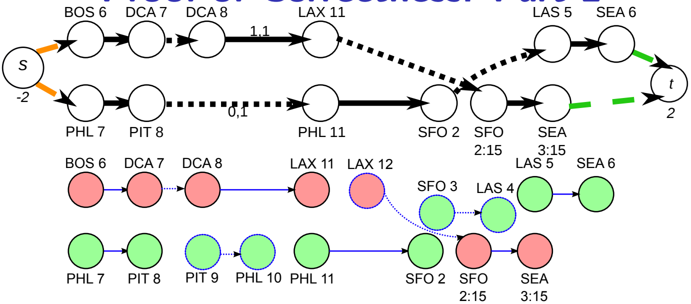

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  - Output these paths. Paths define extra flight segments automatically.