Coping with NP-Completeness

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Examples of Hard Computational Problems

(*from Kevin Wayne's slides at Princeton University*)

- Aerospace engineering: optimal mesh partitioning for finite elements.
- Biology: protein folding.
- Chemical engineering: heat exchanger network synthesis.
- Civil engineering: equilibrium of urban traffic flow.
- Economics: computation of arbitrage in financial markets with friction.
- Electrical engineering: VLSI layout.
- Environmental engineering: optimal placement of contaminant sensors.
- Financial engineering: find minimum risk portfolio of given return.
- Game theory: find Nash equilibrium that maximizes social welfare.
- Genomics: phylogeny reconstruction.
- Mechanical engineering: structure of turbulence in sheared flows.
- Medicine: reconstructing 3-D shape from biplane angiocardiogram.
- Operations research: optimal resource allocation.
- Physics: partition function of 3-D Ising model in statistical mechanics.
- Politics: Shapley-Shubik voting power.
- Pop culture: Minesweeper consistency.
- Statistics: optimal experimental design.

How Do We Tackle an $\mathcal{NP}$-Complete Problem?

"I can't find an efficient algorithm, but neither can all these famous people."

(Garey and Johnson, *Computers and Intractability*)

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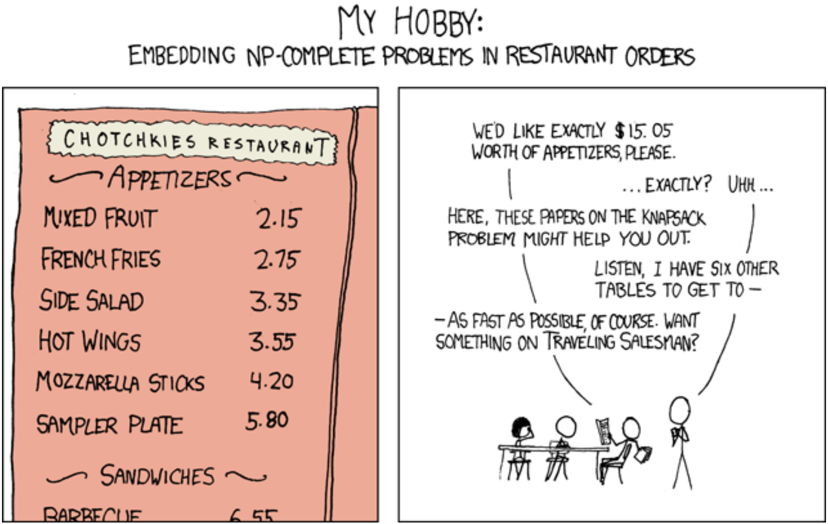

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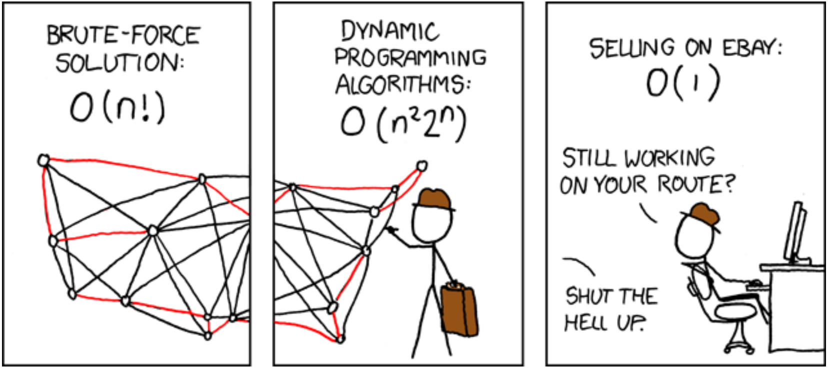

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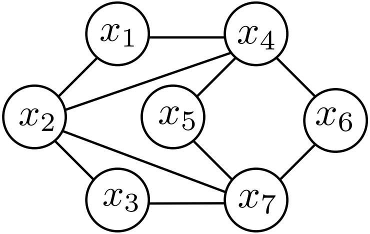
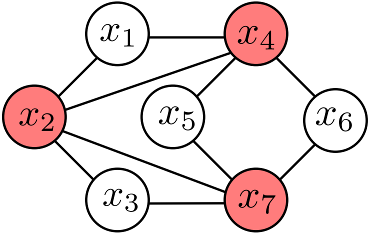
- These problems come up in real life.
- $\mathcal{NP}$-Complete means that a problem is hard to solve in the *worst case*. Can we come up with better solutions at least in *some* cases?
  - Develop algorithms that are exponential in one parameter in the problem.
  - Consider special cases of the input, e.g., graphs that "look like" trees.
  - Develop algorithms that can provably compute a solution close to the optimal.

Vertex Cover Problem





VERTEX COVER

INSTANCE: Undirected graph G and an integer k

QUESTION: Does G contain a vertex cover of size at most k?

- The problem has two parameters: k and n, the number of nodes in G.
- What is the running time of a brute-force algorithm?

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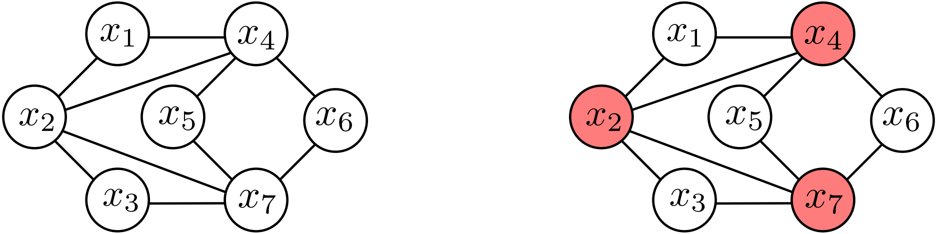

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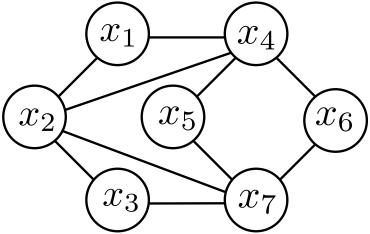

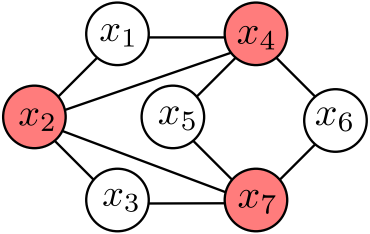

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- What is the running time of a brute-force algorithm? $O(kn\binom{n}{k}) = O(kn^{k+1})$.
- Can we devise an algorithm whose running time is exponential in k but polynomial in n, e.g., $O(2^k n)$?

Designing the Vertex Cover Algorithm

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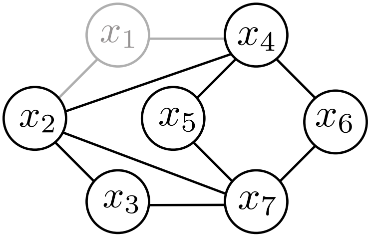

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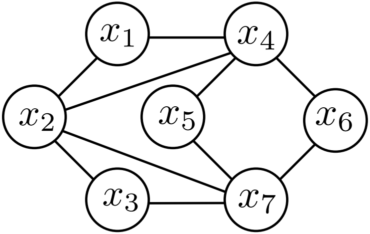

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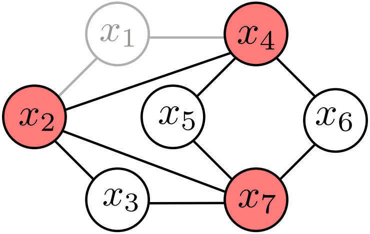
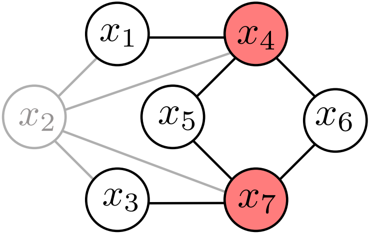
- Easy part of algorithm: Return `no` if G has more than kn edges.
- $G - \{u\}$ is the graph G without node u and the edges incident on u.
- Consider an edge (u, v). Either u or v must be in the vertex cover.
- Claim: G has a vertex cover of size at most k iff for any edge (u, v) either $G - \{u\}$ or $G - \{v\}$ has a vertex cover of size at most $k - 1$.





Vertex Cover Algorithm

```
To search for a k-node vertex cover in G:
  If G contains no edges, then the empty set is a vertex cover
  If G contains> k |V| edges, then it has no k-node vertex cover
  Else let e=(u,v) be an edge of G
    Recursively check if either of G−{u} or G−{v}
                has a vertex cover of size k−1
    If neither of them does, then G has no k-node vertex cover
    Else, one of them (say, G−{u}) has a (k−1)-node vertex cover T
       In this case, T∪{u} is a k-node vertex cover of G
    Endif
  Endif
```

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- $T(n, k) \leq 2T(n, k-1) + ckn$.
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Analysing the Vertex Cover Algorithm

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- $T(n, k) \leq 2T(n, k-1) + ckn$.
  - We need $O(kn)$ time to count the number of edges.
- Claim: $T(n, k) = O(2^k kn)$.

Solving $\mathcal{NP}$-Hard Problems on Trees

- "$\mathcal{NP}$-Hard": at least as hard as $\mathcal{NP}$-Complete. We will use $\mathcal{NP}$-Hard to refer to optimisation versions of decision problems.

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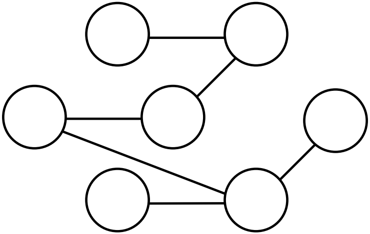
- "$\mathcal{NP}$-Hard": at least as hard as $\mathcal{NP}$-Complete. We will use $\mathcal{NP}$-Hard to refer to optimisation versions of decision problems.
- Many $\mathcal{NP}$-Hard problems can be solved efficiently on trees.
- Intuition: subtree rooted at any node v of the tree "interacts" with the rest of tree only through v. Therefore, depending on whether we include v in the solution or not, we can decouple solving the problem in v's subtree from the rest of the tree.

Designing Greedy Algorithm for Independent Set



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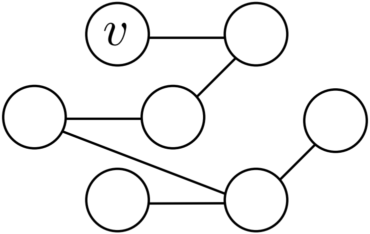

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- Claim: Every tree $T(V, E)$ has a *leaf*, a node with degree 1.
- Claim: If a tree T has a leaf v, then there exists a maximum-size independent set in T that contains v.

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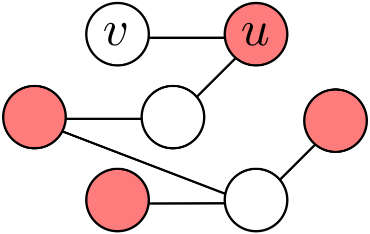

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  - Let S be a maximum-size independent set that does not contain v.
  - Let v be connected to u.
  - u must be in S; otherwise, we can add v to S, which means S is not maximum size.
  - Since u is in S, we can swap u and v.

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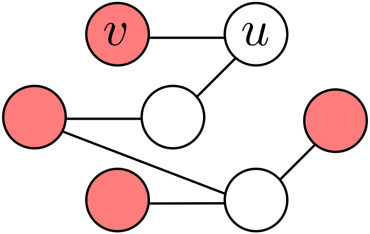

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- Claim: If a tree T has a a leaf v, then a maximum-size independent set in T is v and a maximum-size independent set in $T - \{v\}$.

Greedy Algorithm for Independent Set

- A *forest* is a graph where every connected component is a tree.

```
To find a maximum-size independent set in a forest F:
  Let S be the independent set to be constructed (initially empty)
  While F has at least one edge
     Let e = (u, v) be an edge of F such that v is a leaf
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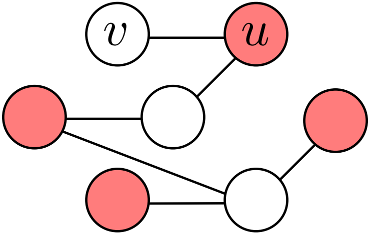

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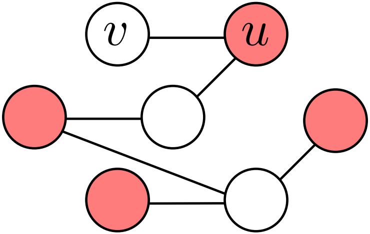

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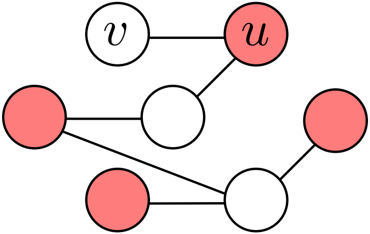

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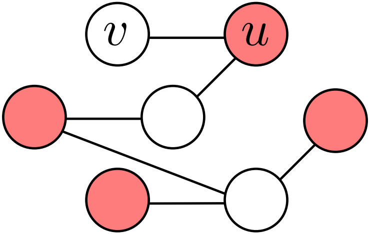

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- But there are still only two possibilities: either include u in the independent set or include *all* neighbours of u that are leaves.
- Suggests dynamic programming algorithm.

Designing Dynamic Programming Algorithm

- Dynamic programming algorithm needs a set of sub-problems, recursion to combine sub-problems, and order over sub-problems.
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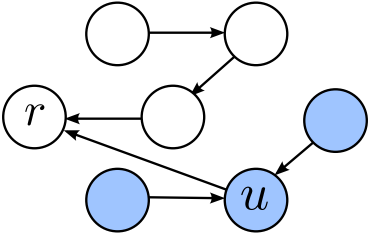
  - *parent* $p(u)$ of a node u is the node adjacent to u along the path to r.
  - Sub-problems are T_u: subtree induced by u and all its descendants.



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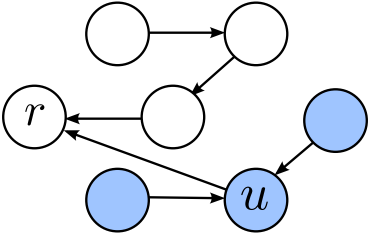
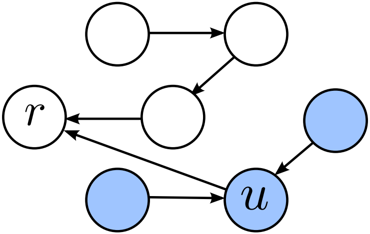
- Ordering the sub-problems: start at leaves and work our way up to the root.



Recursion for Dynamic Programming Algorithm



- Either we include u in an optimal solution or exclude u.
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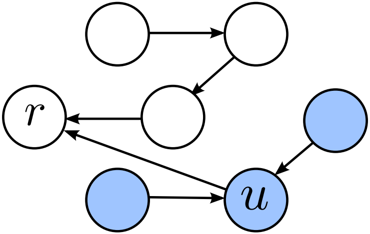

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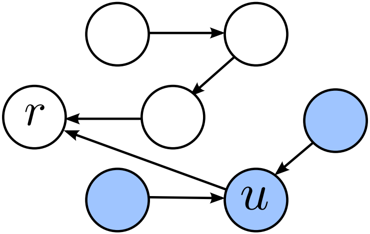

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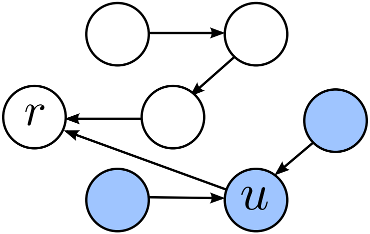
- Base cases: For a leaf u, $\mathrm{OPT}_{\mathrm{in}}(u) = w_u$ and $\mathrm{OPT}_{\mathrm{out}}(u) = 0$.
- Recurrence: Include u or exclude u.

Recursion for Dynamic Programming Algorithm



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  1. If we include u, all children must be excluded.
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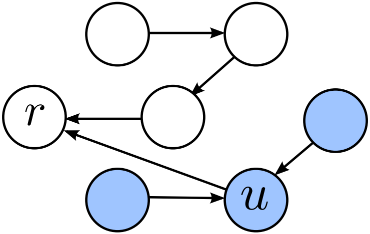

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     $\text{OPT}_{\text{in}}(u) = w_u + \sum_{v \in \text{children}(u)} \text{OPT}_{\text{out}}(v)$
  2. If we exclude u, a child may or may not be excluded.
     $\text{OPT}_{\text{out}}(u) = \sum_{v \in \text{children}(u)} \max\left(\text{OPT}_{\text{in}}(v), \text{OPT}_{\text{out}}(v)\right)$

Dynamic Programming Algorithm

```
To find a maximum-weight independent set of a tree T:
    Root the tree at a node r
    For all nodes u of T in post-order
        If u is a leaf then set the values:
```

$$M_{out}[u] = 0$$

$$M_{in}[u] = w_u$$

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        Else set the values:
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$$M_{out}[u] = \sum_{v \in children(u)} \max(M_{out}[v], \; M_{in}[v])$$

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        Endif
    Endfor
    Return max(M_out[r], M_in[r])
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Return $\max(M_{out}[r], M_{in}[r])$

- Running time of the algorithm is $O(n)$.

Approximation Algorithms

- Methods for optimisation versions of $\mathcal{NP}$-Complete problems.
- Run in polynomial time.
- Solution returned is guaranteed to be within a small factor of the optimal solution

Approximation Algorithm for VertexCover

EasyVertexCover(G) (Gavril, 1974; Yannakakis)

$C \leftarrow \emptyset$ {C will be the vertex cover}
while G has at least one edge **do**
    Let (u, v) be any edge in G
    ______________________________ {Update C using u and/or v}
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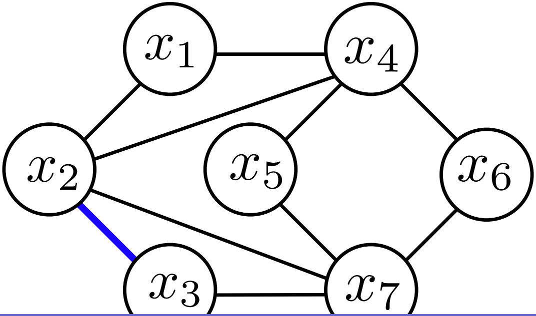
end while
return C



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C ← ∅                    {C will be the vertex cover}
while G has at least one edge do
    Let (u, v) be any edge in G
    Add u and v to C
    ______________________  {Update G using u and/or v}

end while
return C
```

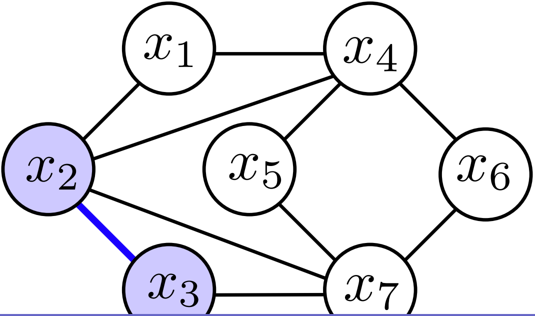

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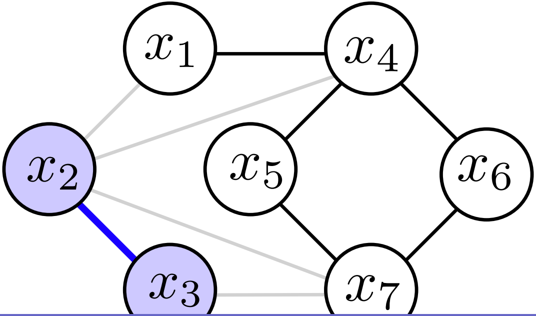

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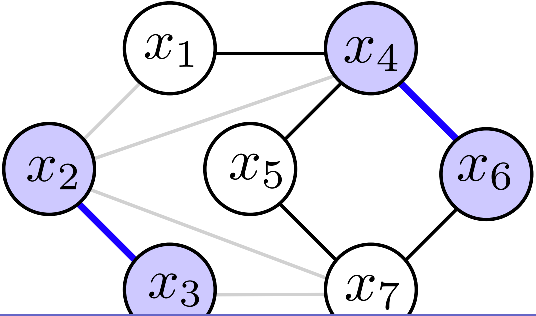

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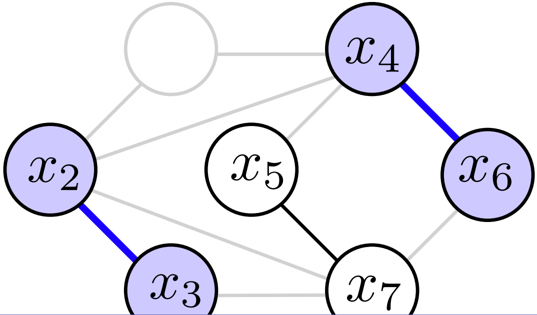

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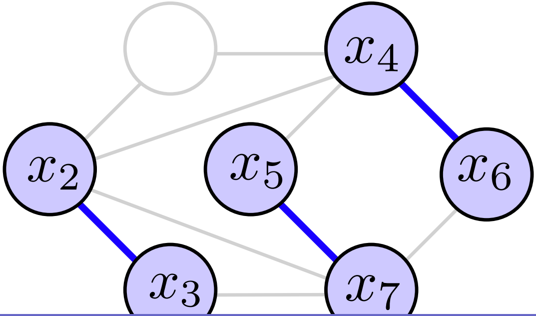

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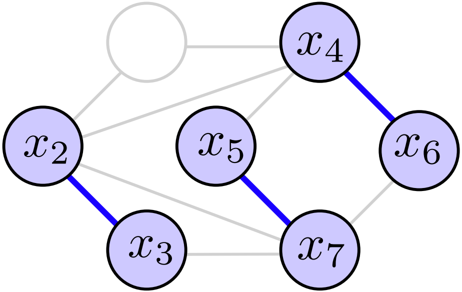

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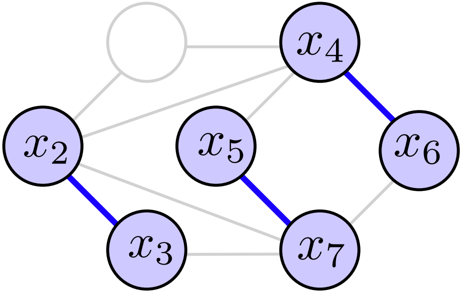

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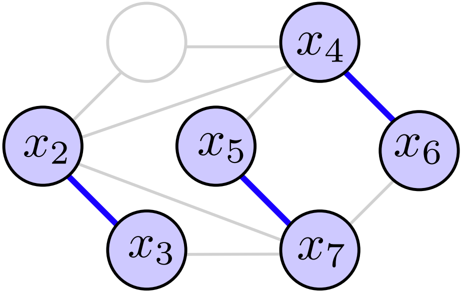

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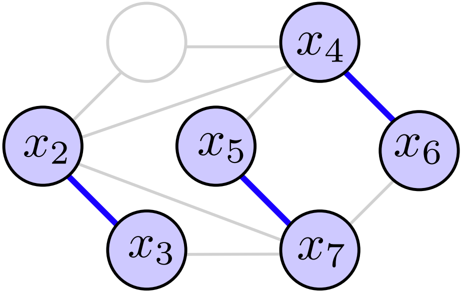


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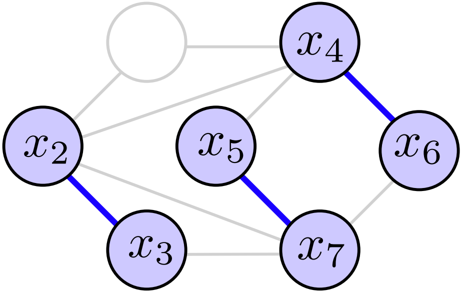

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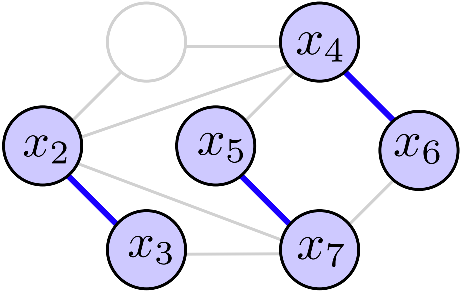
Analysis of EasyVertexCover

EASYVERTEXCOVER(G)

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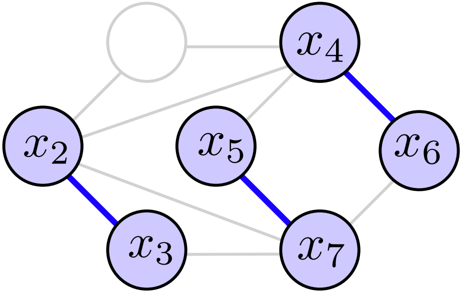

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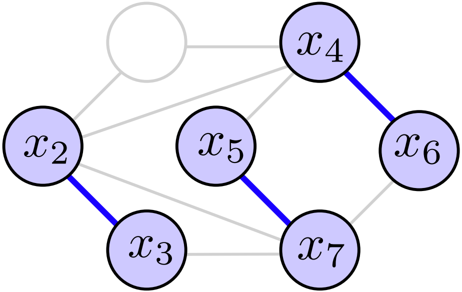

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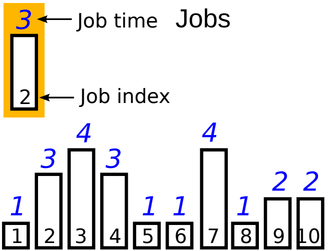
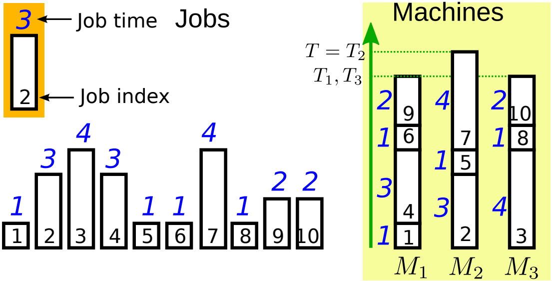
- No approximation algorithm with a factor better than 2 is possible if the "unique games conjecture" is true (Khot and Regev, 2008).

Load Balancing Problem



- Given set of m machines $M_1, M_2, \ldots M_m$.
- Given a set of n jobs: job j has processing time t_j.
- Assign each job to one machine so that the total time spent is minimised.

Load Balancing Problem



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- Total time spent on machine i is $T_i = \sum_{k \in A(i)} t_k$.
- Minimise *makespan* $T = \max_i T_i$, the largest load on any machine.

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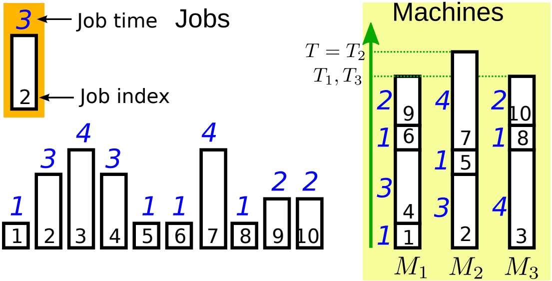

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- Minimise *makespan* $T = \max_i T_i$, the largest load on any machine.
- Minimising makespan is $\mathcal{NP}$-Complete.

Greedy-Balance Algorithm

- Adopt a greedy approach (Graham, 1966).
- Process jobs in *any* order.
- Assign next job to the processor that has smallest total load so far.

Greedy-Balance:

Start with no jobs assigned

Set $T_i = 0$ and $A(i) = \emptyset$ for all machines M_i

For $j = 1, \ldots, n$

  Let M_i be a machine that achieves the minimum $\min_k T_k$

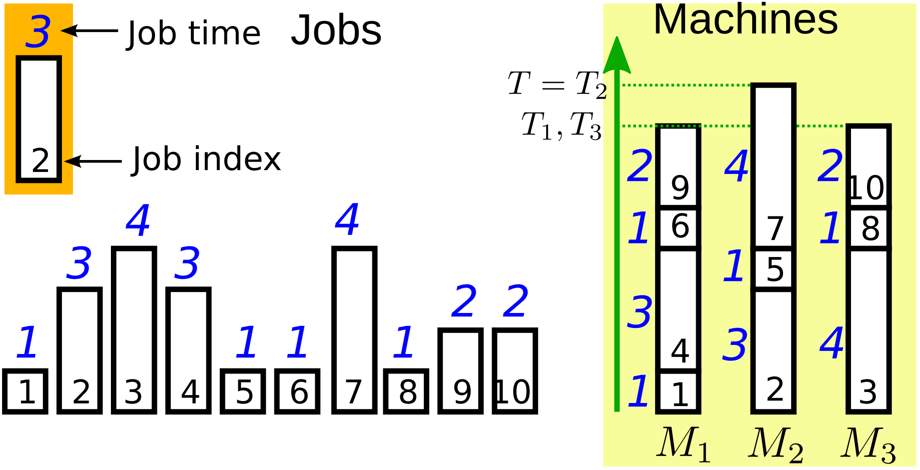
  Assign job j to machine M_i

  Set $A(i) \leftarrow A(i) \cup \{j\}$

  Set $T_i \leftarrow T_i + t_j$

EndFor

Example of Greedy-Balance Algorithm



Lower Bounds on the Optimal Makespan

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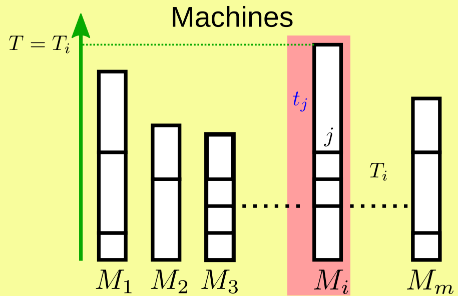
- The two bounds below will suffice:

$$T^* \geq \frac{1}{m} \sum_j t_j$$

$$T^* \geq \max_j t_j$$

Analysing Greedy-Balance



- Claim: Computed makespan $T \leq 2T^*$.

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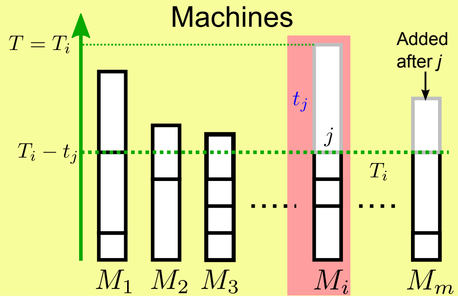

- Claim: Computed makespan $T \leq 2T^*$.
- Let M_i be the machine whose load is T and j be the last job placed on M_i.
- What was the situation just before placing this job?

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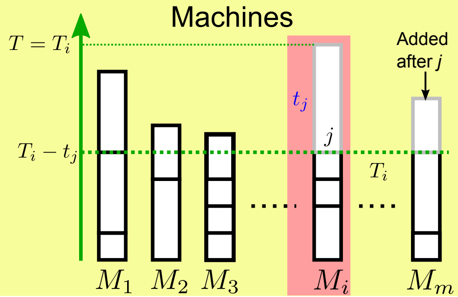
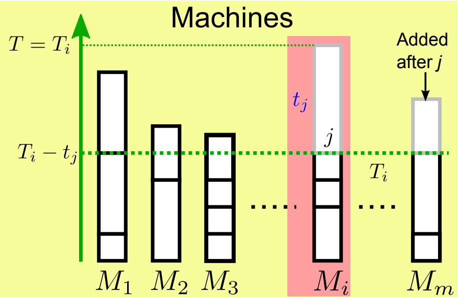


- Claim: Computed makespan $T \leq 2T^*$.
- Let M_i be the machine whose load is T and j be the last job placed on M_i.
- What was the situation just before placing this job?
- M_i had the smallest load and its load was $T - t_j$.
- For every machine M_k, load $T_k \geq T - t_j$.

Analysing Greedy-Balance



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$$\sum_k T_k \geq m(T - t_j), \text{ where } k \text{ ranges over all machines}$$

$$\sum_j t_j \geq m(T - t_j), \text{ where } j \text{ ranges over all jobs}$$

$$T - t_j \leq 1/m \sum_j t_j \leq T^*$$

$$T \leq 2T^*, \text{ since } t_j \leq T^*$$

Improving the Bound

- It is easy to construct an example for which the greedy algorithm produces a solution close to a factor of 2 away from optimal.

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- It is easy to construct an example for which the greedy algorithm produces a solution close to a factor of 2 away from optimal.
- How can we improve the algorithm?
- What if we process the jobs in decreasing order of processing time? (Graham, 1969)

Sorted-Balance Algorithm

Sorted-Balance:

Start with no jobs assigned

Set $T_i = 0$ and $A(i) = \emptyset$ for all machines M_i

Sort jobs in decreasing order of processing times t_j

Assume that $t_1 \geq t_2 \geq \ldots \geq t_n$

For $j = 1, \ldots, n$

  Let M_i be the machine that achieves the minimum $\min_k T_k$

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Assume that t_1 ≥ t_2 ≥ ... ≥ t_n
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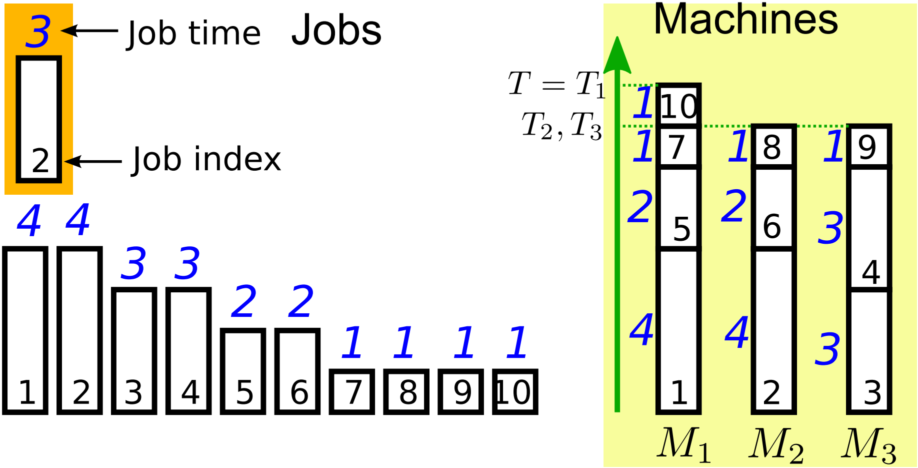
- This algorithm assigns the first m jobs to m distinct machines.

Example of Sorted-Balance Algorithm



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- Claim: if there are fewer than m jobs, algorithm is optimal.
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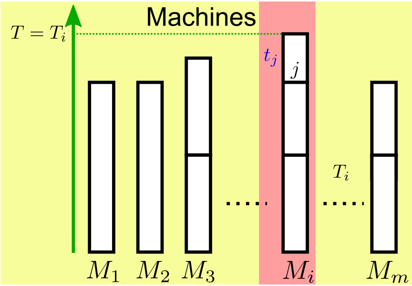

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$$t_j \leq t_{m+1} \leq T^*/2, \text{ since } j \geq m+1$$
$$T - t_j \leq T^*, \text{ GREEDY-BALANCE proof}$$
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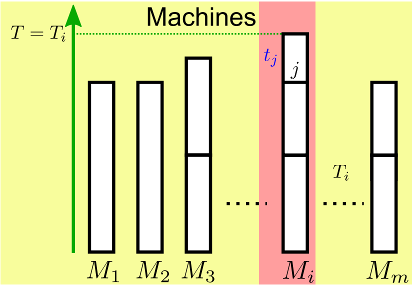

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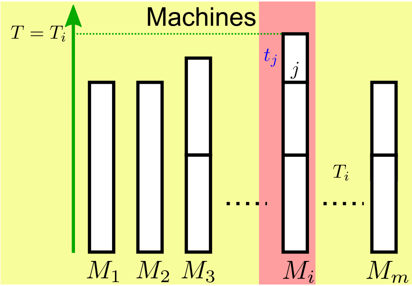
- Better bound: $T \leq 4T^*/3$ (Graham, 1969).



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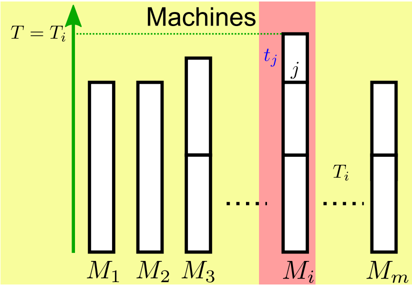

- Better bound: $T \leq 4T^*/3$ (Graham, 1969).
  - *Polynomial-time approximation scheme*: for every $\varepsilon > 0$, compute solution with makespan $T \leq (1+\varepsilon)T^*$ in $O((n/\varepsilon)^{(1/\varepsilon^2)})$ time (Hochbaum and Shmoys, 1987).

The Knapsack Problem

PARTITION
INSTANCE: A set of n natural numbers $w_1, w_2, \ldots, w_n$.
SOLUTION: A subset S of numbers such that $\sum_{i \in S} w_i = \sum_{i \notin S} w_i$.

The Knapsack Project

PARTITION

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SUBSET SUM

INSTANCE: A set of n natural numbers $w_1, w_2, \ldots, w_n$ and a target W.

SOLUTION: A subset S of numbers such that $\sum_{i \in S} w_i$ is maximised subject to the constraint $\sum_{i \in S} w_i \leq W$.

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KNAPSACK

INSTANCE: A set of n elements, with each element i having a weight w_i and a value v_i, and a knapsack capacity W.

SOLUTION: A subset S of items such that $\sum_{i \in S} v_i$ is maximised subject to the constraint $\sum_{i \in S} w_i \leq W$.

The Knapsack Problem

PARTITION

INSTANCE: A set of n natural numbers $w_1, w_2, \ldots, w_n$.

SOLUTION: A subset S of numbers such that $\sum_{i \in S} w_i = \sum_{i \notin S} w_i$.

SUBSET SUM

INSTANCE: A set of n natural numbers $w_1, w_2, \ldots, w_n$ and a target W.

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- 3D MATCHING $\leq_P$ PARTITION $\leq_P$ SUBSET SUM $\leq_P$ KNAPSACK
- All problems have dynamic programming algorithms with pseudo-polynomial running times.

Dynamic Programming for Subset Sum

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- $OPT(i, w)$ is the largest sum possible using only the first i numbers with target w.

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$$\text{OPT}(i, w) = \text{OPT}(i-1, w), \qquad i > 0, w_i > w$$

$$\text{OPT}(i, w) = \max\left(\text{OPT}(i-1, w), w_i + \text{OPT}(i-1, w - w_i)\right), \qquad i > 0, w_i \leq w$$

$$\text{OPT}(0, w) = 0$$

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- Can generalize the dynamic program for SUBSET SUM.
- But we will develop a different dynamic program that will be useful later.
- $OPT(i, v)$ is the smallest knapsack weight so that there is a solution using only the first i items with total value $\geq v$.
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  - Given i, v ranges between 0 and $\sum_{1 \leq j \leq i} v_j$.
  - Largest value of v is $\sum_{1 \leq j \leq n} v_j \leq nv^*$, where $v^* = \max_i v_i$.
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$$\mathrm{OPT}(i, 0) = 0 \qquad \text{for every } i \geq 1$$

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- Can find items in the solution by tracing back.
- Running time is $O(n^2 v^*)$, which is pseudo-polynomial in the input size.

Intuition Underlying Approximation Algorithm

- What is the running time if all values are the same?

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- What is the running time if all values are the same? Polynomial.
- What is the running time if all values are small integers?

Intuition Underlying Approximation Algorithm

- What is the running time if all values are the same? Polynomial.
- What is the running time if all values are small integers? Also polynomial.
- Idea:
  - Round and scale all the values to lie in a smaller range.
  - Run the dynamic programming algorithm with the modified new values.
  - Return the items in this optimal solution.
  - Prove that the value of this solution is not much smaller than the true optimum.

Polynomial-Time Approximation Scheme for Knapsack

- $0 < \varepsilon < 1$ is a "precision" parameter; assume that $1/\varepsilon$ is an integer.
- Scaling factor $\theta = \frac{\varepsilon v^*}{2n}$.
- For every item i, set

$$\tilde{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil \theta, \qquad \hat{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil$$

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Solve the Knapsack problem using the dynamic program with the values $\hat{v}_i$.
Return the set S of items found.

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`Knapsack-Approx(`ε`)`

Solve the Knapsack problem using the dynamic program with the values $\hat{v}_i$.
Return the set S of items found.

- What is the running time of `Knapsack-Approx`?
  $O(n^2 \max_i \hat{v}_i) = O(n^2 v^* / \theta) = O(n^3/\varepsilon)$.
- We need to show that the value of the solution returned by `Knapsack-Approx` is good.

Approximation Guarantee for `Knapsack-Approx`

- Let S be the solution computed by `Knapsack-Approx`.
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- Claim: $(1+\varepsilon)\sum_{i \in S} v_i \geq \sum_{j \in S^*} v_j$. Polynomial-time approximation scheme.
- Since `Knapsack-Approx` is optimal for the values $\tilde{v}_i$,

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$$\sum_{j \in S^*} v_j \leq \sum_{j \in S^*} \tilde{v}_j \leq \sum_{i \in S} \tilde{v}_i \leq \sum_{i \in S} v_i + n\theta = \sum_{i \in S} v_i + \frac{\varepsilon v^*}{2}$$

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- Apply argument to S^* containing only the item with largest value: $v^* \leq \sum_{i \in S} v_i + \frac{\varepsilon v^*}{2} \leq \sum_{i \in S} v_i + \frac{v^*}{2}$, i.e., $v^* \leq 2\sum_{i \in S} v_i$.

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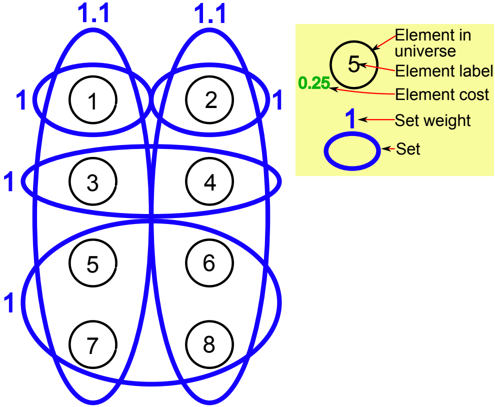
- How can we do better ? Improve running time to $O(n \log_2 \frac{1}{\varepsilon} + \frac{1}{\varepsilon^4})$ (Lawler, 1979).

Set Cover

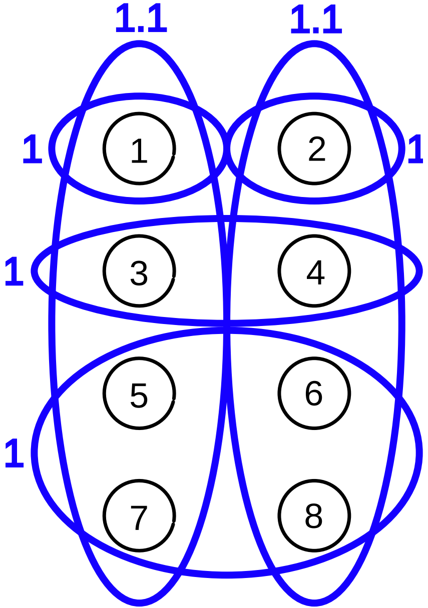
SET COVER

INSTANCE: A set U of n elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, each with an associated weight w.

SOLUTION: A collection $\mathcal{C}$ of sets in the collection such that $\bigcup_{S_i \in C} S_i = U$ and $\sum_{S_i \in C} w_i$ is minimised.



Greedy Approach



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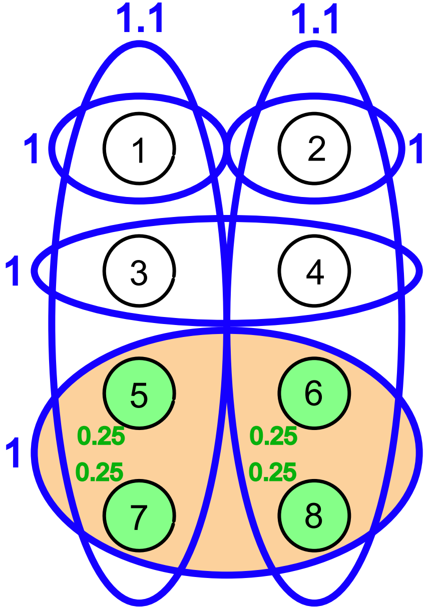

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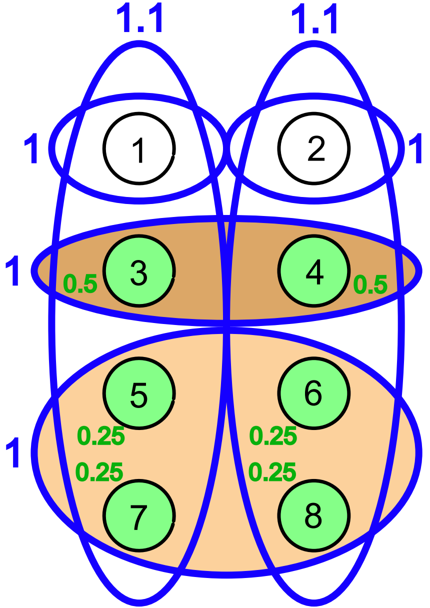

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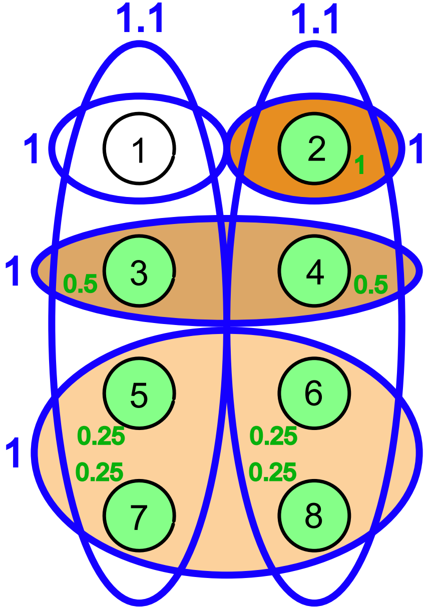

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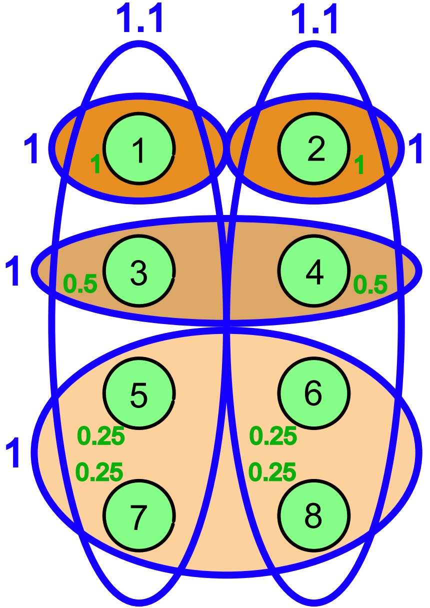

Greedy-Set-Cover

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Greedy-Set-Cover:

Start with $R = U$ and no sets selected

While $R \neq \emptyset$

  Select set S_i that minimizes $w_i/|S_i \cap R|$

  Delete set S_i from R

EndWhile

Return the selected sets

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- The algorithm computes a set cover whose weight is at most $O(\log n)$ times the optimal weight (Johnson 1974, Lovász 1975, Chvatal 1979).

Add Bookkeeping to Greedy-Set-Cover

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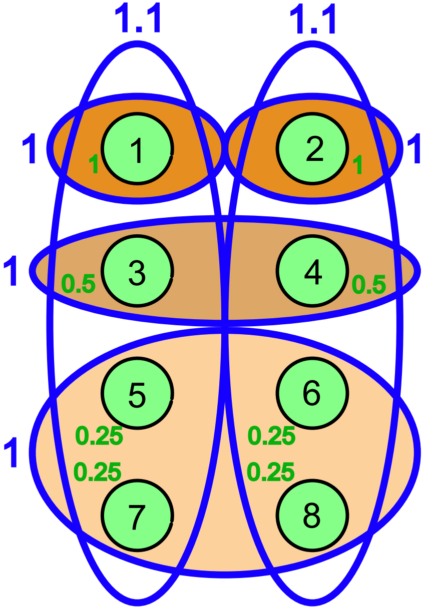
- In the algorithm, after selecting S_i, add the line

  `Define` $c_t = w_i/|S_i \cap R|$ `for all` $t \in S_i \cap R$.
- As each set S_i is selected, distribute its weight over the costs c_t of the *newly*-covered elements.
- Each element in the universe assigned cost exactly once.



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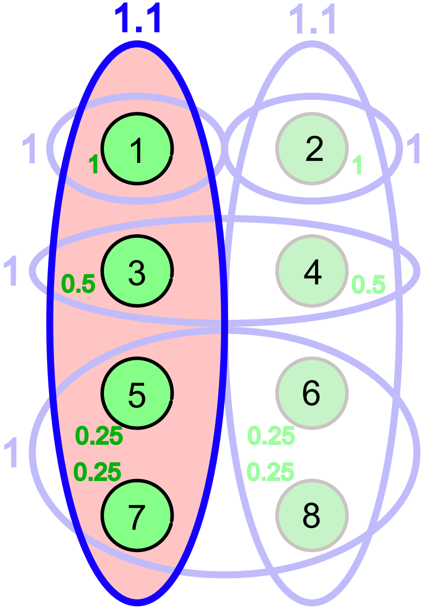

Starting the Analysis of Greedy-Set-Cover

- Let $\mathcal{C}$ be the set cover computed by GREEDY-SET-COVER.
- Claim: $\sum_{S_i \in \mathcal{C}} w_i = \sum_{t \in U} c_s$.

$$\sum_{S_i \in \mathcal{C}} w_i = \sum_{S_i \in \mathcal{C}} \left(\sum_{t \in S_i \cap R} c_s \right), \text{ by definition of } c_s$$

$$= \sum_{t \in U} c_t, \text{ since each element in the universe contributes exactly once}$$

- In other words, the total weight of the solution computed by GREEDY-SET-COVER is the sum of the costs it assigns to the elements in the universe.
- Can "switch" between set-based weight of solution and element-based costs.
- Note: sets have weights whereas GREEDY-SET-COVER assigns costs to elements.

Intuition Behind the Proof

- Suppose $\mathcal{C}^*$ is the optimal set cover: $w^* = \sum_{S_j \in \mathcal{C}^*} w_j$.
- Goal is to relate total weight of sets in $\mathcal{C}$ to total weight of sets in $\mathcal{C}^*$.

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- What is the total cost assigned by GREEDY-SET-COVER to the elements in the sets in the optimal cover $\mathcal{C}^*$?
- Since $\mathcal{C}^*$ is a set cover, $\sum_{S_j \in \mathcal{C}^*} \left(\sum_{t \in S_j} c_t \right) \geq \sum_{t \in U} c_t = \sum_{S_i \in \mathcal{C}} w_i = w$.

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- For every set S_k in the input, goal is to prove an upper bound on $\dfrac{\sum_{t \in S_k} c_t}{w_k}$.

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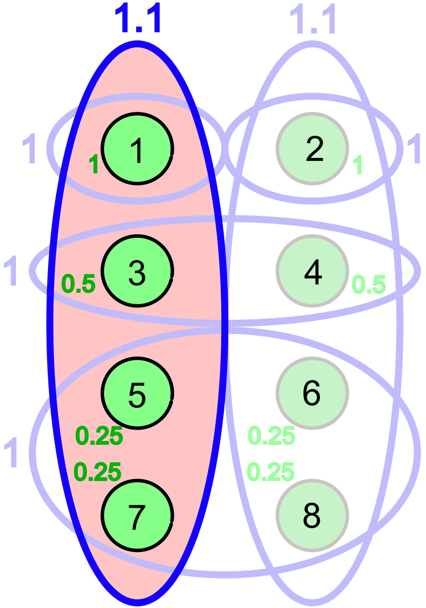
- Claim: For every set S_k, the sum $\sum_{t \in S_k} c_t \leq H(|S_k|)w_k$.



Renumbering Elements in S_k

- Renumber elements in U so that elements in S_k are the first $d = |S_k|$ elements of U, i.e., $S_k = \{t_1, t_2, \ldots, t_d\}$.
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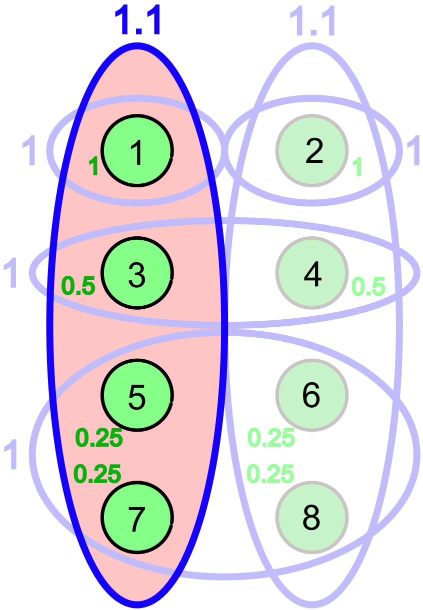

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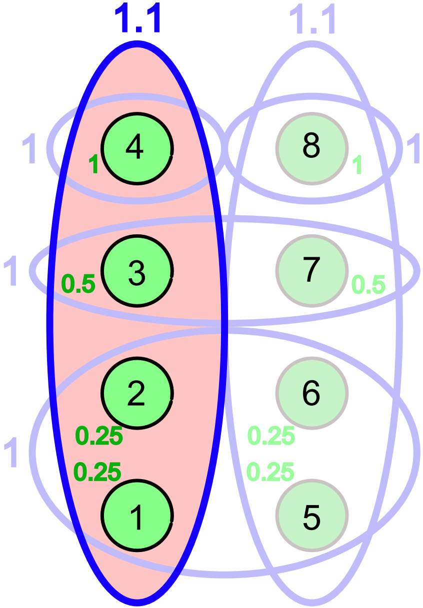

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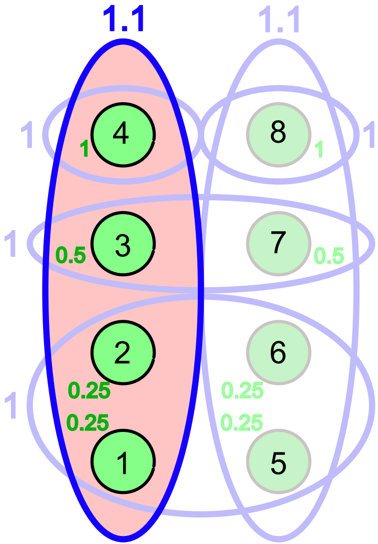

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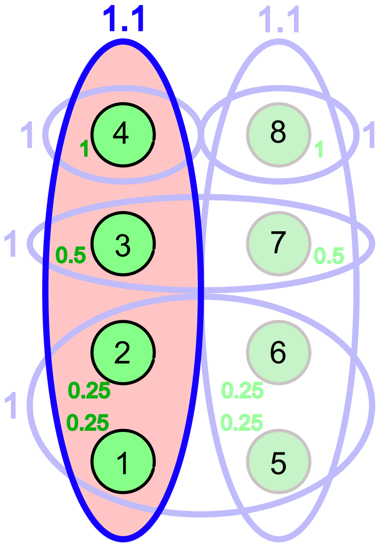

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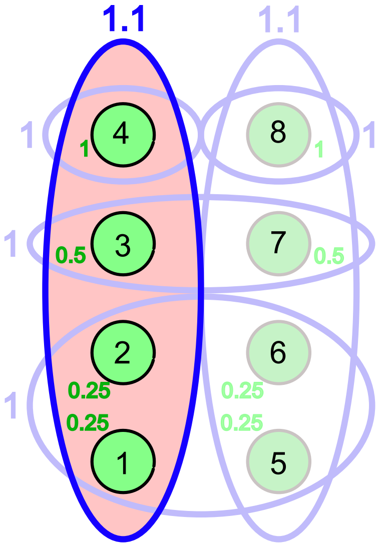

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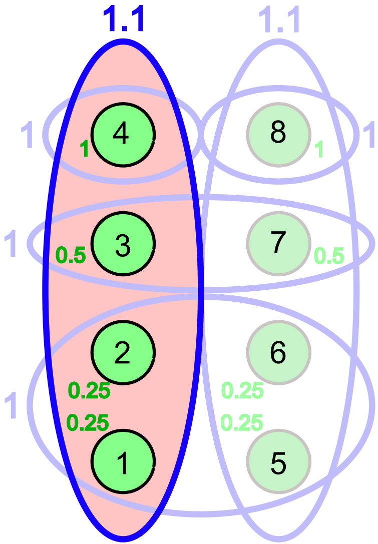
- We are done!

$$\sum_{t \in S_k} c_t = \sum_{j=1}^{d} c_{s_j} \leq \sum_{j=1}^{d} \frac{w_k}{d - j + 1} = H(d) w_k.$$



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- We have proven that Greedy-Set-Cover computes a set cover whose weight is at most $H(d^*)$ times the optimal weight.

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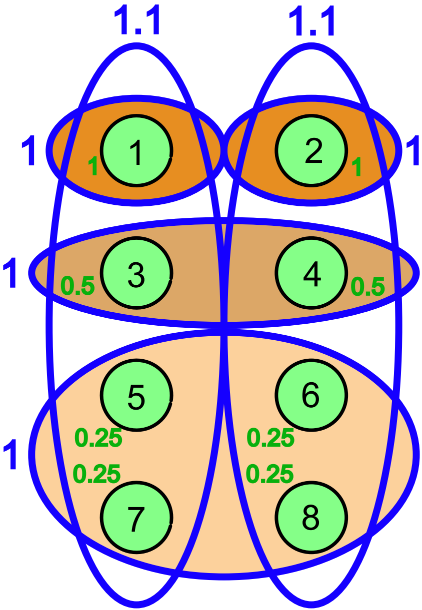

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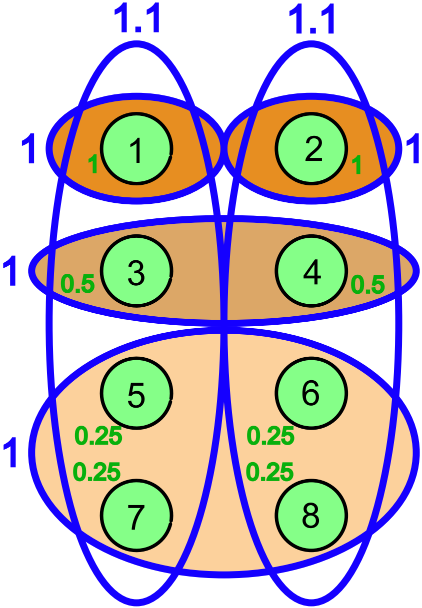

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- More complex constructions show greedy algorithm incurs a weight close to $H(n)$ times the optimal weight.
- No polynomial time algorithm can achieve an approximation bound better than $(1 - \Omega(1)) \ln n$ times optimal unless $\mathcal{P} = \mathcal{NP}$ (Dinur and Steurer, 2014)

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- Euclidean TSP (distances defined by points in d dimensions): PTAS in $O(n(\log n)^{1/\varepsilon})$ time (Arora, 1997; Mithcell, 1999) (second algorithm is slower).

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- Edit distance (sequence alighment) between two strings of length n: If it can be computed in $O(n^{2-\delta})$ time for some constant $\delta >$), then SAT with n variables and m clauses can be solved in $m^{O(1)}2^{(1-\varepsilon)n}$ time, for some $\varepsilon > 0$ (Backurs, Indyk, 2015).