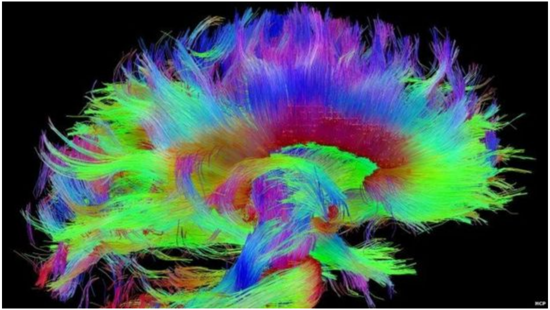
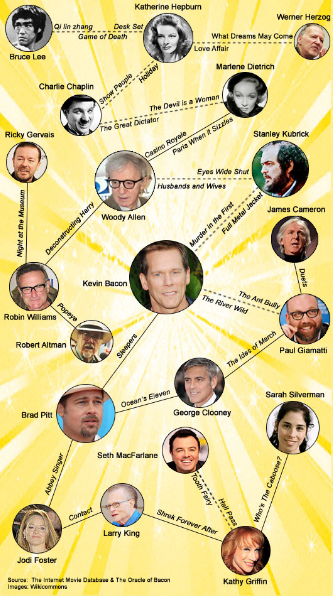
CS 4884: Introduction to Graphs

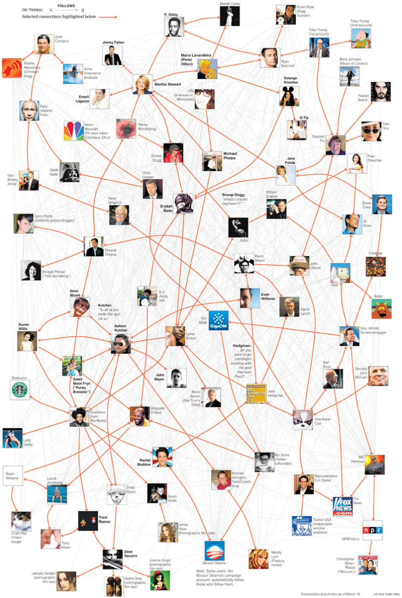
T. M. Murali

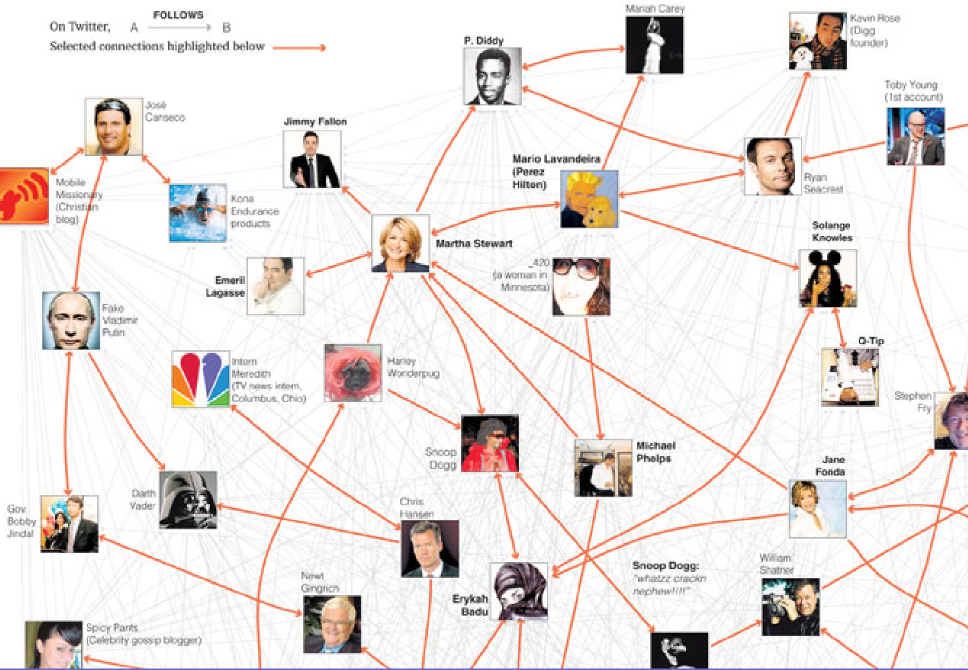
January 28, 2020





The Oracle of Bacon


On Twitter, A FOLLOWS B
Selected connections highlighted below
P. Diddy
Mariah Carey
Kevin Rose (Digg founder)
Toby Young (1st account)
Toby Young (2nd account)
José Canseco
Jimmy Fallon
Mobile Missionary (Christian blog)
Kona Endurance products
Mario Lavandeira (Perez Hilton)
Ryan Seacrest
Boris Johnson (Mayor of London)
Martha Stewart
Emeril Lagasse
Solange Knowles
Russell Brand
Fake Vladimir Putin
Intern Meredith (TV news intern, Columbus, Ohio)
Harley Wonderpug
Q-Tip
Yoko Ono
Stephen Fry
Snoop Dogg
Michael Phelps
Fran Drescher
Darth Vader
Gov. Bobby Jindal
Jane Fonda
Chris Hansen
Erykah Badu
Snoop Dogg: "whatzz crackn nephew!!!!"
William Shatner
Steve Case
Newt Gingrich
Al Gore
Spicy Pants (Celebrity gossip blogger)
Xzibit
Deepak Chopra
Rainn Wilson
John Cleese
Coldplay
Bridget Pilloud ("Pets are talking")
God
Demi Moore
D.J. Jazzy Jeff
Evan Williams
Satan
David Lynch
Kutcher: "Tu es la plu belle fille que j'ai vu"
Om Malik
Rumer Willis
Ashton Kutcher
LeVar Burton
Gov. Arnold Schwarzenegger
Hodgman: "... do you want to go candlepin bowling with me and Harrison Ford?"
Starbucks
Soleil Moon Frye ("Punky Brewster")
John Mayer
Karl Rove
Senator John McCain
Brent Spiner (Star Trek's Data)
John Hodgman
Questlove (from The Roots)
Shaquille O'Neal
Ana Marie Cox
Lady GaGa
Rachel Maddow
Biz Stone (Twitter cofounder)
MC Hammer
Robin Williams
Lance Armstrong
Michael Arrington, TechCrunch blog
Representative Eric Cantor
Fred Durst
Kevin Smith
Fox News
I Can Has Cheezburger
Trent Reznor
Tucker USA (makes pole window washers)
NPRPolitics
Tony Hawk
Jenna Jameson (Pornographic film star)
Dave Navarro
Barack Obama
Mandy Lynn (Playboy model)
Christopher Mintz-Plasse ("McLovin")
January Seraph (pornographic film star)
Joanna Angel (pornographic film star)
Sasha Grey (pornographic film star)
Note. Some users, like Barack Obama's campaign account, automatically follow those who follow them.
Connections and photos as of March 18.
THE NEW YORK TIMES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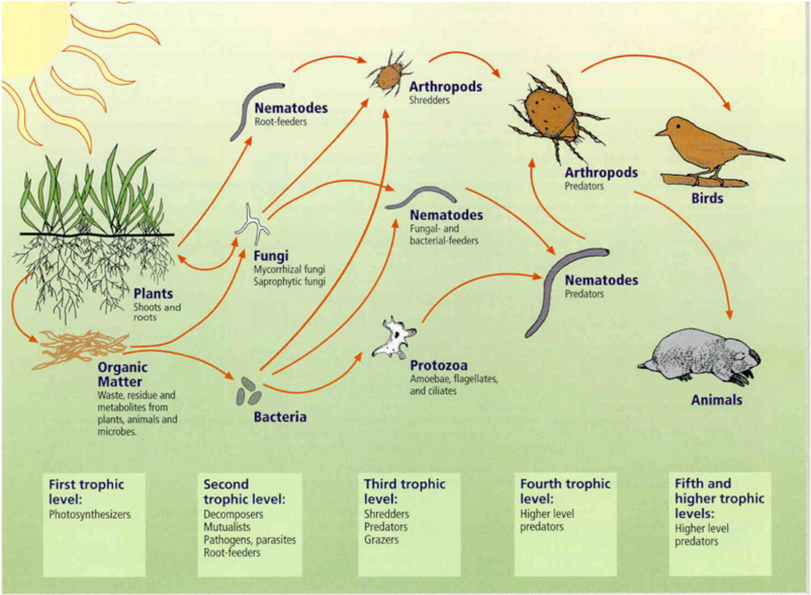
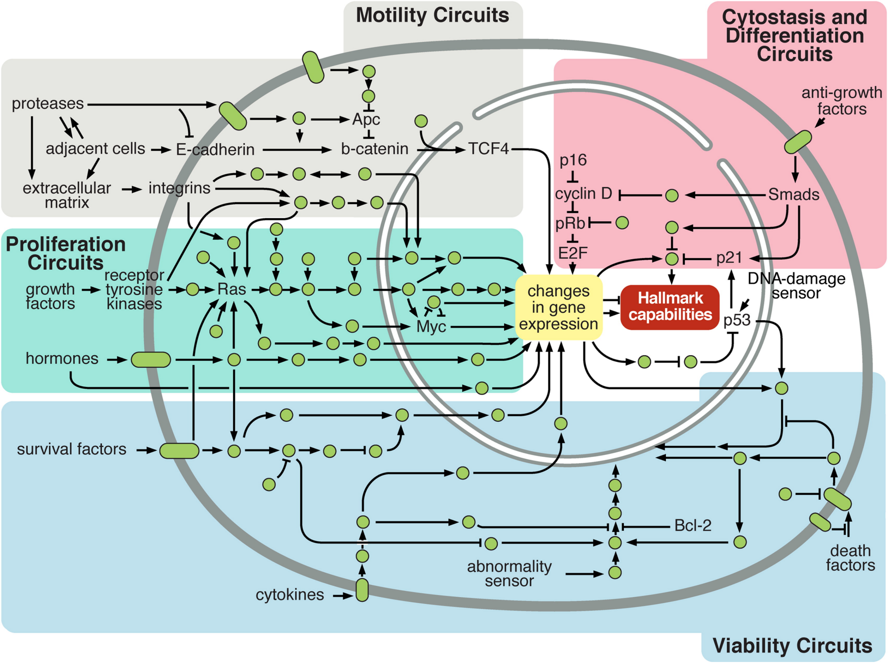

Nematodes
Root-feeders
Arthropods
Shredders
Arthropods
Predators
Birds
Fungi
Mycorrhizal fungi
Saprophytic fungi
Nematodes
Fungal- and
bacterial-feeders
Plants
Shoots and
roots
Nematodes
Predators
Organic
Matter
Waste, residue and
metabolites from
plants, animals and
microbes.
Protozoa
Amoebae, flagellates,
and ciliates
Animals
Bacteria
First trophic
level:
Photosynthesizers
Second
trophic level:
Decomposers
Mutualists
Pathogens, parasites
Root-feeders
Third trophic
level:
Shredders
Predators
Grazers
Fourth trophic
level:
Higher level
predators
Fifth and
higher trophic
levels:
Higher level
predators


Motility Circuits
Cytostasis and Differentiation Circuits
proteases
adjacent cells
E-cadherin
Apc
b-catenin
TCF4
extracellular matrix
integrins
anti-growth factors
p16
cyclin D
Smads
pRb
E2F
Proliferation Circuits
growth factors
receptor tyrosine kinases
Ras
Myc
changes in gene expression
Hallmark capabilities
p21
DNA-damage sensor
p53
hormones
survival factors
Bcl-2
abnormality sensor
cytokines
death factors
Viability Circuits

Graphs

Graph $\equiv$ Network

- Model pairwise relationships (edges) between objects (nodes).

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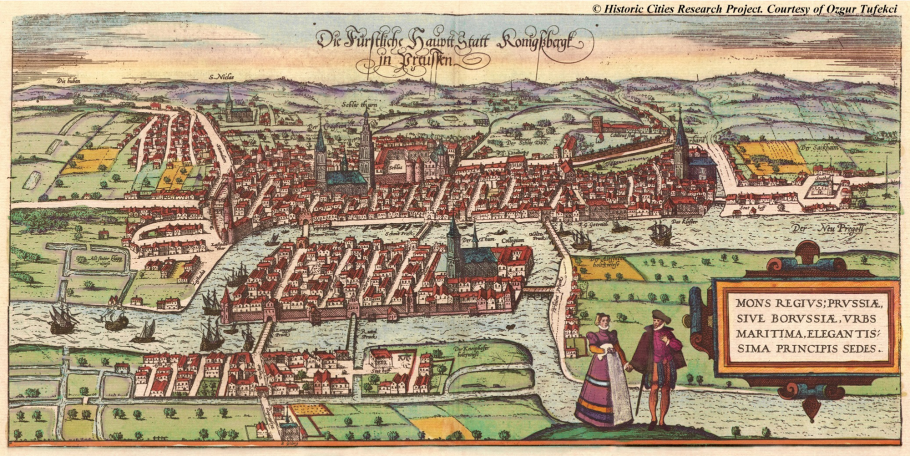
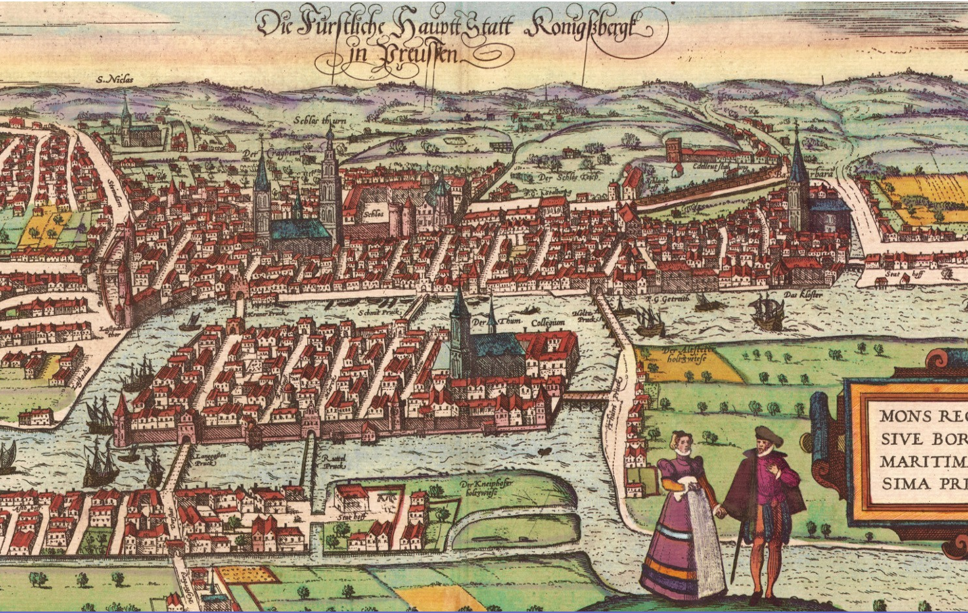
- Problems involving graphs have a rich history dating back to Euler.

Euler and Graphs



Devise a walk through the city that
crosses each of the seven bridges exactly once.

Euler and Graphs



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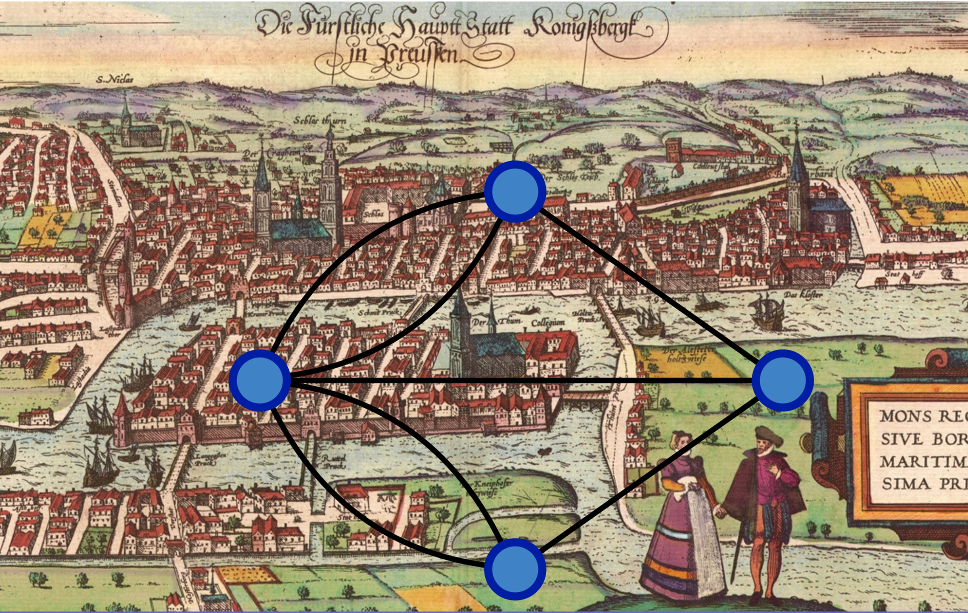
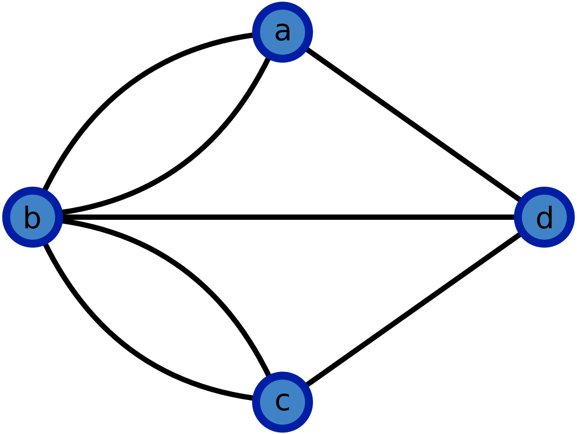


Definition of an Undirected Graph

- *Undirected graph* $G = (V, E)$: set V of nodes and set E of edges.
  - Each element of E is an unordered pair of nodes.
  - Edge (u, v) is *incident* on u, v; u and v are *neighbours* of each other.
  - G contains no self loops.



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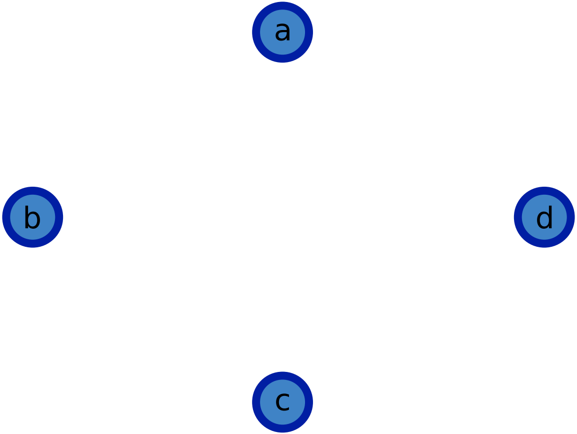

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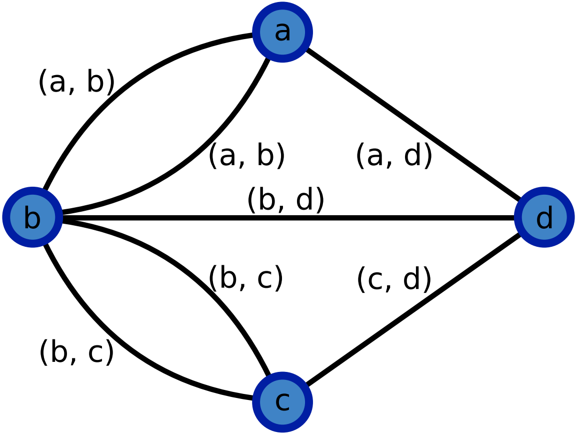
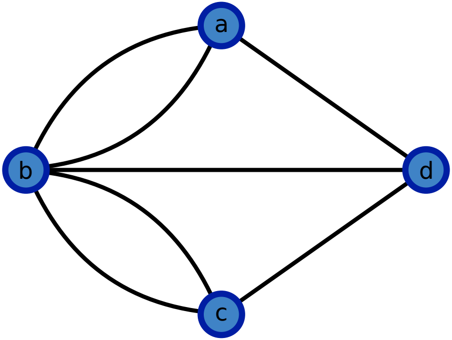
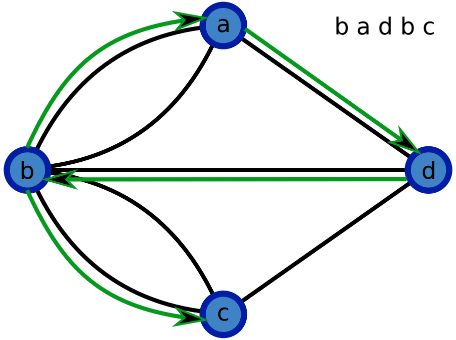


Paths and Cycles in Graphs



- A *v_1-v_k path* in an undirected graph $G = (V, E)$ is a sequence of nodes $v_1, v_2, \ldots, v_{k-1}, v_k \in V$ such that for every $i, 1 \leq i < k$, (v_i, v_{i+1}) is an edge in E.

Paths and Cycles in Graphs



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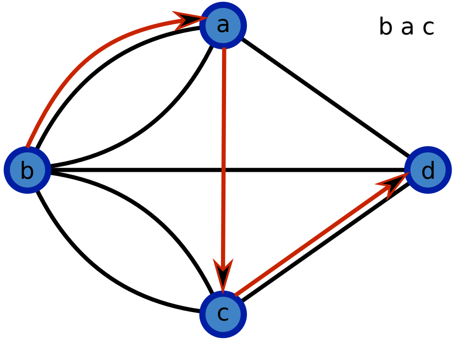

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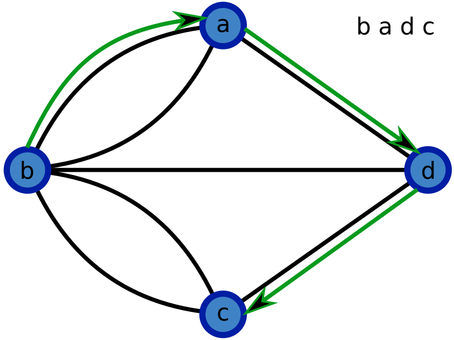

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- A path is *simple* if all its nodes are distinct.

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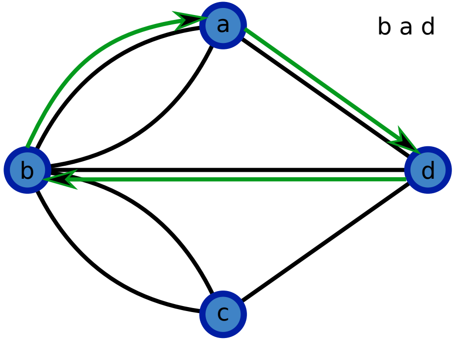

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- A *cycle* is a path where the first $k - 1$ nodes are distinct and $v_1 = v_k$.

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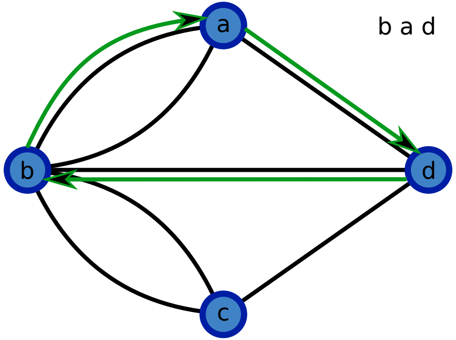

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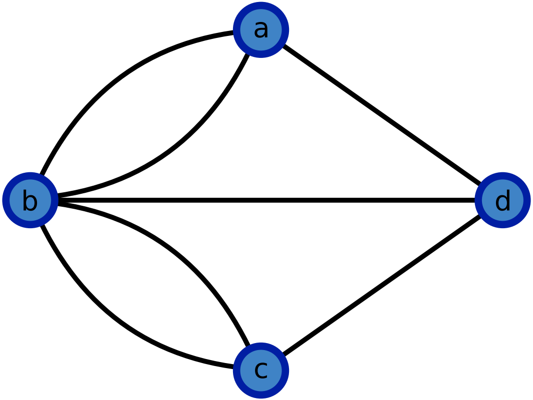
- A path is *simple* if all its nodes are distinct.
- A *cycle* is a path where the first $k - 1$ nodes are distinct and $v_1 = v_k$.
- An undirected graph G is *connected* if for every pair of nodes $u, v \in V$, there is a u-v path in G.

Bridges to Graphs

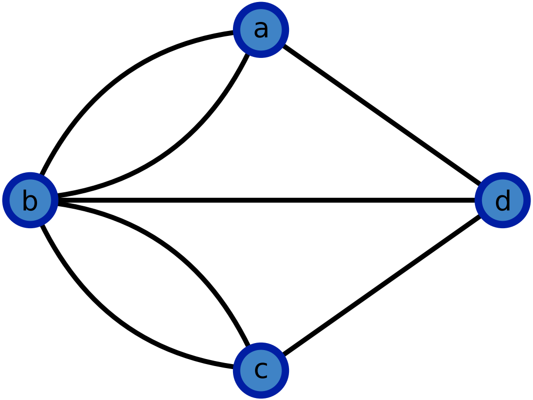


EULERIAN TOUR

Given an undirected graph $G(V, E)$,

construct an *Eulerian tour*, i.e., a path in G that traverses each edge in E exactly once, .

Bridges to Graphs



Eulerian tour

Given an undirected graph $G(V, E)$,

construct an *Eulerian tour*, i.e., a path in G that traverses each edge in E exactly once, if such a tour exists.

What Euler Proved

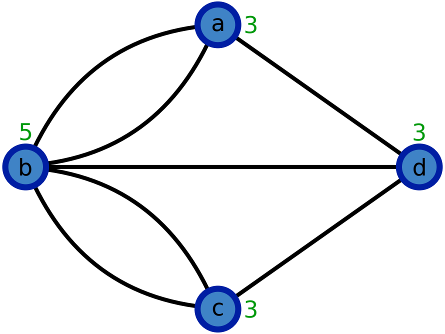
AD GEOMETRIAM SITVS PERTINENTIS. 139

§. 19. Praeterea ſi duo tantum numeri litteris A, B, C etc. adſcripti fuerint impares, reliqui vero omnes pares, tum ſemper deſideratus tranſitus ſuccedet, ſi modo curſus ex regione ad quam pontium impar numerus tendit incipiatur. Si enim pares numeri biſecentur atque etiam impares vnitate aucti, vti praeceptum eſt, ſumma harum medietatum vnitate erit maior quam numerus pontium, ideoque aequalis ipſi numero praefixo. Ex hocque porro perſpicitur, ſi quatuor vel ſex vel octo etc. fuerint numeri impares in ſecunda columna, tum ſummam numerorum tertiae columnae maiorem fore numero praefixo, eumque excedere vel vnitate, vel binario vel ternario etc. et idcirco tranſitus fieri nequit.

§. 20. Caſu ergo quocunque propoſito ſtatim facillime poterit cognoſci, vtrum tranſitus per omnes pontes ſemel inſtitui queat an non, ope huius regulae. Si fuerint plures duabus regiones, ad quas ducentium pontium numerus eſt impar, tum certo affirmari poteſt, talem tranſitum non dari. Si autem ad duas tantum regiones ducentium pontium numerus eſt impar, tunc tranſitus fieri poterit, ſi modo curſus in altera harum regionum incipiatur. Si denique nulla omnino fuerit regio, ad quam pontes numero impares conducant, tum tranſitus deſiderato modo inſtitui poterit, in quacunque regione ambulandi initium ponatur. Hac igitur data regula problemati propoſito pleniſſime ſatisfit.

S 2 §. 21.

What Euler Proved (in English)



- *Degree $d(v)$ of a node v* is the number of edges incident on it.

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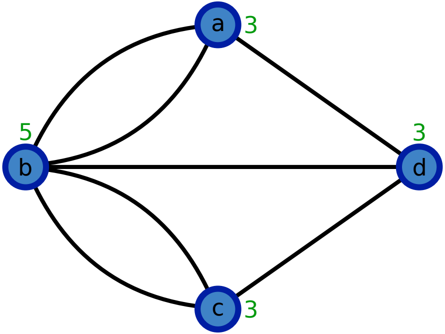

- *Degree $d(v)$ of a node v* is the number of edges incident on it.
- Euler's conclusion:
  1. If there are more than two nodes with odd degree, then the graph has no Eulerian tour.
  2. If exactly two nodes in the graph have odd degree, then there exists a tour that starts at one of these nodes and ends at the other node.
  3. If all nodes have even degree, then there exists a tour starting at any node.

What Didn't Euler Prove?

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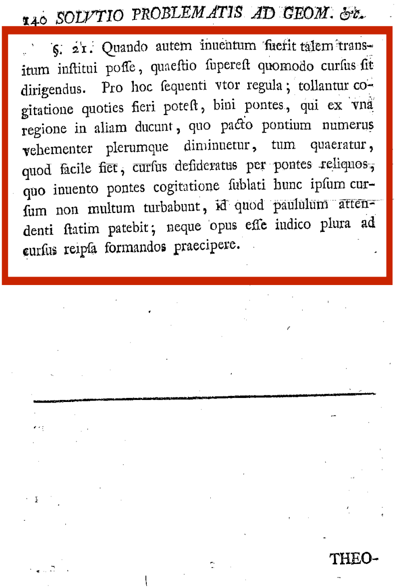

140 *SOLVTIO PROBLEMATIS AD GEOM. &c.*

§. 21. Quando autem inuentum fuerit talem tranſitum inſtitui poſſe, quaeſtio ſupereſt quomodo curſus ſit dirigendus. Pro hoc ſequenti vtor regula; tollantur cogitatione quoties fieri poteſt, bini pontes, qui ex vna regione in aliam ducunt, quo pacto pontium numerus vehementer plerumque diminuetur, tum quaeratur, quod facile fiet, curſus deſideratus per pontes reliquos, quo inuento pontes cogitatione ſublati hunc ipſum curſum non multum turbabunt, id quod paululum attendenti ſtatim patebit; neque opus eſſe indico plura ad curſus reipſa formandos praecipere.

THEO-

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  - Method to accomplish this was trivial, and Euler did not want to spend a great deal of time on it.

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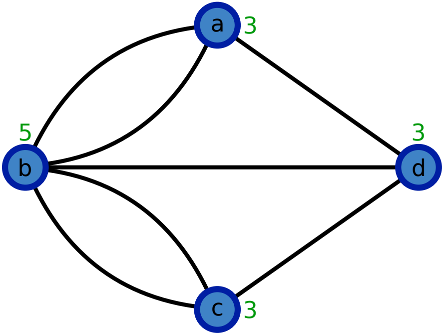
- Hierholzer provided an algorithm.

Hierholzer's Algorithm



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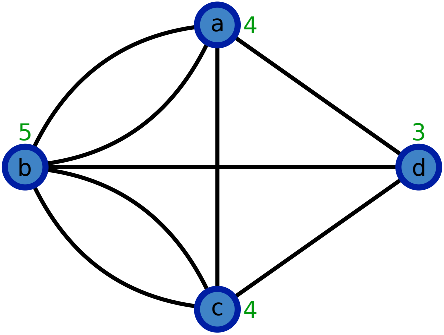

- If there are two nodes in G with odd degree, call them s and t.
- Otherwise, let s be any node in G.

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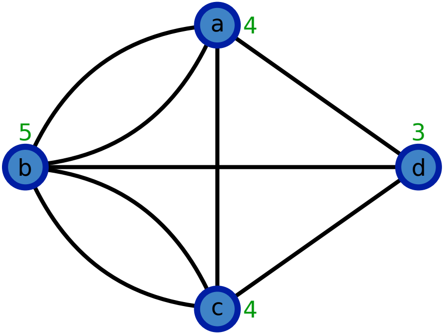

- If there are two nodes in G with odd degree, call them s and t.
- Otherwise, let s be any node in G.

$u \leftarrow s$ #u denotes the currently-visited node.
while $d(u) > 0$ **do**
  Output u.
  Let v be a neighbour of u.
  Delete the edge (u, v) from G.
  $u \leftarrow v$
end while

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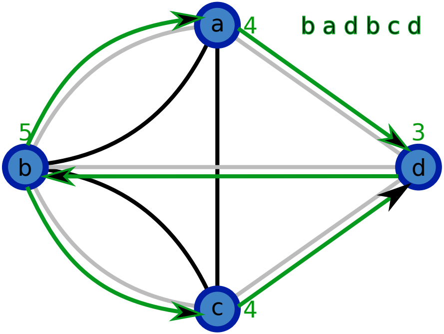

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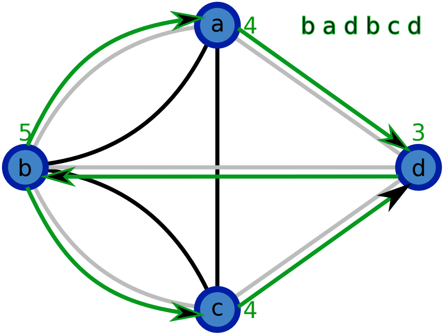
Properties of Heilholzer's Algorithm



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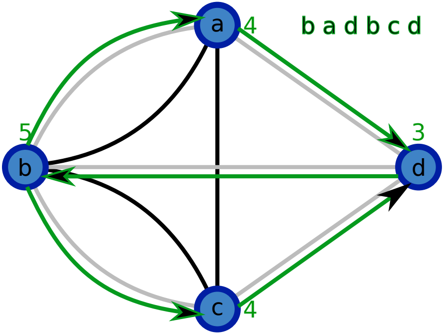

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- If it terminates, what can we say about node u at termination?
- Will all edges of G have been traversed upon termination?

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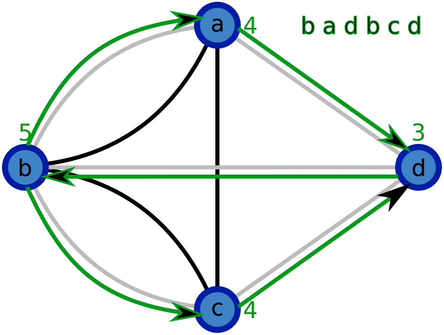


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end while
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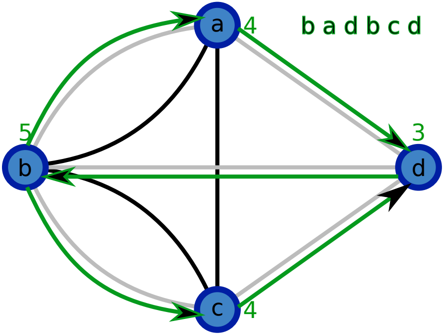

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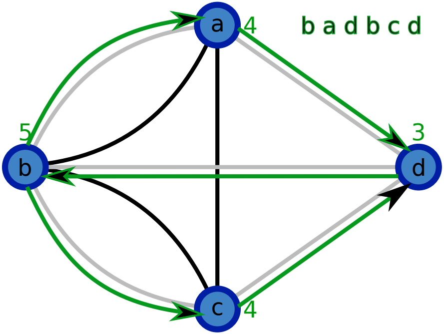
- If it terminates, what can we say about node u at termination?
  - If G had no nodes of odd degree, then $u = s$.
  - If G had two nodes of odd degree, then $u = t$.
- Will all edges of G have been traversed upon termination?

Properties of Heilholzer's Algorithm



$u \leftarrow s$
while $d(u) > 0$ **do**
  Output u.
  Let v be a neighbour of u.
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end while

- Will the algorithm terminate? Yes, because we traverse a new edge in each iteration.
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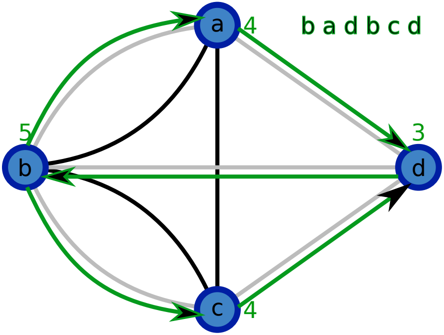

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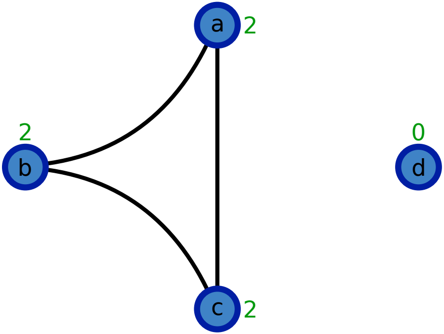

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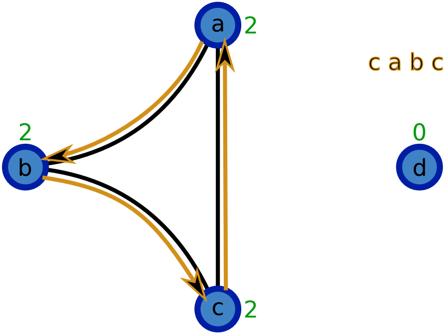

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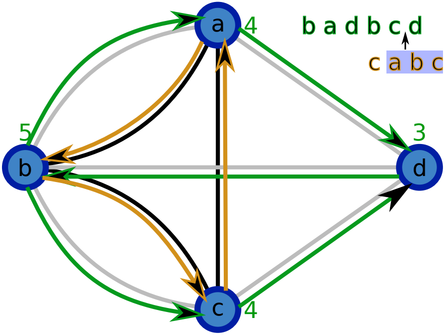


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u ← s
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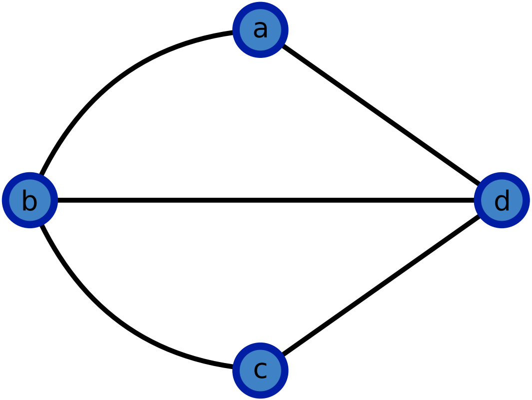
- Algorithm's running time is $O(|V| + |E|)$, i.e., linear in the size of G.

Visiting Nodes Rather than Edges



EULERIAN TOUR

Given an undirected graph $G(V, E)$,

construct an *Eulerian tour*, i.e., a path in G that traverses each edge in E exactly once, if such a tour exists.

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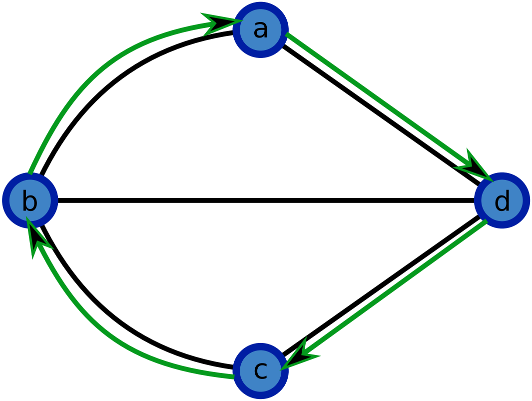

HAMILTONIAN CYCLE

Given an undirected graph $G(V, E)$,

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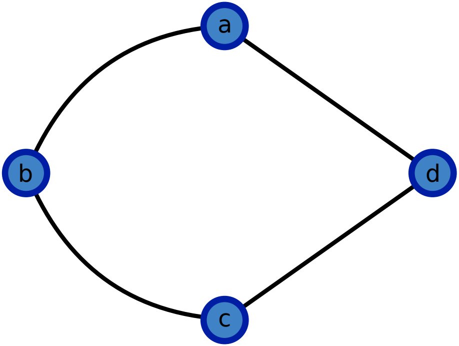
Conditions for Existence of Hamiltonian Cycle



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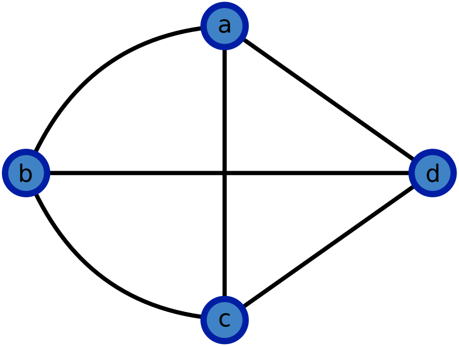

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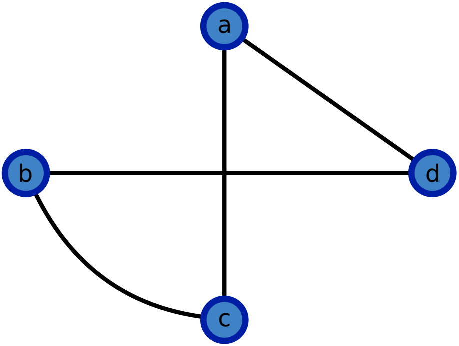

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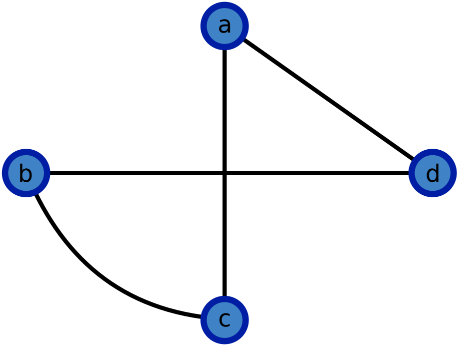

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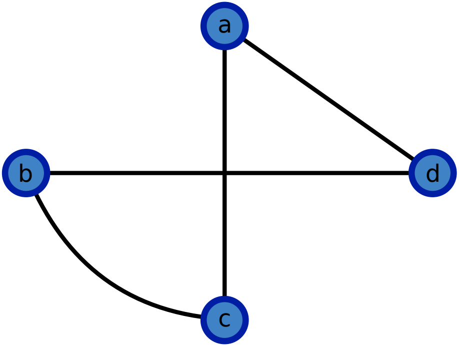

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  - two disconnected nodes with sum of degrees $\geq n$ (Ore, 1952).

Status of Hamiltonian Cycle Problem

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  - Dynamic programming: running time of $O(n^2 2^n)$ (Held and Karp 1962).
  - Fastest known algorithm runs in time $O(1.657^n)$ (Björklund 2010).