Greedy Algorithms

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Algorithm Design

- Start discussion of different ways of designing algorithms.
- Greedy algorithms, divide and conquer, dynamic programming.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.

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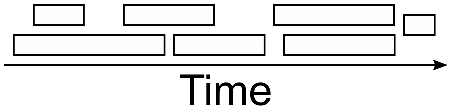
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.
- Greedy algorithms: make the current best choice.

- Input: Start and end time of each ride.
- Constraint: Cannot be in two places at one time.
- Goal: Compute the largest number of rides you can be on in one day.

Interval Scheduling



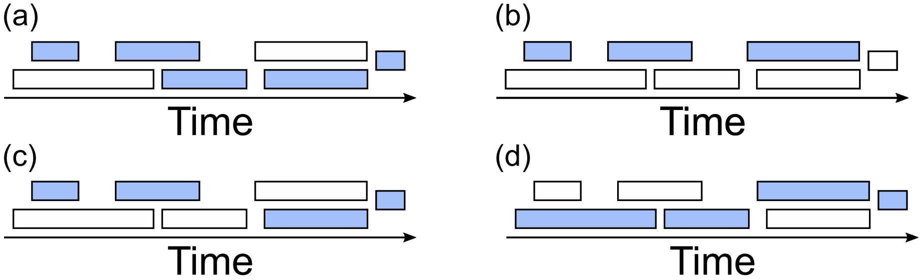
INTERVAL SCHEDULING

INSTANCE: Set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION: The largest subset of mutually compatible jobs.

- Two jobs are *compatible* if they do not overlap.
- This problem models the situation where you have a resource, a set of fixed jobs, and you want to schedule as many jobs as possible.
- For any input set of jobs, algorithm must provably compute the largest set of compatible jobs.

Interval Scheduling Example



Poll

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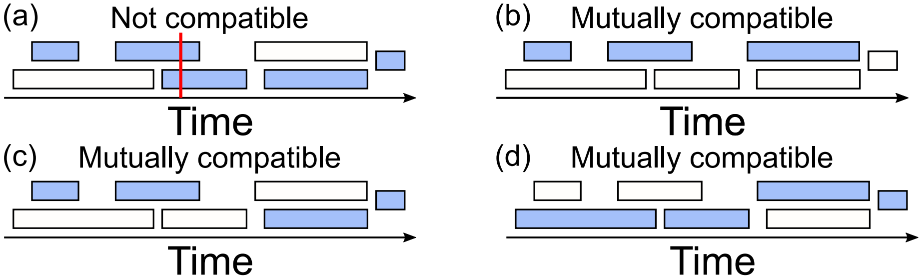

- Solutions (c) and (d) are optimal.
  - Each contains four jobs.
  - No set of mutually compatible jobs can contain more than four jobs.

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- Process jobs in some order. Add next job to the result if it is compatible with the jobs already in the result.
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  Earliest start time Increasing order of start time $s(i)$.
  Earliest finish time Increasing order of finish time $f(i)$.
  Shortest interval Increasing order of length $f(i) - s(i)$.
  Fewest conflicts Increasing order of the number of conflicting jobs.

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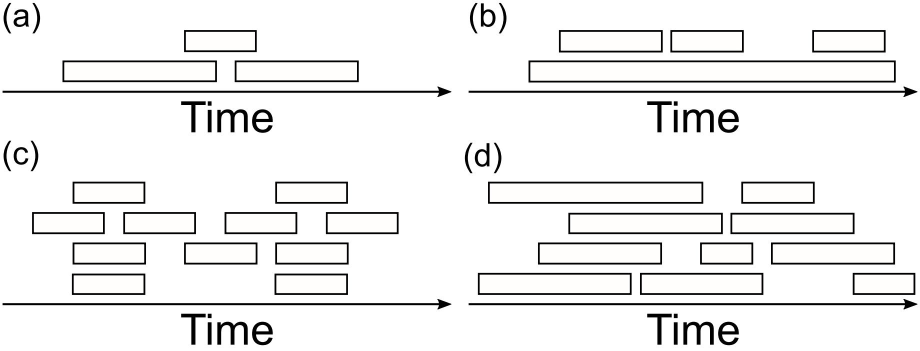

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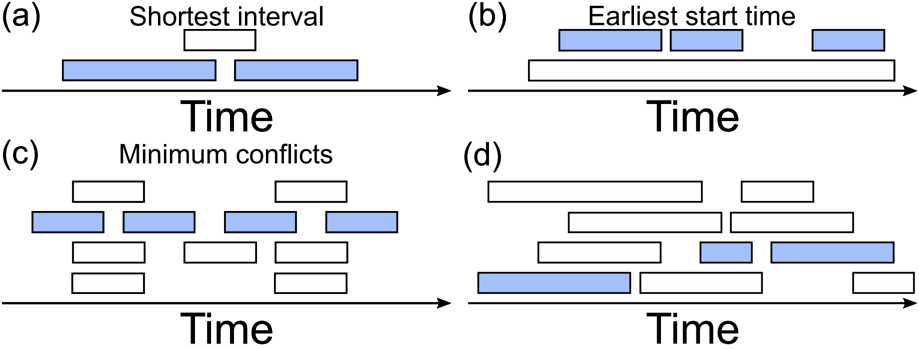

Interval Scheduling Algorithm: Earliest Finish Time

- Schedule jobs in order of earliest finish time (EFT).

Initially let R be the set of all requests, and let A be empty
While R is not yet empty
  Choose a request $i \in R$ that has the smallest finishing time
  Add request i to A
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Return the set A as the set of accepted requests

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- Claim: A is a compatible set of jobs. Proof follows by construction, i.e., the algorithm computes a compatible set of jobs.

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  - What does "better" mean?
  - How do we measure progress of the algorithm?
- Basic idea of proof:
  - We can sort jobs in any solution in increasing order of their finishing time.
  - Finishing time of job number r selected by $A \leq$ finishing time of job number r selected by any other algorithm.

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- Let $i_1, i_2, \ldots, i_k$ be the set of jobs in A in order.
- Let $j_1, j_2, \ldots, j_m$ be the set of jobs in O in order, $m \geq k$.
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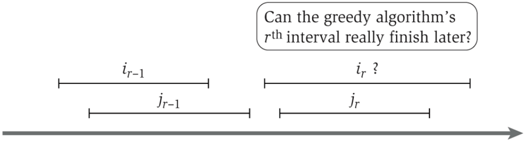

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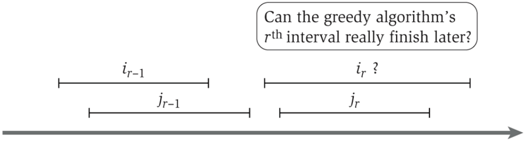

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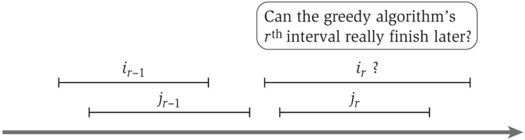
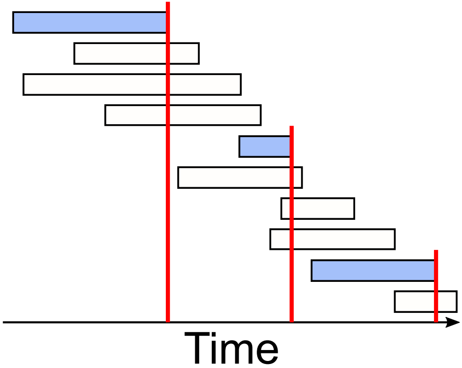


Figure 4.3 The inductive step in the proof that the greedy algorithm stays ahead.

- Claim: $m = k$.
- Claim: The greedy algorithm returns an optimal set A.

Implementing the EFT Algorithm

1. Reorder jobs so that they are in increasing order of finish time.
2. Store starting time of jobs in an array S.
3. $k = 1$.
4. While $k \leq |S|$,
   1. Output job k.
   2. Let finish time of job k be f.
   3. Iterate over S from index k onwards to find the first index i such that $S[i] \geq f$.
   4. $k = i$



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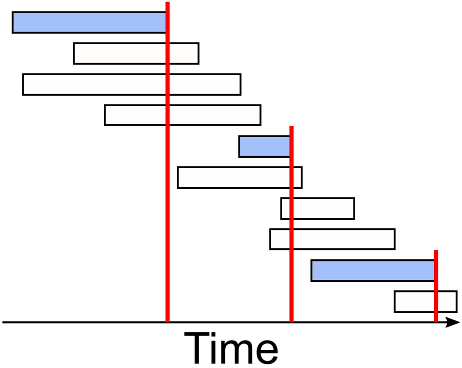

- Must be careful to iterate over S such that we never scan same index more than once.
- Running time is $O(n \log n)$, dominated by sorting.

12616	CS-4104	Data and Algorithm Analysis	L	3	75	CA Shaffer	T R	2:00PM	3:15PM	SURGE 107	14T
18154	CS-4104	Data and Algorithm Analysis	L	3	70	TM Murali	M W	2:30PM	3:45PM	SURGE 104C	14M
12617	CS-4114	Formal Languages	L	3	75	L Zhang	T R	9:30AM	10:45AM	MCB 129	09T
18155	CS-4204	Computer Graphics	L	3	36	D Gracanin	T R	11:00AM	12:15PM	MCB 224	11T
19593	CS-4264	Principles Computer Security	L	3	50	KE Giles	M W	2:30PM	3:45PM	GOODW 135	14M
12618	CS-4284	Systems & Networking Capstone	L	3	40	GV Back	M W	2:30PM	3:45PM	MCB 238	14M
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12620	CS-4604	Int Data Base Mgt Sys	L	3	55	RJ Quintin	M W	4:00PM	5:15PM	SURGE 109	16M
12621	CS-4624	Multimedia/Hypertext	L	3	70	EA Fox	T R	3:30PM	4:45PM	SURGE 109	15T
12622	CS-4644	Creative Computing Studio	L	3	25	SR Harrison	W	2:30PM	5:15PM	MAC 253A	14W
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12623	CS-4654	Intermed Data Analytics & ML	L	3	50	RB Gramacy	M W	4:00PM	5:15PM	SEITZ 313	16M
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	Comments for CRN 12624:	Prerequisite: C or better in CS 3704 OR CS 3714									
12625	CS-4784	Human-Computer Interact Capstn	L	3	30	AL Kavanaugh	F	1:00PM	3:45PM	MAC 253A	13F
	Comments for CRN 12625:	Prerequisite: CS 3724 required; CS 3714 or 3744 recommended									
19924	CS-4784	Human-Computer Interact Capstn	L	3	0	DS McCrickard	F	12:30PM	3:15PM	MCB 655	12F

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- Input: Start and end time of each class.
- Constraint: Cannot schedule two overlapping classes to the same room.
- Output: Assign each class to a room and use smallest number of rooms possible.

Interval Partitioning

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INSTANCE: Set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION: A partition of the jobs into k sets, where each set of jobs is mutually compatible, and k is minimised.

- This problem models the situation where you a set of fixed jobs, and you want to schedule all jobs using as few resources as possible.

Depth of Intervals

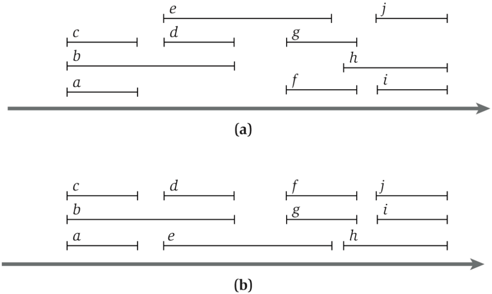
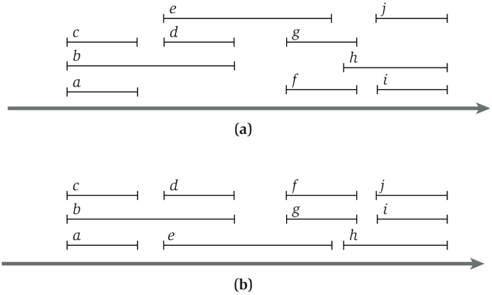


Figure 4.4 (a) An instance of the Interval Partitioning Problem with ten intervals (*a* through *j*). (b) A solution in which all intervals are scheduled using three resources: each row represents a set of intervals that can all be scheduled on a single resource.

- The *depth* of a set of intervals is the maximum number of intervals that contain any time point.

Depth of Intervals



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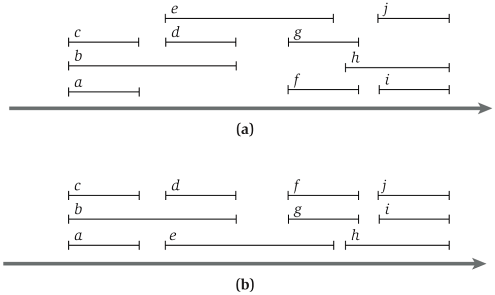

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- The *depth* of a set of intervals is the maximum number of intervals that contain any time point.
- Claim: In any instance of Interval Partitioning, $k \geq$ depth.
- Is it possible to compute the depth efficiently? Is $k =$ depth?

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1. Sort the start times and finish times of the jobs into a single list L.
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3. For i ranging from 1 to $2n$
   1. If L_i is a start time, increment d by 1.
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- Algorithm runs in $O(n \log n)$ time.

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Let I_1, I_2, ..., I_n denote the intervals in this order
For j = 1, 2, 3, ..., n
  For each interval I_i that precedes I_j in sorted order and overlaps it
     Exclude the label of I_i from consideration for I_j
  Endfor
  If there is any label from {1, 2, ..., d} that has not been excluded then
    Assign a nonexcluded label to I_j
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    Leave I_j unlabeled
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- Claim: Every interval gets a label and no pair of overlapping intervals get the same label.

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- Claim: The greedy algorithm is optimal.
- The running time of the algorithm is $O(n \log n)$. Can modify algorithm for computing depth to maintain set of available labels and to assign them efficiently.

Scheduling to Minimise Lateness

- Study different model: job i has a length $t(i)$ and a deadline $d(i)$.
- We want to schedule all n jobs on one resource.
- Our goal is to assign a starting time $s(i)$ to each job such that each job is delayed as little as possible.

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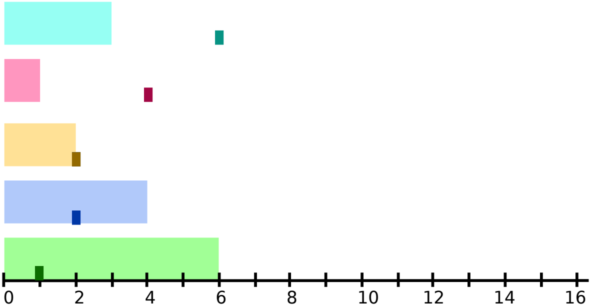
- Our goal is to assign a starting time $s(i)$ to each job such that each job is delayed as little as possible.
- A job i is *delayed* if $f(i) > d(i)$; the *lateness of the job* is

$$\max(0, f(i) - d(i)).$$

- The *lateness of a schedule* is

$$\max_{1 \leq i \leq n} \Big(\max \big(0, f(i) - d(i)\big)\Big).$$

Examples of Lateness



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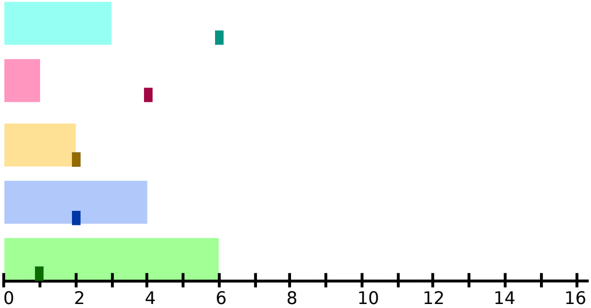

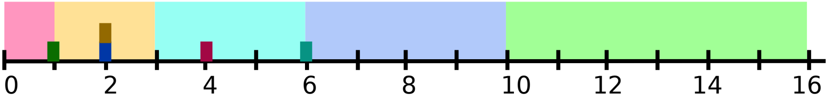

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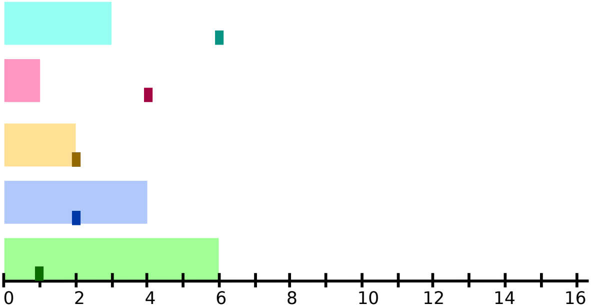

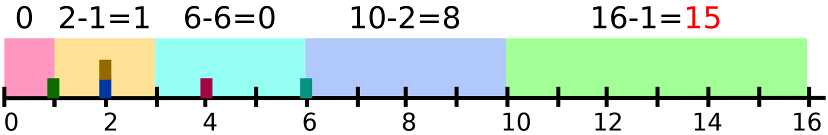

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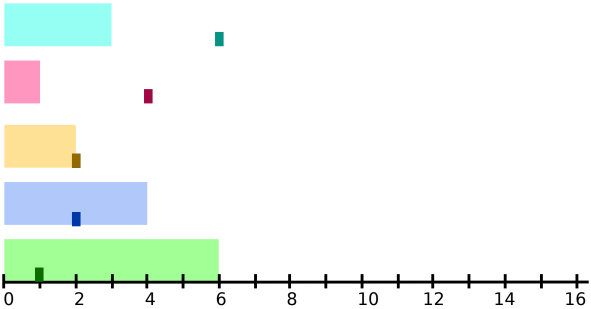
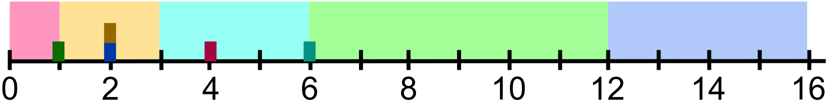
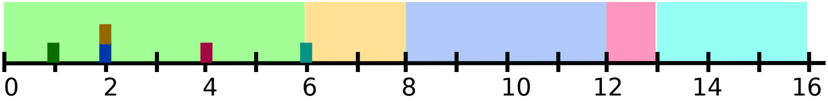


Poll





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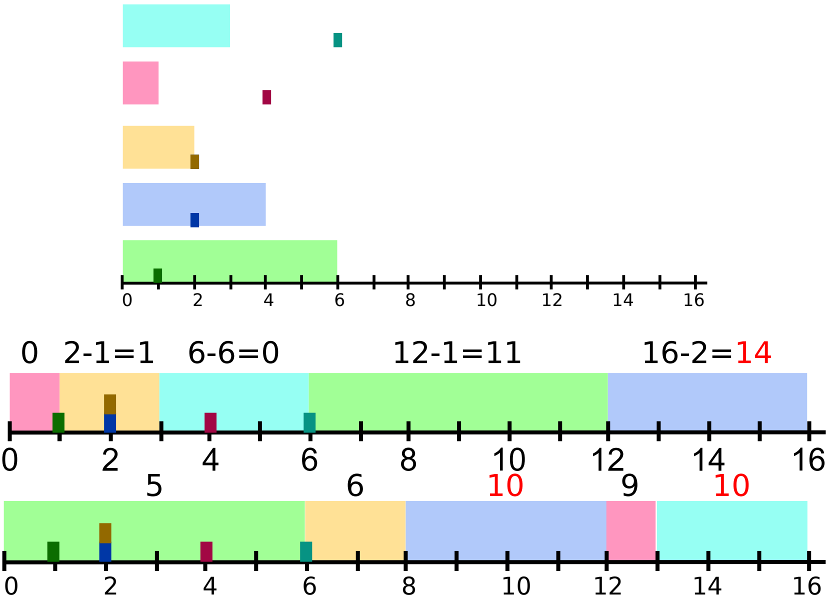
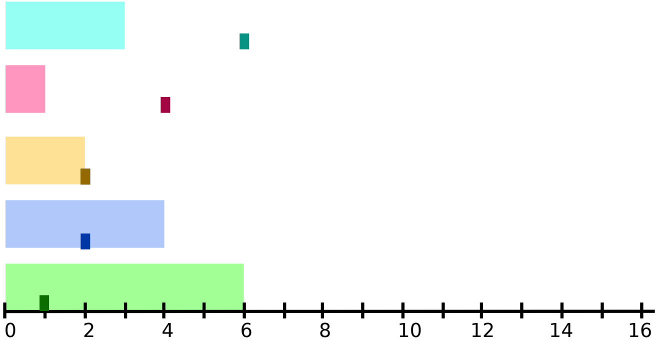


Scheduling to Minimise Lateness

Minimise Lateness

INSTANCE: Set $\{(t(i), d(i)), 1 \leq i \leq n\}$ of lengths and deadlines of n jobs.

SOLUTION: Set $\{s(i), 1 \leq i \leq n\}$ of start times such that $\max_{1 \leq i \leq n} \left(\max \left(0, s(i) + t(i) - d(i)\right)\right)$ is as small as possible.



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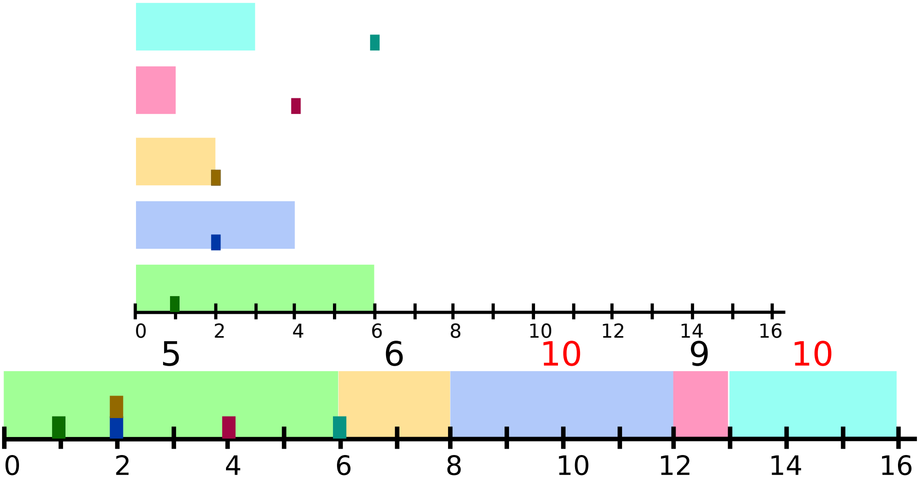

Template for Greedy Algorithm

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  Shortest length Increasing order of length $t(i)$.

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- Key question: In what order should we schedule the jobs?

  Shortest length Increasing order of length $t(i)$. Ignores deadlines completely! Shortest job may have a very late deadline.

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$d(i)$	100	10

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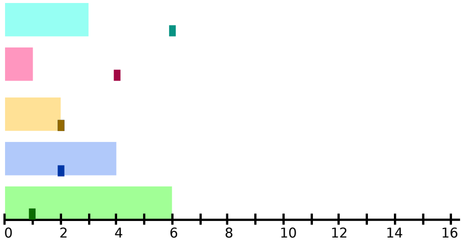
Earliest deadline Increasing order of deadline $d(i)$. Correct? Does it make sense to tackle jobs with earliest deadlines first?

Minimising Lateness: Earliest Deadline First

```
Order the jobs in order of their deadlines
Assume for simplicity of notation that d_1 ≤ ... ≤ d_n
Initially, f = s
Consider the jobs i = 1, ..., n in this order
   Assign job i to the time interval from s(i) = f to f(i) = f + t_i
   Let f = f + t_i
End
Return the set of scheduled intervals [s(i), f(i)] for i = 1, ..., n
```



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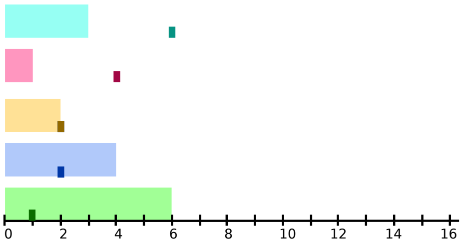

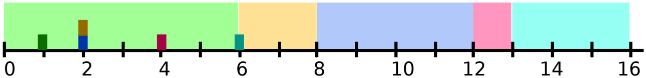

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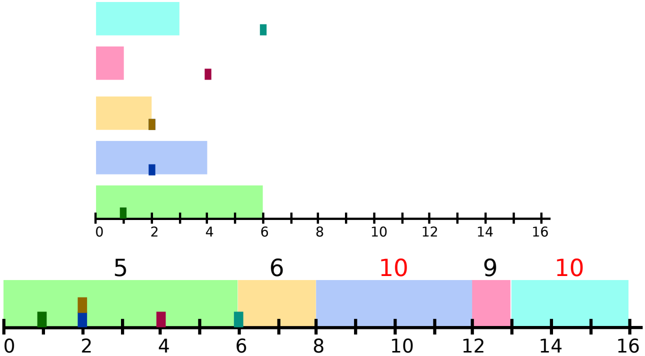
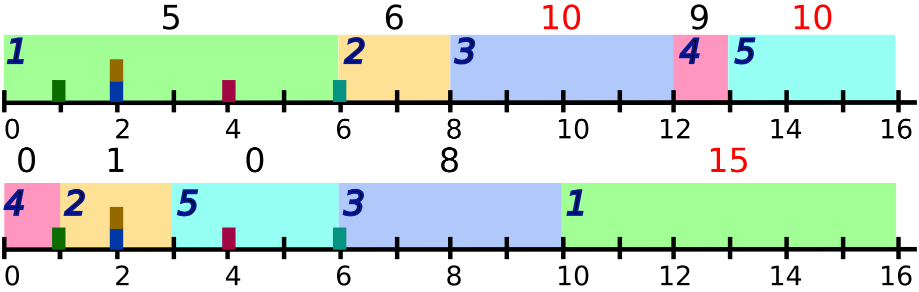


Inversions



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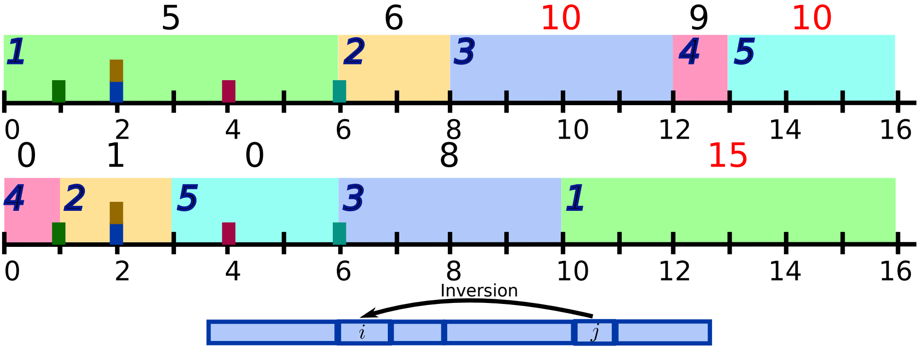

$d(j) < d(i)$ but $s(i) < s(j)$

- A schedule has an *inversion* if a job i with deadline $d(i)$ is scheduled before a job j with an earlier deadline $d(j)$, i.e., $d(j) < d(i)$ and $s(i) < s(j)$.
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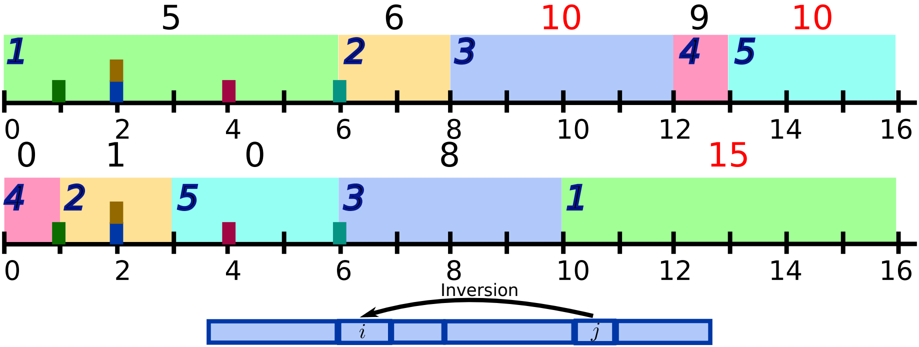
  - Examples: Poll

Inversions



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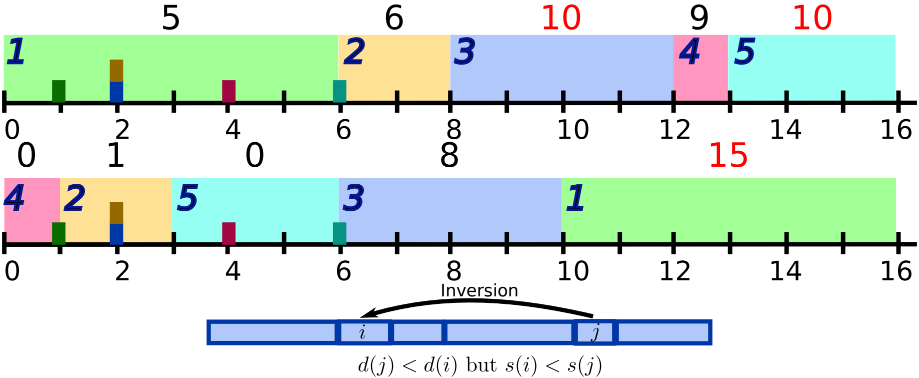
Inversions



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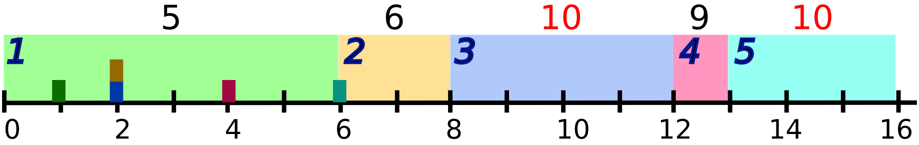
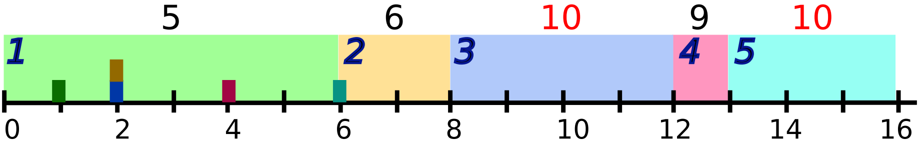
  - If i and j have the same deadlines, they cannot cause an inversion.
  - Examples: 2 and 1, 3 and 1, 4 and 1, 5 and 1, 4 and 2, 4 and 3, 5 and 3.
- Claim: If a schedule has an inversion, then there is a pair of *consecutive* jobs with an inversion, i.e., there are jobs i and j such that j is scheduled immediately after i and $d(j) < d(i)$.

Properties of Schedules



- Claim 1: The algorithm produces a schedule with no inversions and no idle time.

Properties of Schedules



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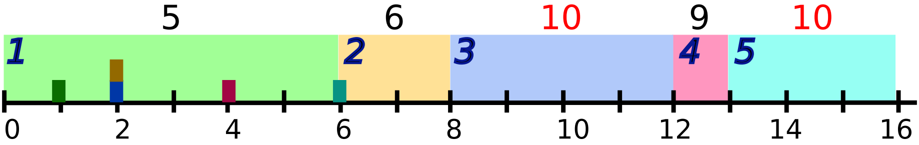

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Poll

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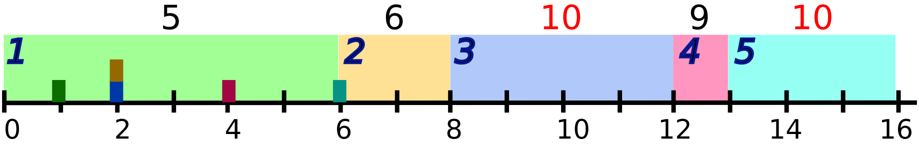

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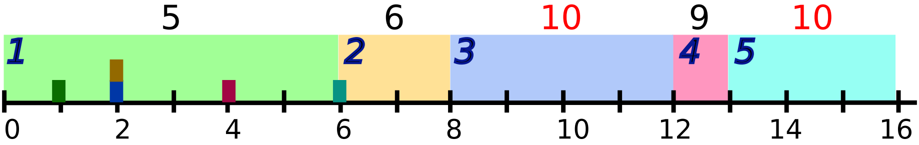

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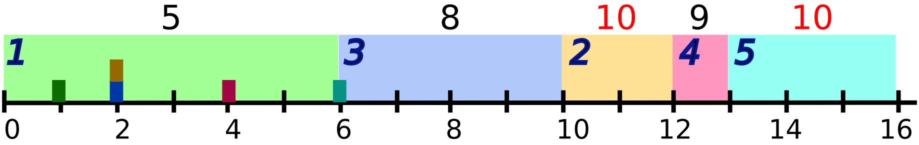

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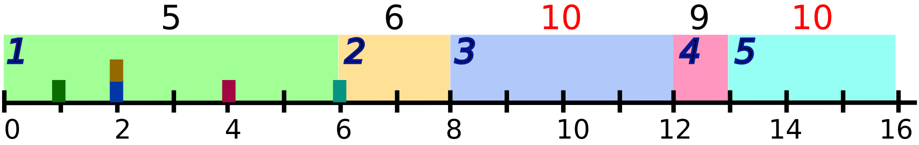

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- Claim 3: There is an optimal schedule with no idle time.

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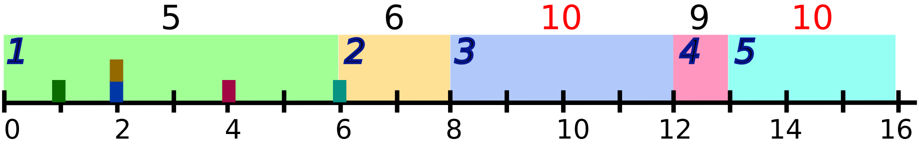

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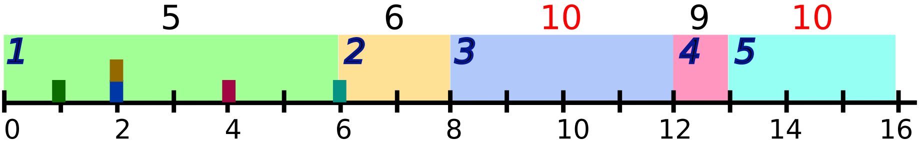
- Claim 4: There is an optimal schedule with no inversions and no idle time.

Properties of Schedules



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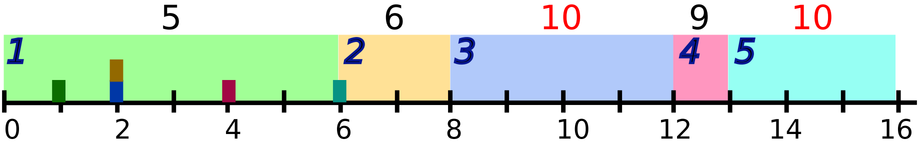

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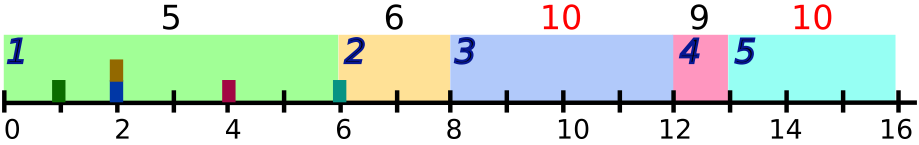

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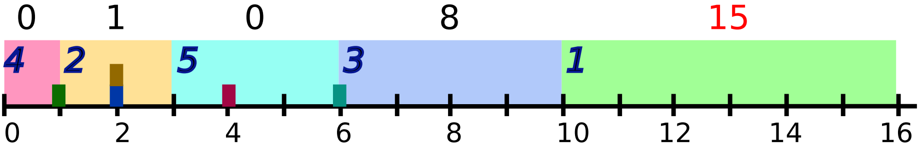
Proving Claim 4



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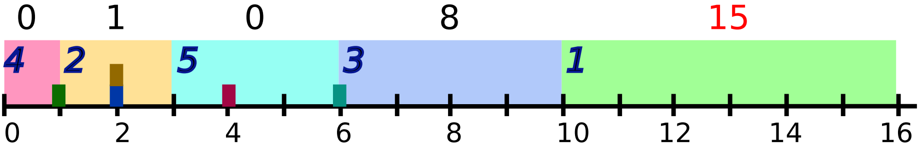

- Claim 4: There is an optimal schedule with no inversions and no idle time.
- Proof: Start with an optimal schedule O (that may have inversions) and use an *exchange argument* to convert O into a schedule that satisfies Claim 4 and has lateness not larger than O.

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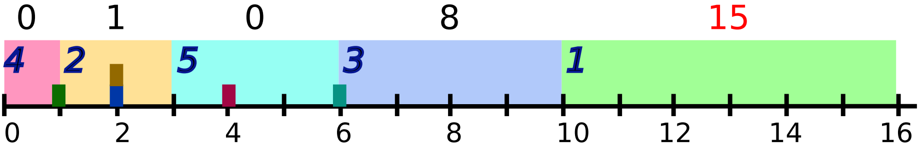

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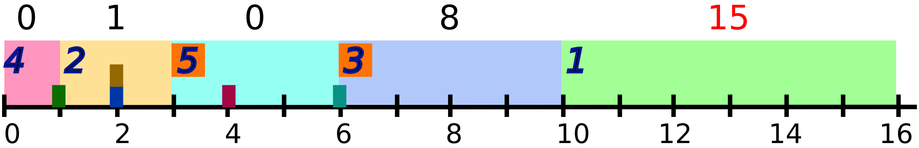

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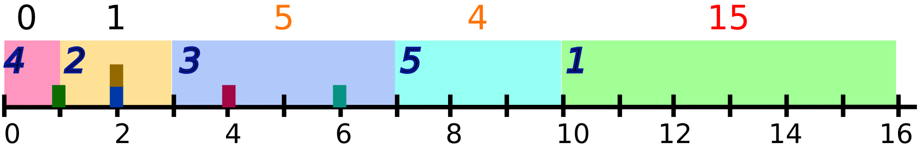
Proving Claim 4



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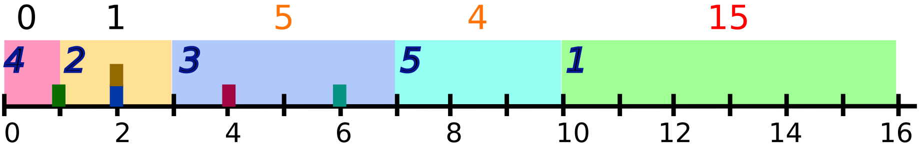

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- It is enough to prove the last item, since after $\binom{n}{2}$ swaps, we obtain a schedule with no inversions whose lateness is no larger than that of O.

Swapping Inverted Jobs

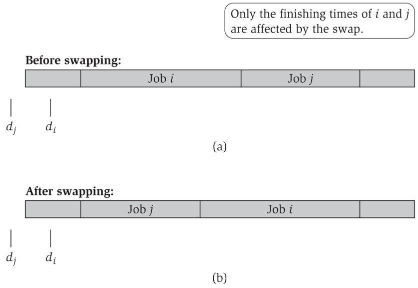
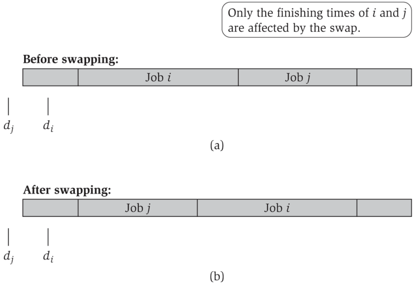


Figure 4.6 The effect of swapping two consecutive, inverted jobs.

- In O, assume each job r is scheduled for the interval $[s(r), f(r)]$ and has lateness $l(r)$. For O', let the lateness of job r be $l'(r)$.

Swapping Inverted Jobs



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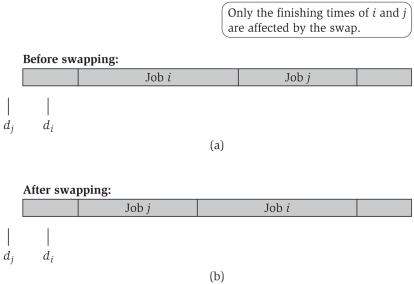

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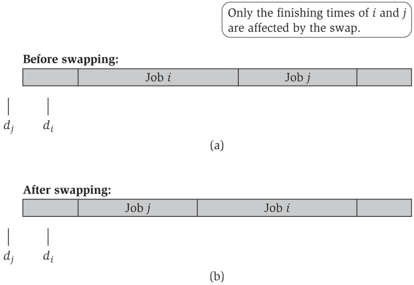

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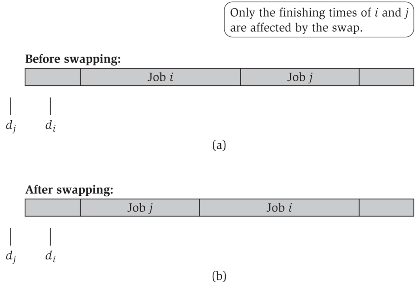
Swapping Inverted Jobs



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- Claim: $l'(k) = l(k)$, for all $k \neq i, j$.
- Claim: $l'(j) \leq l(j)$.
- Claim: $l'(i) \leq l(j)$ because $l'(i) = f(j) - d_i \leq f(j) - d_j = l(j)$.

Summary of Proof

1. Think of a schedule as a 2D point: x-coordinate is the number of inversions in the schedule and y-coordinate is the lateness of the schedule.

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3. Where does some other schedule B with no inversions lie?

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2. Where does the schedule A produced by the algorithm lie? Somewhere on the y-axis since it has no inversions, say $(0, l_A)$.
3. Where does some other schedule B with no inversions lie? Also at $(0, l_A)$ since all schedules with no inversions have the same lateness.

Summary of Proof

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9. We have a contradiction!
10. Lateness of A cannot be larger than that of O!

Common Mistakes with Exchange Arguments

- Wrong: start with algorithm's schedule A and argue that A cannot be improved by swapping two jobs.
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- Wrong: Swap two jobs that are not neighbouring in O. Pitfall is that the completion times of all intervening jobs changes.
- Correct: Show that an inversion exists between two neighbouring jobs and swap them.

Summary

- Greedy algorithms make local decisions.
- Three analysis strategies:

  Greedy algorithm stays ahead Show that after each step in the greedy algorithm, its solution is at least as good as that produced by any other algorithm.

  Structural bound First, discover a property that must be satisfied by every possible solution. Then show that the (greedy) algorithm produces a solution with this property.

  Exchange argument Transform the optimal solution in steps into the solution by the greedy algorithm without worsening the quality of the optimal solution.