Greedy Graph Algorithms

T. M. Murali

September 21, 23, 28, 2021
Algorithm Design

- Start discussion of different ways of designing algorithms.
- Greedy algorithms, divide and conquer, dynamic programming, flow-based approaches.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.
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- Greedy algorithms, divide and conquer, dynamic programming, flow-based approaches.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.
- Greedy algorithms: make the current best choice.
  - First discussed greedy algorithms for scheduling (Chapters 4.1 to 4.3).
  - Now we will discuss greedy graph algorithms.
Shortest Paths Problem

- $G(V, E)$ is a connected directed graph. Each edge $e$ has a length $l(e) \geq 0$.
- *Length of a path* $P$ is the sum of the lengths of the edges in $P$. 

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![Graph Diagram]

**INSTANCE:** A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$

**SOLUTION:** A set $\{P_u, u \in V\}$ of paths, where $P_u$ is the shortest path in $G$ from $s$ to $u$. 

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- Goal: compute the shortest path from a specified start node $s$ to each node in $V$.

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Shortest Paths Problem

- $G(V, E)$ is a connected directed graph. Each edge $e$ has a length $l(e) \geq 0$.
- **Length of a path** $P$ is the sum of the lengths of the edges in $P$.
- Goal: compute the shortest path from a specified start node $s$ to each node in $V$.
- Aside: If $G$ is undirected, convert to a directed graph by replacing each edge in $G$ by two directed edges.

**Shortest Paths**

**INSTANCE:** A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$

**SOLUTION:** A set $\{P_u, u \in V\}$ of paths, where $P_u$ is the shortest path in $G$ from $s$ to $u$. 
Shortest Paths Problem Instance

![Graph with nodes and edges labeled with weights]

- Nodes: s, e, a, b, c, f
- Edges with weights: s-a (1), a-c (3), s-e (4), e-c (1), e-b (2), b-a (1), b-f (3), a-f (2), b-c (2), c-f (3)

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Generalizing BFS

Graph representation with nodes labeled as s, b, c, e, f, and edges labeled with 1.
Generalizing BFS

Unweighted graph: Use BFS. Process nodes in non-decreasing order of distance.
Generalizing BFS

Weighted graph: Edge weights are integers. Can we make the graph unweighted?
Generalizing BFS

Add dummy nodes: Edge of weight $w$ gets $w - 1$ nodes.
Generalizing BFS

Dummy nodes: BFS computes shortest paths correctly. Running time is
Generalizing BFS

Dummy nodes: BFS computes shortest paths correctly. Running time is $O(m + n + \sum_{e \in E} l(e))$. *Pseudo-polynomial time*: depends on input values.
Generalizing BFS to Dijkstra’s Algorithm

Like BFS: explore nodes in non-increasing order of distance from $s$. Once a node is explored, its distance is fixed.
Generalizing BFS to Dijkstra’s Algorithm

Unlike BFS: Layers are not uniform. Which node to process next? Candidates are nodes with an edge from an explored node.
Generalizing BFS to Dijkstra’s Algorithm

For each unexplored node, determine “best” preceding explored node.
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Generalizing BFS to Dijkstra’s Algorithm

For each unexplored node, determine “best” preceding explored node. Record shortest path length only through explored nodes.
Generalizing BFS to Dijkstra’s Algorithm

For each unexplored node, determine “best” preceding explored node.
Generalizing BFS to Dijkstra’s Algorithm

Explore node with smallest path length only through explored nodes.
Generalizing BFS to Dijkstra’s Algorithm

Like BFS: Record previous node in the computed path.
Generalizing BFS to Dijkstra’s Algorithm

Follow previous nodes to compute shortest path. Like BFS: these edges form a tree.
Idea Underlying Dijkstra’s Algorithm

- Maintain a set $S$ of explored nodes.
  - For each node $u \in S$, compute a value $d(u)$, which (we will prove) is the length of the shortest path from $s$ to $u$.
  - For each node $x \not\in S$, maintain a value $d'(x)$, which is the length of the shortest path from $s$ to $x$ using only the nodes in $S$ (and $x$, of course).
Idea Underlying Dijkstra’s Algorithm

Maintain a set $S$ of explored nodes.

- For each node $u \in S$, compute a value $d(u)$, which (we will prove) is the length of the shortest path from $s$ to $u$.
- For each node $x \notin S$, maintain a value $d'(x)$, which is the length of the shortest path from $s$ to $x$ using only the nodes in $S$ (and $x$, of course).

“Greedily” add a node $v$ to $S$ that has the smallest value of $d'(v)$ (is closest to $s$ using only nodes in $S$).
Dijkstra’s Algorithm

Dijkstra’s Algorithm\((G, l, s)\)

1: \( S = \{s\} \) and \( d(s) = 0 \)
2: \textbf{while} \( S \neq V \) \textbf{do}
3: \textbf{for} every node \( x \in V - S \) \textbf{do}
4: \hspace{1em} Set \( d'(x) = \min_{u \in S} (d(u) + l(u, x)) \)
5: \hspace{1em} Set \( v = \arg \min_{x \in V - S} d'(x) \)
6: \hspace{1em} Add \( v \) to \( S \) and set \( d(v) = d'(v) \)

How do we parse \( v = \arg \min_{x \in V - S} d'(x) \)?

- Run over all (unexplored) nodes \( x \) in \( V - S \).
- Examine the \( d' \) values for these nodes.
- Return the argument (i.e., the node) that has the smallest value of \( d' \).

To compute the shortest paths: when adding a node \( v \) to \( S \), store the predecessor \( u \) that minimises \( d'(v) \).

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![Dijkstra's Algorithm diagram](image)
Dijkstra’s Algorithm

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  - Argument of min runs over all edges of the type \((u, x)\), where \(u\) is in \(S\) (i.e., \(u\) is explored).
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  - The algorithm is examining a particular (unexplored) node $x$ in $V - S$.
  - Argument of min runs over all edges of the type $(u, x)$, where $u$ is in $S$ (i.e., $u$ is explored).
  - For each such edge, we compute the length of the shortest path from $s$ to $x$ via $u$, which is $d(u) + l(u, x)$.
### Dijkstra’s Algorithm

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  - For each such edge, we compute the length of the shortest path from \(s\) to \(x\) via \(u\), which is \(d(u) + I(u, x)\).
  - We store the smallest of these values in \(d'(x)\).
Dijkstra’s Algorithm

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- How do we parse \(v = \arg \min_{x \in V - S} d'(x)\)?
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### Dijkstra’s Algorithm

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**Diagram:**
- Red nodes are explored.
- Light blue nodes are unexplored.
- Green edges represent the distances.

**Question:**

How do we parse \(v = \arg \min_{x \in V - S} d'(x)\)?

- Run over all (unexplored) nodes \(x\) in \(V - S\).
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**Note:** To compute the shortest paths: when adding a node \(v\) to \(S\), store the predecessor \(u\) that minimises \(d'(v)\).
Proof of Correctness

- Let $P_u$ be the path computed by the algorithm for an arbitrary node $u$.
- Claim: $P_u$ is the shortest path from $s$ to $u$.
- Prove by induction on the size of $S$. 
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  - Base case: $|S| = 1$. The only node in $S$ is $s$. 
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- Base case: $|S| = 1$. The only node in $S$ is $s$. 
- Inductive hypothesis: $|S| = k$, for some $k \geq 1$. 

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- Inductive hypothesis: $|S| = k$, for some $k \geq 1$. The algorithm has correctly computed $P_u$ for every node $u \in S$. Strong induction.
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  - Inductive step: $|S| = k + 1$ because we add the node $v$ to $S$. Could there be a shorter path $P$ from $s$ to $v$? We must prove this cannot be the case.
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![Diagram](image)

**Figure 4.8** The shortest path $P_v$ and an alternate $s$-$v$ path $P$ through the node $y$. 
Comments about Dijkstra’s Algorithm

- Algorithm cannot handle negative edge lengths. We will discuss the Bellman-Ford algorithm in a few weeks.
- Union of shortest paths output by Dijkstra’s algorithm forms a tree. Why?
Comments about Dijkstra’s Algorithm

- Algorithm cannot handle negative edge lengths. We will discuss the Bellman-Ford algorithm in a few weeks.
- Union of shortest paths output by Dijkstra’s algorithm forms a tree. Why?
- Union of shortest paths from a fixed source $s$ forms a tree; paths not necessarily computed by Dijkstra’s algorithm.
Running time of Dijkstra’s Algorithm

Dijkstra’s Algorithm \((G, l, s)\)

1: \(S = \{s\}\) and \(d(s) = 0\)
2: while \(S \neq V\) do
3:   for every node \(x \in V - S\) do
4:     Set \(d'(x) = \min_{(u, x) \in S} (d(u) + l(u, x))\)
5:   Set \(v = \arg \min_{x \in V - S} d'(x)\)
6:   Add \(v\) to \(S\) and set \(d(v) = d'(v)\)

- \(V\) has \(n\) nodes and \(E\) has \(m\) edges.
- How many iterations are there of the while loop?
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- In each iteration, for each node \(x \in V - S\), compute

\[
d'(x) = \min_{(u, x), u \in S} (d(u) + l(u, x))
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- $V$ has $n$ nodes and $E$ has $m$ edges.
- How many iterations are there of the while loop? $n – 1$.
- In each iteration, for each node $x \in V – S$, compute
  $$d'(x) = \min_{(u, x), u \in S} (d(u) + l(u, x))$$
- Running time per iteration is

\[O(m)\]
Running time of Dijkstra’s Algorithm

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- \( V \) has \( n \) nodes and \( E \) has \( m \) edges.
- How many iterations are there of the while loop? \( n - 1 \).
- In each iteration, for each node \( x \in V - S \), compute

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d'(x) = \min_{(u, x)} \left( d(u) + l(u, x) \right)
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- Running time per iteration is \( O(m) \), since the algorithm processes each edge \((u, x)\) in the graph exactly once (when computing \( d'(x) \)).
- The overall running time is \( O(nm) \).
A Faster implementation of Dijkstra’s Algorithm

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---

Observation: If we add $v$ to $S$, $d'(x)$ changes only if $(v, x)$ is an edge in $G$ and $x$ is not in $S$.

Idea: For each node $x \in V - S$, store the current value of $d'(x)$. Upon adding a node $v$ to $S$, update $d'(x)$ only for neighbours of $v$ that are not in $S$.

How do we efficiently compute $v = \arg \min_{x \in V - S} d'(x)$?

Use a priority queue!
A Faster implementation of Dijkstra’s Algorithm

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6. Set \(d'(x) = \min_{(u,x):u \in S}(d(u) + l(u,x))\)

- Observation: If we add \(v\) to \(S\), \(d'(x)\) changes only if \((v,x)\) is an edge in \(G\) and \(x\) is not in \(S\).
- Idea: For each node \(x \in V - S\), store the current value of \(d'(x)\). Upon adding a node \(v\) to \(S\), update \(d'()\) only for neighbours of \(v\) that are not in \(S\).
A Faster implementation of Dijkstra’s Algorithm

Dijkstra’s Algorithm \((G, l, s)\)

1. \(S = \{s\}\) and \(d(s) = 0\)
2. \(\textbf{while } S \neq V \textbf{ do} \)
   3. Set \(v = \arg \min_{x \in V - S} d'(x)\)
   4. Add \(v\) to \(S\) and set \(d(v) = d'(v)\)
   5. \(\textbf{for every node } x \in V - S \textbf{ do} \)
   6. Set \(d'(x) = \min_{(u, x)}: u \in S (d(u) + l(u, x))\)

- Observation: If we add \(v\) to \(S\), \(d'(x)\) changes only if \((v, x)\) is an edge in \(G\) and \(x\) is not in \(S\).
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- How do we efficiently compute \(v = \arg \min_{x \in V - S} d'(x)\)?
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**Dijkstra’s Algorithm** \((G, l, s)\)

1. \(S = \{s\}\) and \(d(s) = 0\)
2. while \(S \neq V\) do
3.   Set \(v = \arg \min_{x \in V - S} d'(x)\)
4.   Add \(v\) to \(S\) and set \(d(v) = d'(v)\)
5.   for every node \(x \in V - S\) do
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- Observation: If we add \(v\) to \(S\), \(d'(x)\) changes only if \((v, x)\) is an edge in \(G\) and \(x\) is not in \(S\).
- Idea: For each node \(x \in V - S\), store the current value of \(d'(x)\). Upon adding a node \(v\) to \(S\), update \(d'()\) only for neighbours of \(v\) that are not in \(S\).
- How do we efficiently compute \(v = \arg \min_{x \in V - S} d'(x)\)?:
- Use a priority queue!
Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm** \((G, l, s)\)

1: **Insert** \((Q, s, 0)\).
2: **while** \(S \neq V\) **do**
3: \((v, d'(v)) = \text{ExtractMin}(Q)\)
4: Add \(v\) to \(S\) and set \(d(v) = d'(v)\)
5: **for** every node \(x \in V - S\) such that \((v, x)\) is an edge in \(G\) **do**
6: \[\text{if } d(v) + l(v, x) < d'(x) \text{ then}\]
7: \[d'(x) = d(v) + l(v, x)\]
8: \[\text{ChangeKey}(Q, x, d'(x))\]

- For each node \(x \in V - S\), store the pair \((x, d'(x))\) in a priority queue \(Q\) with \(d'(x)\) as the key.
- Determine the next node \(v\) to add to \(S\) using \text{ExtractMin} (line 3).
- After adding \(v\) to \(S\), for each node \(x \in V - S\) such that there is an edge from \(v\) to \(x\), check if \(d'(x)\) should be updated, i.e., if there is a shortest path from \(s\) to \(x\) via \(v\) (lines 5–8).
- In line 8, if \(x\) is not in \(Q\), simply insert it.
Running Time of Faster Dijkstra’s Algorithm

Dijkstra’s Algorithm \((G, l, s)\)

1: Insert \((Q, s, 0)\).
2: while \(S \neq V\) do
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8: \(\text{ChangeKey}(Q, x, d'(x))\)

- How many times does the algorithm invoke \(\text{ExtractMin}\)?
# Running Time of Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm** ($G, l, s$)

1. **Insert** ($Q, s, 0$).
2. while $S \neq V$ do
   3. $(v, d'(v)) = \text{ExtractMin}(Q)$
   4. Add $v$ to $S$ and set $d(v) = d'(v)$
   5. for every node $x \in V - S$ such that $(v, x)$ is an edge in $G$ do
      6. if $d(v) + l(v, x) < d'(x)$ then
         7. $d'(x) = d(v) + l(v, x)$
      8. $\text{ChangeKey}(Q, x, d'(x))$

- How many times does the algorithm invoke $\text{ExtractMin}$? $n - 1$ times.
## Running Time of Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm**($G, l, s$)

1. **Insert**($Q, s, 0$).
2. **while** $S \neq V$ **do**
3. \hspace{1em} $(v, d'(v)) = \text{ExtractMin}(Q)$
4. \hspace{1em} Add $v$ to $S$ and set $d(v) = d'(v)$
5. \hspace{1em} **for** every node $x \in V - S$ such that $(v, x)$ is an edge in $G$ **do**
6. \hspace{2em} **if** $d(v) + l(v, x) < d'(x)$ **then**
7. \hspace{3em} $d'(x) = d(v) + l(v, x)$
8. \hspace{3em} **ChangeKey**($Q, x, d'(x)$)

- How many times does the algorithm invoke $\text{ExtractMin}$? $n - 1$ times.
- For every node $v$, what is the running time of step 5?
Running Time of Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm** \((G, l, s)\)

1. **\text{INSERT}(Q, s, 0).**
2. **while** \(S \neq V\) **do**
3. \((v, d'(v)) = \text{EXTRACTMIN}(Q)\)
4. **Add** \(v\) **to** \(S\) **and set** \(d(v) = d'(v)\)
5. **for** every node \(x \in V - S\) **such that** \((v, x)\) **is an edge in** \(G\) **do**
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8. **\text{CHANGEKEY}(Q, x, d'(x))**

- How many times does the algorithm invoke **\text{EXTRACTMIN}?** \(n - 1\) times.
- For every node \(v\), what is the running time of step 5? \(O(d_v)\), the number of *outgoing* neighbours of \(v\).
Running Time of Faster Dijkstra’s Algorithm

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- For every node \(v\), what is the running time of step 5? \(O(d_v)\), the number of *outgoing* neighbours of \(v\).
- What is the total running time of step 5?
Running Time of Faster Dijkstra’s Algorithm

Dijkstra’s Algorithm\((G, l, s)\)

1: \textbf{Insert}(Q, s, 0).
2: \textbf{while } S \neq V \textbf{ do}
3: \quad (v, d'(v)) = \textbf{ExtractMin}(Q)
4: \quad \textbf{Add } v \textbf{ to } S \textbf{ and set } d(v) = d'(v)
5: \quad \textbf{for } \text{ every node } x \in V - S \textbf{ such that } (v, x) \textbf{ is an edge in } G \textbf{ do}
6: \quad \quad \textbf{if } d(v) + l(v, x) < d'(x) \textbf{ then}
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- How many times does the algorithm invoke \textbf{ExtractMin}? \(n - 1\) times.
- For every node \(v\), what is the running time of step 5? \(O(d_v)\), the number of \textit{outgoing} neighbours of \(v\).
- What is the total running time of step 5? \(\sum_{v \in V} O(d_v) = O(m)\).
Running Time of Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm** $(G, l, s)$

1. **Insert** $(Q, s, 0)$.
2. **while** $S \neq V$ **do**
3.  $(v, d'(v)) = \text{ExtractMin}(Q)$
4.  Add $v$ to $S$ and set $d(v) = d'(v)$
5.  **for** every node $x \in V - S$ such that $(v, x)$ is an edge in $G$ **do**
6.    **if** $d(v) + l(v, x) < d'(x)$ **then**
7.      $d'(x) = d(v) + l(v, x)$
8.      **CHANGE KEY** $(Q, x, d'(x))$

- How many times does the algorithm invoke `ExtractMin`? $n - 1$ times.
- For every node $v$, what is the running time of step 5? $O(d_v)$, the number of outgoing neighbours of $v$.
- What is the total running time of step 5? $\sum_{v \in V} O(d_v) = O(m)$.
- How many times does the algorithm invoke `CHANGE KEY`?
### Running Time of Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm** \((G, l, s)\)

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- How many times does the algorithm invoke **ExtractMin**? \(n - 1\) times.
- For every node \(v\), what is the running time of step 5? \(O(d_v)\), the number of *outgoing* neighbours of \(v\).
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- How many times does the algorithm invoke **CHANGEKEY**? At most \(m\) times.
**Running Time of Faster Dijkstra’s Algorithm**

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- How many times does the algorithm invoke \(\text{ChangeKey}\)? At most \(m\) times.
- What is total running time of the algorithm?

T. M. Murali September 21, 23, 28, 2021 Greedy Graph Algorithms
Running Time of Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm** \( (G, l, s) \)

1. **Insert** \((Q, s, 0)\).
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- How many times does the algorithm invoke \( \text{ExtractMin} \)? \( n - 1 \) times.
- For every node \( v \), what is the running time of step 5? \( O(d_v) \), the number of \emph{outgoing} neighbours of \( v \).
- What is the total running time of step 5? \( \sum_{v \in V} O(d_v) = O(m) \).
- How many times does the algorithm invoke \( \text{ChangeKey} \)? At most \( m \) times.
- What is total running time of the algorithm? \( O(m \log n) \).

---

State of the art: Fibonacci heaps achieve a running time of \( O(m) \) for all \( \text{ChangeKey} \) operations, for a running time of \( O(n \log n + m) \).
Running Time of Faster Dijkstra’s Algorithm

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- How many times does the algorithm invoke **ChangeKey**? At most \(m\) times.
- What is total running time of the algorithm? \(O(m \log n)\).
- State of the art: Fibonacci heaps achieve a running time of \(O(m)\) for all **ChangeKey** operations, for a running time of \(O(n \log n + m)\).
Network Design

- Connect a set of nodes using a set of edges with certain properties.
- Input is usually a graph and the desired network (the output) should use subset of edges in the graph.
- Example: connect all nodes using a cycle of shortest total length.
Network Design

- Connect a set of nodes using a set of edges with certain properties.
- Input is usually a graph and the desired network (the output) should use subset of edges in the graph.
- Example: connect all nodes using a cycle of shortest total length. This problem is the NP-complete traveling salesman problem.
Minimum Spanning Tree (MST)

- Given an undirected graph $G(V, E)$ with a cost $c(e) > 0$ associated with each edge $e \in E$.
- Find a subset $T$ of edges such that the graph $(V, T)$ is connected and the cost $\sum_{e \in T} c(e)$ is as small as possible.
Minimum Spanning Tree (MST)

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(a) (b) (c) (d) (e) (f)
Minimum Spanning Tree (MST)

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(a) Not connected
(b) Not smallest cost
(c) Not smallest cost
(d) Not smallest cost
(e) Not smallest cost
(f) Smallest cost
Minimum Spanning Tree (MST)

**Minimum Spanning Tree**

**INSTANCE:** An undirected graph $G(V, E)$ and a function $c : E \to \mathbb{R}^+$

**SOLUTION:** A set $T \subseteq E$ of edges such that $(V, T)$ is connected and the cost $\sum_{e \in T} c(e)$ is as small as possible.

Claim: If $T$ is a minimum-cost solution to this problem then $(V, T)$ is a tree.

A subset $T$ of $E$ is a spanning tree of $G$ if $(V, T)$ is a tree.
Minimum Spanning Tree (MST)

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A subset $T$ of $E$ is a *spanning tree* of $G$ if $(V, T)$ is a tree.
Greedy Algorithm for the MST Problem

- Template: process edges in some order. Add an edge to $T$ if tree property is not violated.
Greedy Algorithm for the MST Problem

- **Template**: process edges in some order. Add an edge to $T$ if tree property is not violated.

  - **Increasing cost order**  Process edges in increasing order of cost. Discard an edge if it creates a cycle.

  - **Dijkstra-like**  Start from a node $s$ and grow $T$ outward from $s$: add the node that can be attached most cheaply to current tree.

  - **Decreasing cost order**  Delete edges in order of decreasing cost as long as graph remains connected.
Greedy Algorithm for the MST Problem

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- Which of these algorithms works?
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- Which of these algorithms works? All of them!
Greedy Algorithm for the MST Problem

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  - **Decreasing cost order**: Delete edges in order of decreasing cost as long as graph remains connected. *Reverse-Delete algorithm*

- Which of these algorithms works? All of them!

- Simplifying assumption: all edge costs are distinct.
Characterising MSTs

Does the edge of smallest cost belong to an MST?
Characterising MSTs

- Does the edge of smallest cost belong to an MST? Yes. Why?

Wrong proof: because Kruskal's algorithm adds it. We have not yet proved correctness of Kruskal's algorithm!

Correct proof: will work it out soon.
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- Which edges cannot belong to an MST?
  - What happens when we add an edge to an MST?
  - We obtain a cycle.
  - Which edge in the cycle can we be sure does not belong to an MST?
Graph Cuts

A cut in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).
Graph Cuts

- A *cut* in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).

![Graph](image)
A cut in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).

Every set $S \subset V$ ($S$ cannot be empty or the entire set $V$) has a corresponding cut: $\text{cut}(S)$ is the set of edges $(v, w)$ such that $v \in S$ and $w \in V - S$.

$\text{cut}(S)$ is a “cut” because deleting the edges in $\text{cut}(S)$ disconnects $S$ from $V - S$. 
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(a) Not cut($\{a, b, d\}$): (c, g)
(b) Is cut($\{a, b, d\}$)
(c) Not cut($\{a, b, d\}$): (a, b)
(d) Not cut($\{a, b, d\}$): (b, e)
Graph Cuts

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![Diagrams](a) (b) (c) (d)
Graph Cuts

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(a) Not cut({a, e, g}): (b, c) Not cut({a, g, e}): (f, h) Is cut({a, e, g}) Not cut({a, e, g}): (d, e)
Cut Property

When is it safe to include an edge in an MST?

Claim: For every $S \subset V$, $S \neq \emptyset$, every MST contains the cheapest edge in cut($S$).

Proof by contradiction using exchange argument. How do you state the contradiction to the claim?

Let $e = (u, v)$ be the cheapest edge in cut($S$).

Proof strategy: If $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$. 

The diagram shows a graph with labeled edges, but the text focuses on the cut property and the proof method.
**Cut Property**

- When is it safe to include an edge in an MST?
- Claim: For every $S \subset V, S \neq \emptyset$, every MST contains the cheapest edge in cut($S$).

![Diagram of a graph with labeled edges and nodes representing the cut property example.]
Cut Property

- When is it safe to include an edge in an MST?
- Claim: For every $S \subset V, S \neq \emptyset$, every MST contains the cheapest edge in cut($S$).
- Proof by contradiction using exchange argument.

How do you state the contradiction to the claim?

$\text{Claim: For every } S \subset V, S \neq \emptyset, \text{ every MST contains the cheapest edge in cut}(S).$

Proof strategy: If $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$.
Cut Property

- When is it safe to include an edge in an MST?
- Claim: For every $S \subset V, S \neq \emptyset$, every MST contains the cheapest edge in cut($S$).
- Proof by contradiction using exchange argument.
- How do you state the contradiction to the claim?

![Graph Diagram]

**Proof strategy:** If $T$ does not contain $e = (u, v)$, show that there is a tree with smaller cost than $T$ that contains $e$. 

**Diagram:**

- Vertices: a, b, c, d, e, f, g, h
- Edges and Weights:
  - a to b: 4
  - a to g: 5
  - a to c: 15
  - b to c: 11
  - c to h: 7
  - d to e: 12
  - d to e: 3
  - e to f: 20
  - e to d: 8
Cut Property

- When is it safe to include an edge in an MST?
- Claim: For every $S \subset V, S \neq \emptyset$, every MST contains the cheapest edge in $\text{cut}(S)$.
- Proof by contradiction using exchange argument.
- How do you state the contradiction to the claim? There is a set $S \subset V$ and an MST $T$ such that $T$ does not contain the cheapest edge in $\text{cut}(S)$. 
Cut Property

- When is it safe to include an edge in an MST?
- Claim: For every $S \subset V$, $S \neq \emptyset$, every MST contains the cheapest edge in $\text{cut}(S)$.
- Proof by contradiction using exchange argument.
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  - Let $e = (u, v)$ be the cheapest edge in $\text{cut}(S)$. 

![Graph](image)
Cut Property

- When is it safe to include an edge in an MST?
- Claim: For every $S \subset V, S \neq \emptyset$, every MST contains the cheapest edge in $\text{cut}(S)$.
- Proof by contradiction using exchange argument.
- How do you state the contradiction to the claim? There is a set $S \subset V$ and an MST $T$ such that $T$ does not contain the cheapest edge in $\text{cut}(S)$.
  - Let $e = (u, v)$ be the cheapest edge in $\text{cut}(S)$.
- Proof strategy: If $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$. 

![Graph Image]
Proof of Cut Property

- There is a set $S \subset V$ and an MST $T$ such that $T$ does not contain the cheapest edge in $\text{cut}(S)$.
- Proof strategy: If $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$. 

\[
\begin{align*}
\text{Wrong proof:} & \quad \text{Since } T \text{ is spanning, it must contain some edge, e.g., } f. \\
& \quad T - \{f\} \cup \{e\} \text{ has smaller cost than } T \text{ but may not be a spanning tree.}
\end{align*}
\]

\[
\begin{align*}
\text{Correct proof:} & \quad \text{Add } e \text{ to } T \text{ forming a cycle.} \\
& \quad \text{This cycle must contain an edge } e' \text{ in } \text{cut}(S). \\
& \quad \text{Poll } T - \{e'\} \cup \{e\} \text{ has smaller cost than } T \text{ and is a spanning tree.}
\end{align*}
\]
Proof of Cut Property

- There is a set $S \subseteq V$ and an MST $T$ such that $T$ does not contain the cheapest edge in $\text{cut}(S)$.
- Proof strategy: If $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$.

Wrong proof:
- Since $T$ is spanning, it must contain some edge, e.g., $f$, in $\text{cut}(S)$.
- $T - \{f\} \cup \{e\}$ has smaller cost than $T$ but...
**Proof of Cut Property**

- There is a set $S \subset V$ and an MST $T$ such that $T$ does not contain the cheapest edge in $\text{cut}(S)$.
- Proof strategy: If $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$.

Wrong proof:
- Since $T$ is spanning, it must contain some edge, e.g., $f$, in $\text{cut}(S)$.
- $T - \{f\} \cup \{e\}$ has smaller cost than $T$ but may not be a spanning tree.

Correct proof:
- Add $e$ to $T$ forming a cycle.
- This cycle must contain an edge $e'$ in $\text{cut}(S)$.
- Poll
- $T - \{e'\} \cup \{e\}$ has smaller cost than $T$ and is a spanning tree.
Proof of Cut Property

There is a set $S \subset V$ and an MST $T$ such that $T$ does not contain the cheapest edge in cut($S$).

Proof strategy: If $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$.

Wrong proof:
- Since $T$ is spanning, it must contain some edge, e.g., $f$, in cut($S$).
- $T - \{f\} \cup \{e\}$ has smaller cost than $T$ but may not be a spanning tree.

Correct proof:
- Add $e$ to $T$ forming a cycle.
Proof of Cut Property

There is a set $S \subseteq V$ and an MST $T$ such that $T$ does not contain the cheapest edge in cut($S$).

Proof strategy: If $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$.

Wrong proof:
- Since $T$ is spanning, it must contain some edge, e.g., $f$, in cut($S$).
- $T - \{f\} \cup \{e\}$ has smaller cost than $T$ but may not be a spanning tree.

Correct proof:
- Add $e$ to $T$ forming a cycle.
- This *cycle* must contain an edge $e'$ in cut($S$).

\[
c(e) < c(e')\]
Proof of Cut Property

There is a set $S \subseteq V$ and an MST $T$ such that $T$ does not contain the cheapest edge in $\text{cut}(S)$.

Proof strategy: If $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$.

Wrong proof:
- Since $T$ is spanning, it must contain some edge, e.g., $f$, in $\text{cut}(S)$.
- $T - \{f\} \cup \{e\}$ has smaller cost than $T$ but may not be a spanning tree.

Correct proof:
- Add $e$ to $T$ forming a cycle.
- This cycle must contain an edge $e'$ in $\text{cut}(S)$.
- $T - \{e'\} \cup \{e\}$ has smaller cost than $T$ and is a spanning tree.
Prim’s Algorithm

- Maintain a tree $(S, T)$, i.e. a set of nodes and a set of edges, which we will show will always be a tree.
- Start with an arbitrary node $s \in S$. 

**Algorithm**

1. Initialize $S = \{s\}$ and $T = \emptyset$.
2. While $S \neq V$
   1. Compute $(u, v) = \arg\min_{u \in S, v \in V - S} c(u, v)$.
   2. Add the node $v$ to $S$ and add the edge $(u, v)$ to $T$.

**Note**

$\arg\min_{(u, v)} \in \text{cut}(S) c(u, v)$

In other words, in each step, Prim’s algorithm computes and adds the cheapest edge in the current value of cut$(S)$. 

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Prim’s Algorithm

- Maintain a tree \((S, T)\), i.e. a set of nodes and a set of edges, which we will show will always be a tree.
- Start with an arbitrary node \(s \in S\).

\[
\text{Prim’s Algorithm}(G, c, s) \\
1: \quad S = \{s\} \text{ and } T = \emptyset \\
2: \quad \textbf{while } S \neq V \textbf{ do} \\
3: \quad \text{Compute } (u, v) = \arg \min_{(u, v): u \in S, v \in V - S} c(u, v) \\
4: \quad \text{Add the node } v \text{ to } S \text{ and add the edge } (u, v) \text{ to } T.
\]
Prim’s Algorithm

- Maintain a tree \((S, T)\), i.e. a set of nodes and a set of edges, which we will show will always be a tree.
- Start with an arbitrary node \(s \in S\).

**Prim’s Algorithm** \((G, c, s)\)

1. \(S = \{s\}\) and \(T = \emptyset\)
2. while \(S \neq V\) do
3. Compute \((u, v) = \arg \min_{u \in S, v \in V - S} c(u, v)\)
4. Add the node \(v\) to \(S\) and add the edge \((u, v)\) to \(T\).

- Note that

\[
\arg \min_{(u,v), u \in S, v \in V - S} c(u, v) \equiv \arg \min_{(u,v) \in \text{cut}(S)} c(u, v).
\]

- In other words, in each step, Prim’s algorithm computes and adds the cheapest edge in the current value of \(\text{cut}(S)\).
Example of Prim’s Algorithm

- **Vertices:** a, b, c, d, e, f, g, h, f
- **Edges:**
  - a to b: 4
  - a to c: 5
  - a to g: 15
  - b to c: 11
  - b to d: 12
  - b to e: 3
  - c to d: 8
  - c to e: 20
  - c to h: 1
  - d to e: 1
  - d to f: 6
  - e to f: 2

**Notes:**
- Prim's Algorithm starts with a single vertex and iteratively adds the lowest-cost edge that connects a vertex in the tree to a vertex outside the tree.
- The algorithm continues until all vertices are included.

**Implementation:**
- Prim's Algorithm is a greedy algorithm that constructs a minimum spanning tree by continuously adding the lowest-cost edge that connects a vertex in the tree to a vertex outside the tree.
Example of Prim’s Algorithm

```
  a
  | 4
 b  | 5  15
  | 11
  +-----
  | 12  3
  d  | 8
  | 20
  +-----
  e  | 6
  | 7
  f
  h
  g
```

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Example of Prim’s Algorithm
Example of Prim’s Algorithm

Graph with weighted edges:

- Vertices: a, b, c, d, e, f, g, h
- Edges and weights:
  - a-b: 4
  - a-c: 5
  - a-g: 11
  - b-c: 12
  - b-h: 1
  - c-f: 20
  - c-g: 6
  - d-c: 3
  - e-f: 2
  - f-g: 15

Algorithm steps:
1. Start with vertex a (minimum weight is 0).
2. Add vertex b (minimum weight is 4).
3. Add vertex c (minimum weight is 5).
4. Add vertex g (minimum weight is 11).
5. Add vertex h (minimum weight is 7).
6. Add vertex f (minimum weight is 6).
7. Add vertex e (minimum weight is 20).

Resulting Minimum Spanning Tree (MST):

- a-b
- a-c
- a-g
- b-c
- c-h
- c-g
- f-g
- f-e
- e-f

Total weight of the MST: 66.
Example of Prim’s Algorithm

- Graph with edges and weights:
  - a to b: 4
  - a to c: 5
  - a to g: 15
  - b to c: 11
  - b to d: 12
  - c to e: 3
  - c to f: 20
  - d to e: 8
  - e to f: 6
  - g to h: 2
Example of Prim’s Algorithm
Example of Prim’s Algorithm
Example of Prim’s Algorithm
Example of Prim’s Algorithm

Graph with edges and weights:
- (a, b) weight 4
- (a, g) weight 5
- (b, c) weight 11
- (b, d) weight 12
- (c, e) weight 3
- (c, h) weight 7
- (d, e) weight 8
- (e, f) weight 20
- (g, h) weight 6

Minimum Spanning Tree algorithm:
1. Start with a single vertex (a, b, c, d, e, f, g, h)
2. Choose the minimum weight edge that connects a new vertex to the tree
3. Repeat until all vertices are included in the tree

Implementation details:
- Greedy approach
- Efficiency: O(E log V)

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Example of Prim’s Algorithm

Diagram showing a graph with nodes labeled a, b, c, d, e, f, g, and h, and edges with weights 4, 5, 15, 2, 11, 12, 3, 1, 8, 20, and 6. The algorithm's progression is indicated with blue edges.
Example of Prim’s Algorithm

Graph with weights:

- a to b: 4
- a to c: 5
- a to g: 15
- b to c: 11
- b to d: 12
- c to e: 3
- c to f: 20
- d to e: 8
- e to f: 6
- g to h: 2

Start with a and iteratively add the minimum weight edge:
1. a
2. b (4)
3. c (4+5=9)
4. d (9+11=20)
5. e (20+3=23)
6. f (23+20=43)
7. g (43+2=45)

Minimum spanning tree includes edges:
- a to b
- a to c
- b to c
- c to e
- d to e
- e to f
- g to h
Example of Prim’s Algorithm

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Example of Prim’s Algorithm
Example of Prim’s Algorithm

Graph with edge weights:

- a to b: 4
- a to c: 5
- a to g: 11
- b to c: 12
- b to e: 3
- c to d: 8
- c to e: 1
- c to h: 7
- d to e: 20
- e to f: 6
- f to g: 2

Minimum Spanning Tree:

- a
- b
- e
- f
- h
Example of Prim’s Algorithm

A graph with edges and weights is shown. The algorithm starts with an arbitrary vertex, say a, and iteratively adds the minimum-weight edge that connects a new vertex to the growing tree. The process continues until all vertices are included in the tree. The diagram illustrates the steps of Prim’s algorithm, showing how the minimum spanning tree is constructed.
Optimality of Prim’s Algorithm

**Prim’s Algorithm** \((G, c, s)\)

1: \(S = \{s\}\) and \(T = \emptyset\)
2: **while** \(S \neq V\) **do**
3: Compute \((u, v) = \arg \min_{(u,v) \in \text{cut}(S)} c(u, v)\)
4: Add the node \(v\) to \(S\) and add the edge \((u, v)\) to \(T\).

- **Claim:** Prim’s algorithm outputs an MST.
Optimality of Prim’s Algorithm

\textbf{Prim's Algorithm} \((G, c, s)\)

1: \(S = \{s\}\) and and \(T = \emptyset\)
2: \textbf{while} \(S \neq V\) \textbf{do}
3: \hspace{1em} Compute \((u, v) = \arg \min_{(u,v) \in \text{cut}(S)} c(u, v)\)
4: \hspace{1em} Add the node \(v\) to \(S\) and add the edge \((u, v)\) to \(T\).

\textbullet \textbf{Claim:} Prim’s algorithm outputs an MST.
\begin{itemize}
  \item [1] Prove that every edge inserted satisfies the cut property.
  \item [2] Prove that the graph constructed is a spanning tree.
\end{itemize}
Optimality of Prim’s Algorithm

**Prim’s Algorithm** \((G, c, s)\)

1. \(S = \{s\}\) and and \(T = \emptyset\)
2. while \(S \neq V\) do
3. \(\text{Compute } (u, v) = \arg \min_{(u, v) \in \text{cut}(S)} c(u, v)\)
4. \(\text{Add the node } v \text{ to } S \text{ and add the edge } (u, v) \text{ to } T.\)

Claim: Prim’s algorithm outputs an MST.

1. Prove that every edge inserted satisfies the cut property.
   - By construction, in each iteration \((u, v)\) is the cheapest edge in \(\text{cut}(S)\) for the current value of \(S\).
2. Prove that the graph constructed is a spanning tree.
Optimality of Prim’s Algorithm

### Prim’s Algorithm \((G, c, s)\)

1. \(S = \{s\}\) and \(T = \emptyset\)
2. while \(S \neq V\) do
3. Compute \((u, v) = \arg\min_{(u,v) \in \text{cut}(S)} c(u, v)\)
4. Add the node \(v\) to \(S\) and add the edge \((u, v)\) to \(T\).

- **Claim:** Prim’s algorithm outputs an MST.
  1. Prove that every edge inserted satisfies the cut property.
    - By construction, in each iteration \((u, v)\) is the cheapest edge in \(\text{cut}(S)\) for the current value of \(S\).
  2. Prove that the graph constructed is a spanning tree.
    - Why are there no cycles in \((V, T)\)?
Optimality of Prim’s Algorithm

**Prim’s Algorithm** \((G, c, s)\)

1. \(S = \{s\} \) and and \(T = \emptyset\)
2. **while** \(S \neq V\) **do**
3. Compute \((u, v) = \arg \min_{(u, v) \in \text{cut}(S)} c(u, v)\)
4. Add the node \(v\) to \(S\) and add the edge \((u, v)\) to \(T\).

- **Claim:** Prim’s algorithm outputs an MST.
  1. Prove that every edge inserted satisfies the cut property.
     - By construction, in each iteration \((u, v)\) is the cheapest edge in \(\text{cut}(S)\) for the current value of \(S\).
  2. Prove that the graph constructed is a spanning tree.
     - Why are there no cycles in \((V, T)\)?
     - Why is \((V, T)\) a spanning tree (edges in \(T\) connect all nodes in \(V\))?
Kruskal’s Algorithm

- Start with an empty set $T$ of edges.
- Process edges in $E$ in increasing order of cost.
- Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle. Discard $e$ if it creates a cycle.
Example of Kruskal’s Algorithm
Example of Kruskal’s Algorithm

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Example of Kruskal’s Algorithm

Graph:

- Vertices: a, b, c, d, e, f, g, h
- Edges with weights:
  - (a, b) = 4
  - (b, c) = 11
  - (c, d) = 3
  - (c, e) = 1
  - (e, f) = 20
  - (d, e) = 2
  - (h, g) = 2

Minimum Spanning Tree:

- The minimum spanning tree includes edges (a, b), (b, c), (c, e), and (e, f) with total weight 38.
Example of Kruskal’s Algorithm

- Nodes: a, b, c, d, e, f, g, h
- Edges with weights:
  - a to b: 4
  - b to c: 11
  - c to d: 8
  - d to e: 3
  - e to f: 20
  - c to e: 1
  - e to g: 7
  - g to h: 6
  - f to h: 2

The minimum spanning tree is formed by the edges with the lowest weights.
Example of Kruskal’s Algorithm

The graph consists of nodes labeled a, b, c, d, e, f, g, and h. The edges and their weights are as follows:

- a to b: 4
- a to c: 5
- a to g: 15
- b to c: 11
- b to d: 12
- c to e: 3
- c to h: 7
- d to e: 8
- e to f: 20
- f to g: 6

The algorithm starts by selecting the edge with the smallest weight and continues until all nodes are included in the tree. The selected edges are shown in blue.
Example of Kruskal’s Algorithm
Example of Kruskal’s Algorithm
Example of Kruskal’s Algorithm

```
a  b  c
|    |
\---|
 |

4  5  15
11
12 3
12
8 3
1
20
6
2
```
Example of Kruskal’s Algorithm
Example of Kruskal’s Algorithm

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Example of Kruskal’s Algorithm

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Example of Kruskal’s Algorithm

Diagram:

- Nodes: a, b, c, d, e, f, g, h
- Edges and Weights:
  - (a, b) = 4
  - (b, c) = 11
  - (c, e) = 1
  - (e, f) = 20
  - (a, d) = 12
  - (d, e) = 8
  - (g, h) = 6
  - (a, f) = 15
  - (b, e) = 3
  - (c, h) = 7
  - (g, c) = 2

The algorithm would select edges 4, 11, 1, 20, 6, 2, 3, 7, and 15 to form the minimum spanning tree.
Example of Kruskal’s Algorithm

The diagram shows a graph with vertices labeled a, b, c, d, e, f, g, and h. The edges are labeled with their weights. The algorithm starts by sorting all the edges by weight and then iteratively adds the lightest edge that does not create a cycle. The selected edges are highlighted in red.

- Edge (b, c) with weight 11
- Edge (e, f) with weight 20
- Edge (b, d) with weight 12
- Edge (c, e) with weight 3
- Edge (c, g) with weight 15
- Edge (a, d) with weight 4
- Edge (c, h) with weight 7
- Edge (f, g) with weight 2

The red edges form a minimum spanning tree of the graph.
Example of Kruskal’s Algorithm

Diagram of a weighted graph with nodes labeled as a, b, c, d, e, f, g, and h, and weights on the edges as follows:
- Edge ab: weight 4
- Edge ac: weight 5
- Edge ad: weight 11
- Edge bc: weight 12
- Edge bd: weight 3
- Edge cd: weight 1
- Edge ce: weight 8
- Edge de: weight 20
- Edge cg: weight 15
- Edge ch: weight 7
- Edge gh: weight 2
- Edge fg: weight 6

Source: T. M. Murali
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Greedy Graph Algorithms
Optimality of Kruskal’s Algorithm

Kruskal’s algorithm:
- Start with an empty set $T$ of edges.
- Process edges in $E$ in increasing order of cost.
- Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle. Discard $e$ if it creates a cycle.

Note: at any iteration, $T$ may contain several connected components and each node in $V$ is in some component.

Claim: Kruskal’s algorithm outputs an MST.
Optimality of Kruskal’s Algorithm

Kruskal’s algorithm:
- Start with an empty set $T$ of edges.
- Process edges in $E$ in increasing order of cost.
- Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle. Discard $e$ if it creates a cycle.

Note: at any iteration, $T$ may contain several connected components and each node in $V$ is in some component.

Claim: Kruskal’s algorithm outputs an MST.
1. For every edge $e$ added, demonstrate the existence of a set $S \subseteq V$ (and $V - S$) such that $e$ and $S$ satisfy the cut property, i.e., $e$ is the cheapest edge in $\text{cut}(S)$.

2. Prove that the algorithm computes a spanning tree.
Optimality of Kruskal’s Algorithm

- **Kruskal’s algorithm:****
  - Start with an empty set $T$ of edges.
  - Process edges in $E$ in increasing order of cost.
  - Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle. Discard $e$ if it creates a cycle.

- **Note:** at any iteration, $T$ may contain several connected components and each node in $V$ is in some component.

- **Claim:** Kruskal’s algorithm outputs an MST.
  1. For every edge $e$ added, demonstrate the existence of a set $S \subseteq V$ (and $V - S$) such that $e$ and $S$ satisfy the cut property, i.e., $e$ is the cheapest edge in $\text{cut}(S)$.
     - If $e = (u, v)$, let $S$ be the set of nodes connected to $u$ in the current graph $T$.
  2. Prove that the algorithm computes a spanning tree.
Optimality of Kruskal’s Algorithm

Kruskal’s algorithm:

▶ Start with an empty set $T$ of edges.
▶ Process edges in $E$ in increasing order of cost.
▶ Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle. Discard $e$ if it creates a cycle.

Note: at any iteration, $T$ may contain several connected components and each node in $V$ is in some component.

Claim: Kruskal’s algorithm outputs an MST.

1. For every edge $e$ added, demonstrate the existence of a set $S \subseteq V$ (and $V - S$) such that $e$ and $S$ satisfy the cut property, i.e., $e$ is the cheapest edge in cut$(S)$.
   - If $e = (u, v)$, let $S$ be the set of nodes connected to $u$ in the current graph $T$.
   - Why is $e$ the cheapest edge in cut$(S)$?

2. Prove that the algorithm computes a spanning tree.
Optimality of Kruskal’s Algorithm

- **Kruskal’s algorithm:**
  - Start with an empty set $T$ of edges.
  - Process edges in $E$ in increasing order of cost.
  - Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle. Discard $e$ if it creates a cycle.

- **Note:** at any iteration, $T$ may contain several connected components and each node in $V$ is in some component.

- **Claim:** Kruskal’s algorithm outputs an MST.
  1. For every edge $e$ added, demonstrate the existence of a set $S \subset V$ (and $V - S$) such that $e$ and $S$ satisfy the cut property, i.e., $e$ is the cheapest edge in $\text{cut}(S)$.
     - If $e = (u, v)$, let $S$ be the set of nodes connected to $u$ in the current graph $T$.
     - Why is $e$ the cheapest edge in $\text{cut}(S)$?
  2. Prove that the algorithm computes a spanning tree.
     - $(V, T)$ contains no cycles by construction.
Optimality of Kruskal’s Algorithm

Kruskal’s algorithm:

- Start with an empty set $T$ of edges.
- Process edges in $E$ in increasing order of cost.
- Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle. Discard $e$ if it creates a cycle.

Note: at any iteration, $T$ may contain several connected components and each node in $V$ is in some component.

Claim: Kruskal’s algorithm outputs an MST.

1. For every edge $e$ added, demonstrate the existence of a set $S \subset V$ (and $V - S$) such that $e$ and $S$ satisfy the cut property, i.e., $e$ is the cheapest edge in $\text{cut}(S)$.
   - If $e = (u, v)$, let $S$ be the set of nodes connected to $u$ in the current graph $T$.
   - Why is $e$ the cheapest edge in $\text{cut}(S)$?

2. Prove that the algorithm computes a spanning tree.
   - $(V, T)$ contains no cycles by construction.
   - If $(V, T)$ is not connected, there exists a subset $S$ of nodes not connected to $V - S$. What is the contradiction?

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Greedy Graph Algorithms
Cycle Property

- When can we be sure that an edge cannot be in any MST?
Cycle Property

- When can we be sure that an edge cannot be in any MST?
- Let $C$ be any cycle in $G$ and let $e = (v, w)$ be the most expensive edge in $C$.
- Claim: $e$ does not belong to any MST of $G$. 

Proof: exchange argument. If a supposed MST $T$ contains $e$, show that there is a tree with smaller cost than $T$ that does not contain $e$. 

T. M. Murali September 21, 23, 28, 2021 Greedy Graph Algorithms
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---

**Figure 4.11** Swapping the edge $e'$ for the edge $e$ in the spanning tree $T$, as described in the proof of (4.20).
Optimality of the Reverse-Delete Algorithm

- Reverse-Delete algorithm: Maintain a set $E'$ of edges.
  - Start with $E' = E$.
  - Process edges in decreasing order of cost.
  - Delete the next edge $e$ from $E'$ only if $(V, E')$ is connected after deletion.
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  1. Show that every edge deleted belongs to no MST.
     - A deleted edge must belong to some cycle $C$.
     - Since the edge is the first encountered by the algorithm, it is the most expensive edge in $C$.
  2. Prove that the graph remaining at the end is a spanning tree.
     - $(V, E')$ is connected at the end, by construction.
     - If $(V, E')$ contains a cycle, consider the costliest edge in that cycle. The algorithm would have deleted that edge.
Comments on MST Algorithms

- To handle multiple edges with the same length, perturb each length by a random infinitesimal amount. Read the textbook.
- *Any* algorithm that constructs a spanning tree by including edges that satisfy the cut property and deleting edges that satisfy the cycle property will yield an MST!
Implementing Prim’s Algorithm

\textbf{Prim’s Algorithm} (\( G, c, s \))

1. \( S = \{s\} \) and \( T = \emptyset \)
2. while \( S \neq V \) do
3. Compute \((u, v) = \arg\min_{(u,v): u \in S, v \in V - S} c(u, v)\)
4. Add the node \( v \) to \( S \) and add the edge \((u, v)\) to \( T \).

- Implementation and analysis are very similar to Dijkstra’s algorithm.
- Maintain \( S \) and store attachment costs \( a(v) = \min_{e \in \text{cut}(S)} c(e) \) for every node \( v \in V - S \) in a priority queue.
- At each step, extract the node \( v \) with the minimum attachment cost from the priority queue and update the attachment costs of the neighbours of \( v \).
Final Version of Prim’s Algorithm

**Prim’s Algorithm** \((G, c, s)\)

1. **Insert** \((Q, s, 0, \emptyset)\)
2. while \(S \neq V\) do
3. \((v, a(v), u) = \text{ExtractMin}(Q)\)
4. Add node \(v\) to \(S\) and edge \((u, v)\) to \(T\).
5. for every node \(x \in V - S\) such that \((v, x)\) is an edge in \(G\) do
6. if \(c(v, x) < a(x)\) then
7. \(a(x) = c(v, x)\)
8. \(\text{ChangeKey}(Q, x, a(x), v)\)

- \(Q\) is a priority queue.
- Each element in \(Q\) is a triple: the node, its attachment cost, and its predecessor in the MST.
- In Step 8, if \(x\) is not already in \(Q\), simply **Insert** \((x, a(x), v)\) into \(Q\).
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- \(Q\) is a priority queue.
- Each element in \(Q\) is a triple: the node, its attachment cost, and its predecessor in the MST.
- In Step 8, if \(x\) is not already in \(Q\), simply **Insert** \((x, a(x), v)\) into \(Q\).
- Total of \(n - 1\) \text{ExtractMin} and \(m\) \text{ChangeKey}/**Insert** operations, yielding a running time of \(O(m \log n)\).
Implementing Kruskal’s Algorithm

- Start with an empty set \( T \) of edges.
- Process edges in \( E \) in increasing order of cost.
- Add the next edge \( e \) to \( T \) only if adding \( e \) does not create a cycle.
Implementing Kruskal’s Algorithm

- Start with an empty set $T$ of edges.
- Process edges in $E$ in increasing order of cost.
- Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle.
- Sorting edges takes $O(m \log n)$ time.
- Key question: “Does adding $e = (u, v)$ to $T$ create a cycle?”
  - Maintain set of connected components of $T$.
  - $\text{Find}(u)$: return the name of the connected component of $T$ that $u$ belongs to.
  - $\text{Union}(A, B)$: merge connected components $A$ and $B$. 
Analysing Kruskal’s Algorithm

How many \texttt{FIND} invocations does Kruskal’s algorithm need?
Analysing Kruskal’s Algorithm

- How many `FIND` invocations does Kruskal’s algorithm need? 2m.
- How many `UNION` invocations does Kruskal’s algorithm need?
Analysing Kruskal’s Algorithm

- How many **FIND** invocations does Kruskal’s algorithm need? $2m$.
- How many **UNION** invocations does Kruskal’s algorithm need? $n - 1$.
Analysing Kruskal’s Algorithm

- How many `FIND` invocations does Kruskal’s algorithm need? \(2m\).
- How many `UNION` invocations does Kruskal’s algorithm need? \(n - 1\).
- Textbook describes two implementations of `UNION-FIND`:
  - Each `FIND` takes \(O(1)\) time, \(k\) invocations of `UNION` take \(O(k \log k)\) time in total.
  - Each `FIND` takes \(O(\log n)\) time and each invocation of `UNION` takes \(O(1)\) time.
Analysing Kruskal’s Algorithm

- How many \texttt{Find} invocations does Kruskal’s algorithm need? $2m$.
- How many \texttt{Union} invocations does Kruskal’s algorithm need? $n - 1$.
- Textbook describes two implementations of \texttt{Union-Find}: (see appendix to this set of slides)
  - Each \texttt{Find} takes $O(1)$ time, $k$ invocations of \texttt{Union} take $O(k \log k)$ time in total.
  - Each \texttt{Find} takes $O(\log n)$ time and each invocation of \texttt{Union} takes $O(1)$ time.
- Total running time of Kruskal’s algorithm is $O(m \log n)$. 
Comments on Union-Find and MST

- The **Union-Find** data structure is useful to maintain the connected components of a graph as edges are added to the graph.
- The data structure does not support edge deletion efficiently.
- Current best algorithm for MST runs in $O(m\alpha(m,n))$ time (Chazelle 2000) and $O(m)$ randomised time (Karger, Klein, and Tarjan, 1995).
- Holy grail: $O(m)$ deterministic algorithm for MST.
Union-Find Data Structure

- Abstraction of the data structure needed by Kruskal’s algorithm.
- Maintain disjoint subsets of elements from a universe $U$ of $n$ elements.
- Each subset has an name. We will set a set’s name to be the identity of some element in it.
- Support three operations:
  1. `MAKEUNIONFIND(U)`: initialise the data structure with elements in $U$.
  2. `FIND(u)`: return the identity of the subset that contains $u$.
  3. `UNION(A, B)`: merge the sets named $A$ and $B$ into one set.
Union-Find Data Structure: Implementation 1

- Store all the elements of $U$ in an array $\text{COMPONENT}$.
  - Assume identities of elements are integers from 1 to $n$.
  - $\text{COMPONENT}[s]$ is the name of the set containing $s$.
- Implementing the operations:
Appendix: Union-Find

Union-Find Data Structure: Implementation 1

- Store all the elements of $U$ in an array $\text{COMPONENT}$.
  - Assume identities of elements are integers from 1 to $n$.
  - $\text{COMPONENT}[s]$ is the name of the set containing $s$.

Implementing the operations:

1. $\text{MAKEUNIONFIND}(U)$: For each $s \in U$, set $\text{COMPONENT}[s] = s$ in $O(n)$ time.
2. $\text{FIND}(s)$: return $\text{COMPONENT}[s]$ in $O(1)$ time.
3. $\text{UNION}(A, B)$: merge $B$ into $A$ by scanning $\text{COMPONENT}$ and updating each index whose value is $B$ to the value $A$. Takes $O(n)$ time.
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- Store all the elements of $U$ in an array $\text{Component}$.
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- $\text{UNION}$ is very slow because
Appendix: Union-Find

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- Store all the elements of $U$ in an array $\text{Component}$.
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- Implementing the operations:
  1. $\text{MakeUnionFind}(U)$: For each $s \in U$, set $\text{Component}[s] = s$ in $O(n)$ time.
  2. $\text{Find}(s)$: return $\text{Component}[s]$ in $O(1)$ time.
  3. $\text{Union}(A, B)$: merge $B$ into $A$ by scanning $\text{Component}$ and updating each index whose value is $B$ to the value $A$. Takes $O(n)$ time.

- $\text{Union}$ is very slow because we cannot efficiently find the elements that belong to a set.
Union-Find Data Structure: Implementation 2

- Optimisation 1: Use an array `ELEMENTS`
  - Indices of `ELEMENTS` range from 1 to \( n \).
  - `ELEMENTS[s]` stores the elements in the subset named \( s \) in a list.

- Execute `UNION(A, B)` by merging \( B \) into \( A \) in two steps:
  1. Updating `COMPONENT` for elements of \( B \) in \( O(|B|) \) time.

- `UNION` takes \( \Omega(n) \) in the worst-case.
Appendix: Union-Find

Union-Find Data Structure: Implementation 2

- Optimisation 1: Use an array \texttt{ELEMENTS}
  - Indices of \texttt{ELEMENTS} range from 1 to \( n \).
  - \texttt{ELEMENTS}[s] stores the elements in the subset named \( s \) in a list.
- Execute \texttt{UNION}(A, B) by merging \( B \) into \( A \) in two steps:
  1. Updating \texttt{COMPONENT} for elements of \( B \) in \( O(|B|) \) time.
  2. Append \texttt{ELEMENTS}[B] to \texttt{ELEMENTS}[A] in \( O(1) \) time.
- \texttt{UNION} takes \( \Omega(n) \) in the worst-case.
- Optimisation 2: Store size of each set in an array (say, \texttt{SIZE}). If \( \texttt{SIZE}[B] \leq \texttt{SIZE}[A] \), merge \( B \) into \( A \). Otherwise merge \( A \) into \( B \). Update \texttt{SIZE}.
Union-Find Data Structure: Analysis of Implementation

- \texttt{MakeUnionFind}(S) \textbf{and} \texttt{Find}(u) \textbf{are as before.}

- Union \texttt{(A, B)}: Running time is proportional to the size of the smaller set, which may be \(\Omega(n)\).

- Any sequence of \(k\) \texttt{Union} operations takes \(O(k \log k)\) time.

- \(k\) \texttt{Union} operations touch at most \(2k\) elements.

- Intuition: running time of \texttt{Union} is dominated by updates to \texttt{Component}.

- Charge each update to the element being updated and bound number of charges per element.

- Consider any element \(s\). Every time \(s\)'s set identity is updated, the size of the set containing \(s\) at least doubles \(\Rightarrow s\)'s set can change at most \(\log(2k)\) times \(\Rightarrow\) the total work done in \(k\) \texttt{Union} operations is \(O(k \log k)\).

- \texttt{Find} is fast in the worst case, \texttt{Union} is fast in an amortised sense. Can we make both operations worst-case efficient?
Union-Find Data Structure: Analysis of Implementation

- \texttt{MakeUnionFind}(S) and \texttt{Find}(u) are as before.
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Union-Find Data Structure: Analysis of Implementation

- **MAKEUNIONFIND(S)** and **FIND(u)** are as before.
- **UNION(A, B):** Running time is proportional to the size of the smaller set, which may be $\Omega(n)$.
- Any sequence of $k$ UNION operations takes $O(k \log k)$ time.
Union-Find Data Structure: Analysis of Implementation

- `MAKEUNIONFIND(S)` and `FIND(u)` are as before.
- `UNION(A, B)`: Running time is proportional to the size of the smaller set, which may be $\Omega(n)$.
- Any sequence of $k$ `UNION` operations takes $O(k \log k)$ time.
  - $k$ `UNION` operations touch at most $2k$ elements.
Union-Find Data Structure: Analysis of Implementation

- **MAKEUNIONFIND(S)** and **FIND(u)** are as before.
- **UNION(A, B):** Running time is proportional to the size of the smaller set, which may be $\Omega(n)$.
- **Any sequence of** $k$ **UNION operations takes** $O(k \log k)$ **time.**
  - $k$ **UNION** operations touch at most $2k$ elements.
  - Intuition: running time of **UNION** is dominated by updates to **COMPONENT**. Charge each update to the element being updated and bound number of charges per element.
Appendix: Union-Find

Union-Find Data Structure: Analysis of Implementation

- **MAKEUNIONFIND**(S) and **FIND**(u) are as before.
- **UNION**(A, B): Running time is proportional to the size of the smaller set, which may be $\Omega(n)$.
- Any sequence of $k$ **UNION** operations takes $O(k \log k)$ time.
  - $k$ **UNION** operations touch at most $2k$ elements.
  - Intuition: running time of **UNION** is dominated by updates to **COMPONENT**. Charge each update to the element being updated and bound number of charges per element.
  - Consider any element $s$. Every time $s$’s set identity is updated, the size of the set containing $s$ at least doubles $\Rightarrow$ $s$’s set can change at most $\log(2k)$ times $\Rightarrow$ the total work done in $k$ **UNION** operations is $O(k \log k)$.

Find is fast in the worst case, **UNION** is fast in an amortised sense. Can we make both operations worst-case efficient?
Union-Find Data Structure: Analysis of Implementation

- MakeUnionFind(S) and Find(u) are as before.
- Union(A, B): Running time is proportional to the size of the smaller set, which may be $\Omega(n)$.
- Any sequence of $k$ Union operations takes $O(k \log k)$ time.
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  - Intuition: running time of Union is dominated by updates to Component. Charge each update to the element being updated and bound number of charges per element.
  - Consider any element s. Every time s’s set identity is updated, the size of the set containing s at least doubles $\Rightarrow$ s’s set can change at most log(2k) times $\Rightarrow$ the total work done in $k$ Union operations is $O(k \log k)$.

- Find is fast in the worst case, Union is fast in an amortised sense. Can we make both operations worst-case efficient?
Union-Find Data Structure: Implementation 3

- Goal: Implement `FIND` in $O(\log n)$ and `UNION` in $O(1)$ worst-case time.
Union-Find Data Structure: Implementation 3

- Goal: Implement `FIND` in $O(\log n)$ and `UNION` in $O(1)$ worst-case time.
- Represent each subset in a tree using pointers:
  - Each tree node contains an element and a pointer to a parent.
  - The identity of the set is the identity of the element at the root.

```
    u
   /\  \\
  s /   v \ / \
 /     \  \  \
 w       t   z
       / \
      /   \
     i   j
```

*Figure 4.12* A Union–Find data structure using pointers. The data structure has only two sets at the moment, named after nodes $v$ and $j$. The dashed arrow from $u$ to $v$ is the result of the last `Union` operation. To answer a `Find` query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query `Find(i)` would involve following the arrows $i$ to $x$, and then $x$ to $j$. 
Union-Find Data Structure: Implementation 3

- Goal: Implement \texttt{FIND} in $O(\log n)$ and \texttt{UNION} in $O(1)$ worst-case time.
- Represent each subset in a tree using pointers:
  - Each tree node contains an element and a pointer to a parent.
  - The identity of the set is the identity of the element at the root.
- Implementing \texttt{FIND}(u): follow pointers from $u$ to the root of $u$'s tree.

![Diagram](image)

\textbf{Figure 4.12} A Union-Find data structure using pointers. The data structure has only two sets at the moment, named after nodes $v$ and $j$. The dashed arrow from $u$ to $v$ is the result of the last Union operation. To answer a \texttt{Find} query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query \texttt{Find}(i) would involve following the arrows $i$ to $x$, and then $x$ to $j$. 
Goal: Implement \texttt{FIND} in $O(\log n)$ and \texttt{UNION} in $O(1)$ worst-case time.

Represent each subset in a tree using pointers:

- Each tree node contains an element and a pointer to a parent.
- The identity of the set is the identity of the element at the root.

Implementing \texttt{FIND}(u): follow pointers from $u$ to the root of $u$'s tree.

Implementing \texttt{UNION}(A, B): make smaller tree’s root a child of the larger tree’s root. Takes $O(1)$ time.

Figure 4.12 A Union–Find data structure using pointers. The data structure has only two sets at the moment, named after nodes $v$ and $j$. The dashed arrow from $u$ to $v$ is the result of the last \texttt{Union} operation. To answer a \texttt{Find} query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query \texttt{Find}(i) would involve following the arrows $i$ to $x$, and then $x$ to $j$. 
Why does \texttt{Find}(u) take \(O(\log n)\) time?

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{union-find.png}
\caption{A Union-Find data structure using pointers. The data structure has only two sets at the moment, named after nodes \(v\) and \(j\). The dashed arrow from \(u\) to \(v\) is the result of the last \texttt{Union} operation. To answer a \texttt{Find} query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query \texttt{Find}(i) would involve following the arrows \(i\) to \(x\), and then \(x\) to \(j\).}
\end{figure}
Why does $\text{FIND}(u)$ take $O(\log n)$ time?

- Number of pointers followed equals the number of times the identity of the set containing $u$ changed.
- Every time $u$’s set’s identity changes, the set at least doubles in size $\Rightarrow$ there are $O(\log n)$ pointers followed.
Every time we invoke $\text{FIND}(u)$, we follow the same set of pointers.
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Path compression: make all nodes visited by $\text{FIND}(u)$ children of the root.
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Path compression: make all nodes visited by \texttt{FIND}(u) children of the root.

Can prove that total time taken by \( n \) \texttt{FIND} operations is \( O(n\alpha(n)) \), where \( \alpha(n) \) is the inverse of the Ackermann function, and grows extremely slowly with \( n \).