Applications of Minimum Spanning Trees

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Minimum Spanning Trees

- We motivated MSTs through the problem of finding a low-cost network connecting a set of nodes.
- MSTs are useful in a number of seemingly disparate applications.
- We will consider two problems: minimum bottleneck graphs (problem 9 in Chapter 4) and clustering (Chapter 4.7).

Minimum Bottleneck Spanning Tree (MBST)

- The MST minimises the total cost of a spanning network.
- Consider another network design criterion:
  - Build a network of roads to connect all cities in a mountainous region but ensure road with highest elevation is as low as possible.
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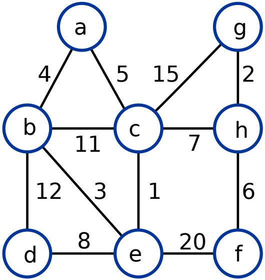
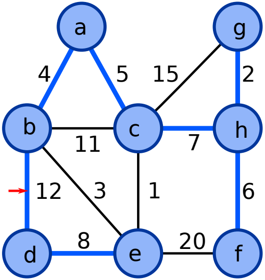
MINIMUM BOTTLENECK SPANNING TREE (MBST)

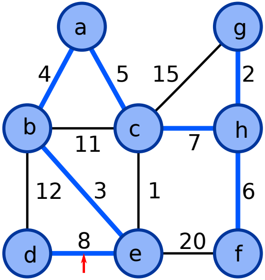
INSTANCE: An undirected graph $G(V, E)$ and a function $c : E \rightarrow \mathbb{R}^+$

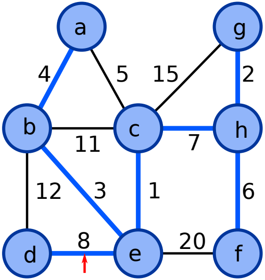
SOLUTION: A set $T \subseteq E$ of edges such that (V, T) is a spanning tree and there is no spanning tree in G with a cheaper bottleneck edge.

MBST Examples









Two Questions on MBSTs

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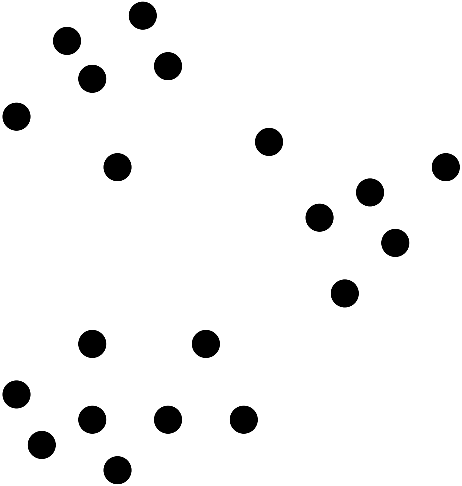
   - Every edge in T' is cheaper than e.
   - Adding e to T' creates a cycle consisting only of edges in T' and e.
   - Since e is the costliest edge in this cycle, by the cycle property, e cannot belong to any MST, which contradicts the fact that T is an MST.

Motivation for Clustering

We did not discuss this problem in class.

- Given a set of objects and distances between them.
- Objects can be images, web pages, people, species . . . .
- Distance function: increasing distance corresponds to decreasing similarity.
- Goal: group objects into clusters, where each cluster is a set of similar objects.

Example of Clustering



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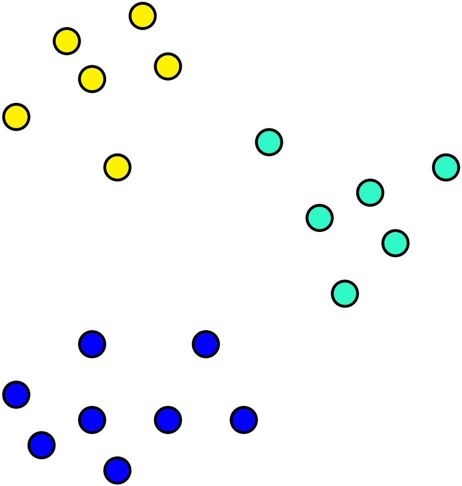

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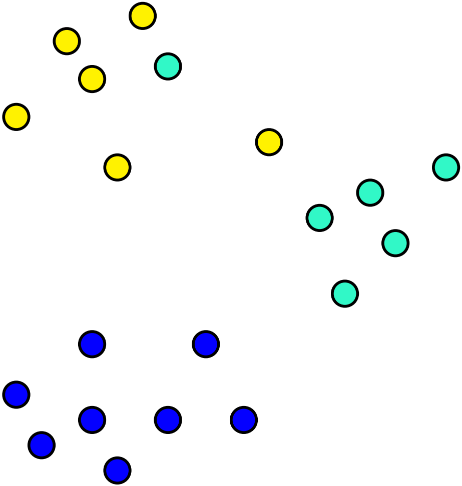

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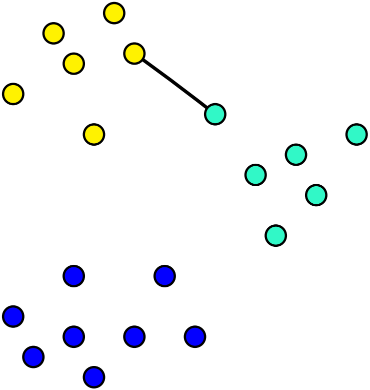

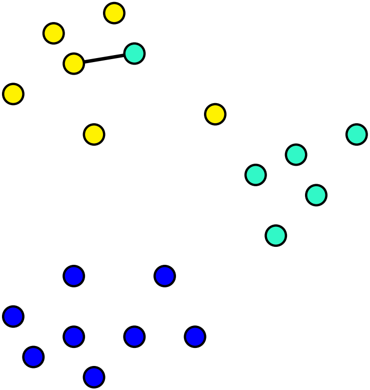

Formalising the Clustering Problem

- Let U be the set of n objects labelled $p_1, p_2, \ldots, p_n$.
- For every pair p_i and p_j, we have a distance $d(p_i, p_j)$.
- We require $d(p_i, p_i) = 0$, $d(p_i, p_j) > 0$, if $i \neq j$, and $d(p_i, p_j) = d(p_j, p_i)$

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- Given a positive integer k, a *k-clustering* of U is a partition of U into k non-empty subsets or "clusters" $C_1, C_2, \ldots C_k$.
- The *spacing* of a clustering is the smallest distance between objects in two different subsets:

$$\text{spacing}(C_1, C_2, \ldots C_k) = \min_{\substack{1 \leq i,j \leq k \\ i \neq j, \\ p \in C_i, q \in C_j}} d(p, q)$$

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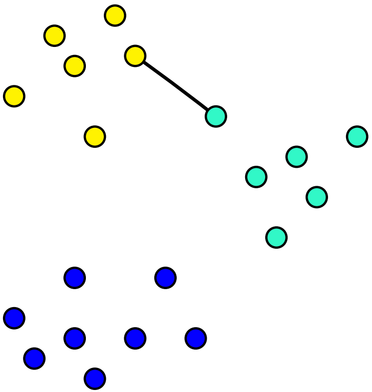
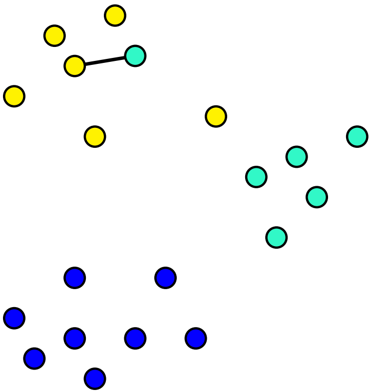
CLUSTERING OF MAXIMUM SPACING

INSTANCE: A set U of objects, a distance function $d : U \times U \rightarrow \mathbb{R}^+$, and a positive integer k

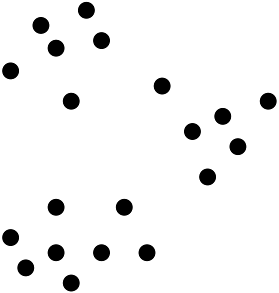
SOLUTION: A k-clustering of U whose spacing is the largest over all possible k-clusterings.

Example of Clustering





Algorithm for Clustering of Maximum Spacing



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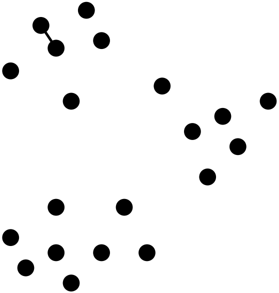

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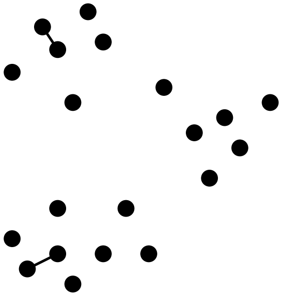

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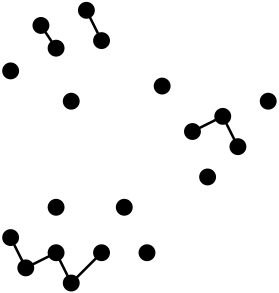

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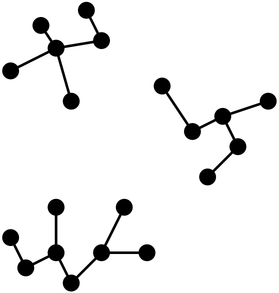

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- Let $\mathcal{C}$ be a set of n clusters, with each object in U in its own cluster.
- Process pairs of objects in increasing order of distance.
  - Let (p, q) be the next pair with $p \in C_p$ and $q \in C_q$.
  - If $C_p \neq C_q$, add new cluster $C_p \cup C_q$ to $\mathcal{C}$, delete C_p and C_q from $\mathcal{C}$.
- Stop when there are k clusters in $\mathcal{C}$.

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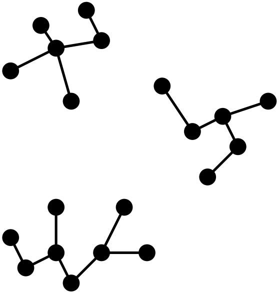

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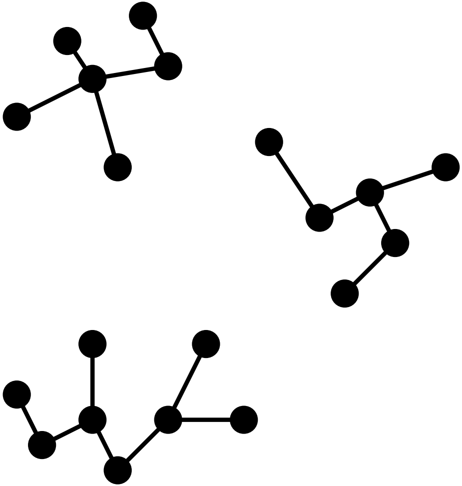
- Stop when there are k clusters in $\mathcal{C}$.
- Same as Kruskal's algorithm but do not add last $k - 1$ edges in MST.

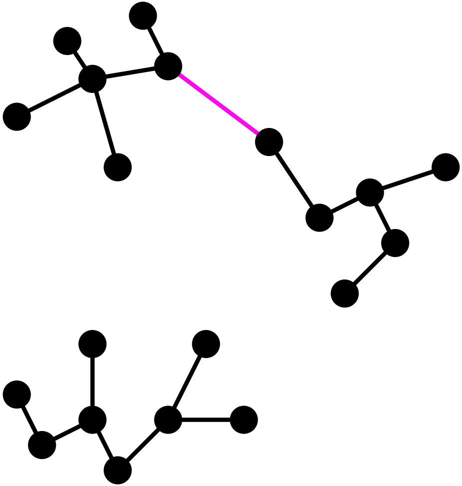
What is the Spacing of the Algorithm's Clustering?

- Let $\mathcal{C}$ be the clustering produced by the algorithm.
- What is spacing($\mathcal{C}$)?

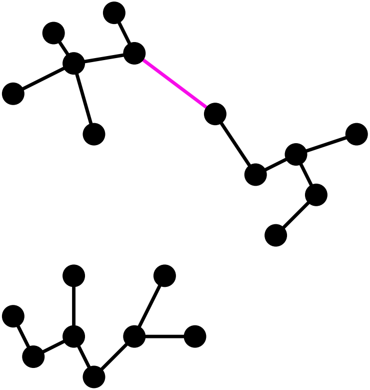


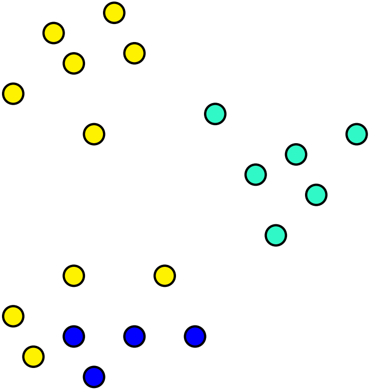
What is the Spacing of the Algorithm's Clustering?

- Let $\mathcal{C}$ be the clustering produced by the algorithm.
- What is spacing($\mathcal{C}$)? It is the cost of the $(k-1)$st most expensive edge in the MST. Let this cost be d^*.



Why Does the Algorithm Compute the Clustering of Largest Spacing?





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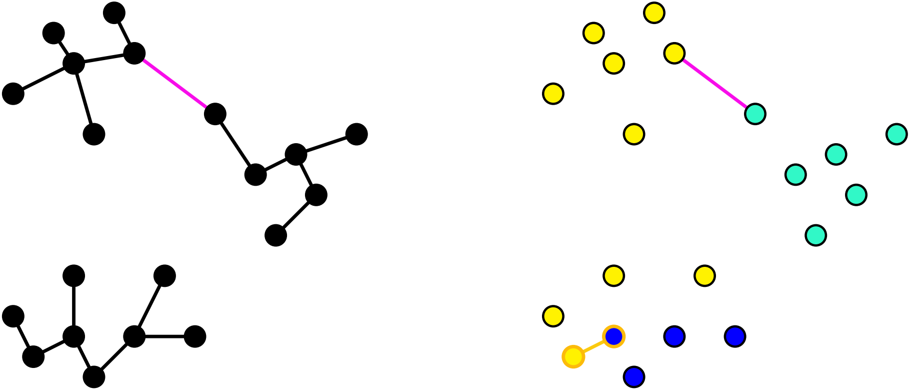

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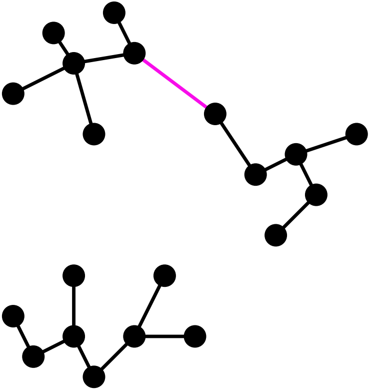
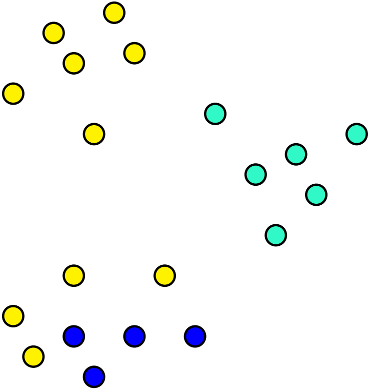
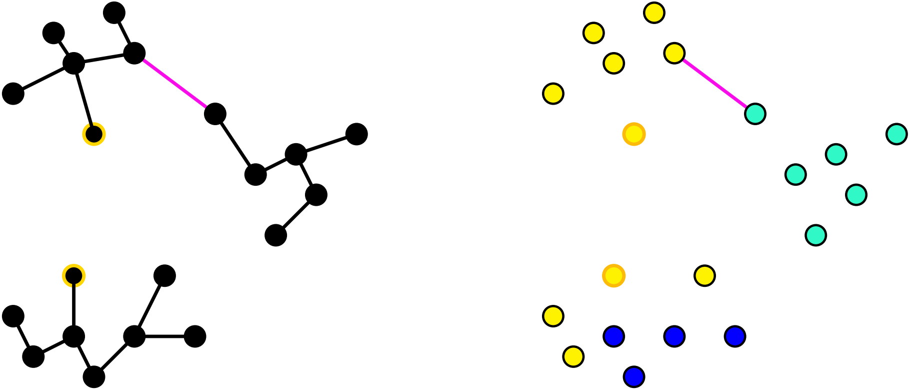
spacing$(\mathcal{C}') \leq d^*$: in Pictures





▸ Poll

spacing($\mathcal{C}'$) $\leq d^*$: in Pictures



There is a pair of objects in the same cluster in $\mathcal{C}'$ but in different clusters in $\mathcal{C}$.

Not useful for proof.

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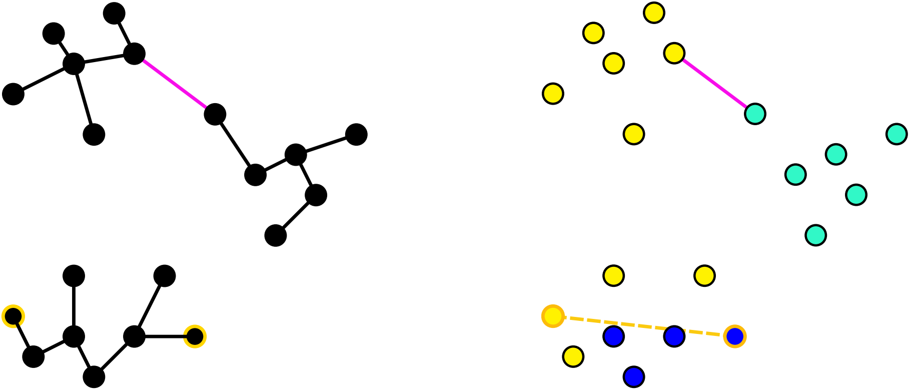
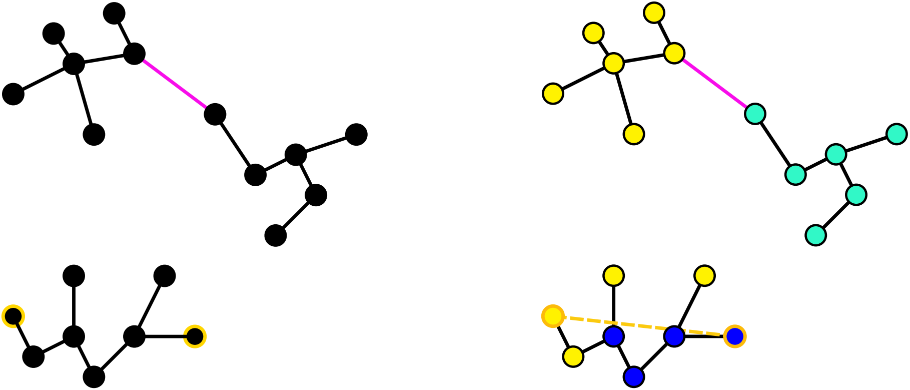


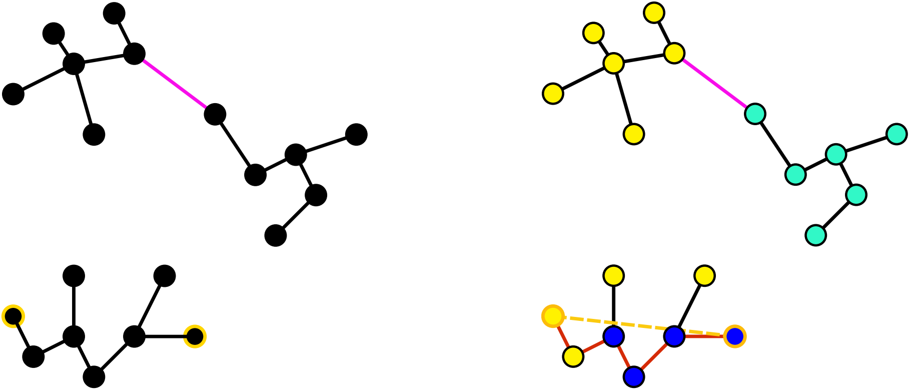
There is a pair of objects in the same cluster in $\mathcal{C}$ but in different clusters in $\mathcal{C}'$.
Can use in proof since they are connected by edges in the tree containing them.

spacing($\mathcal{C}'$) $\leq d^*$: in Pictures



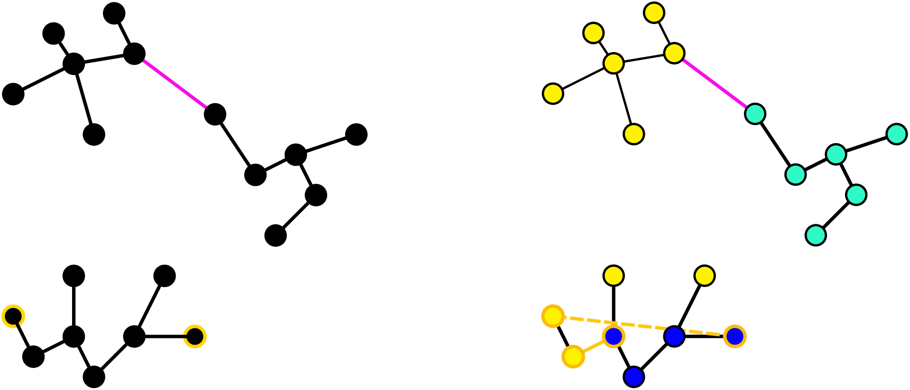
There is a pair of objects in the same cluster in $\mathcal{C}$ but in different clusters in $\mathcal{C}$'.
Consider the MST that the algorithm has already added.

spacing$(\mathcal{C}') \leq d^*$: in Pictures



There is a pair of objects in the same cluster in $\mathcal{C}$ but in different clusters in $\mathcal{C}'$.
A path of MST edges that the algorithm has already added connects these objects.

spacing($\mathcal{C}'$) $\leq d^*$: in Pictures



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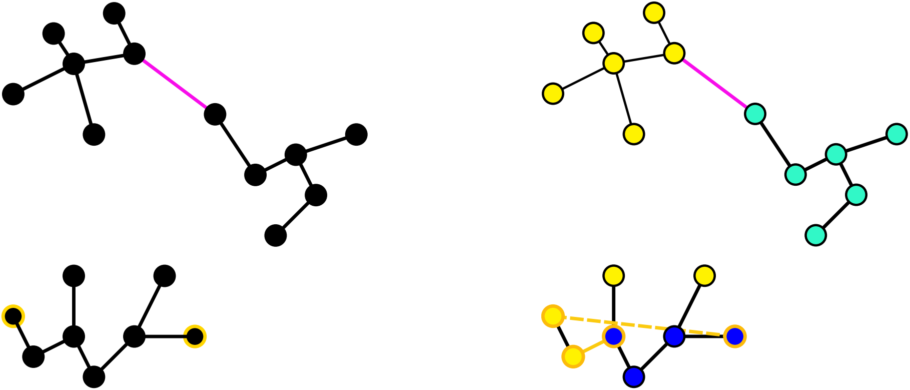

spacing($\mathcal{C}'$) $\leq$ distance between these objects $\leq d^*$

spacing($\mathcal{C}'$) $\leq d^*$: Without pictures

- There must be two objects p_i and p_j that are in the same cluster C_r in $\mathcal{C}$ but in different clusters in $\mathcal{C}'$:

spacing$(\mathcal{C}') \leq d^*$: Without pictures

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- Suppose $p_i \in C'_s$ and $p_j \in C'_t$ in $\mathcal{C}'$.

spacing($\mathcal{C}'$) $\leq d^*$: Without pictures

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- Suppose $p_i \in C'_s$ and $p_j \in C'_t$ in $\mathcal{C}'$.
- All edges in the path Q connecting p_i and p_j in the MST have length $\leq d^*$.
- In particular, there is an object $p \in C'_s$ and an object $p' \notin C'_s$ such that p and p' are adjacent in Q.
- $d(p, p') \leq d^* \Rightarrow$ spacing$(\mathcal{C}') \leq d(p, p') \leq d^*$.

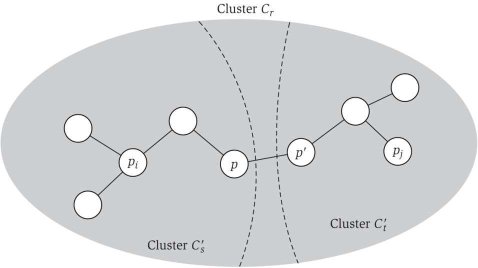

Figure 4.15 An illustration of the proof of (4.26), showing that the spacing of any other clustering can be no larger than that of the clustering found by the single-linkage algorithm.