Divide and Conquer Algorithms

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- Solve each part recursively.
- Solve base cases by brute force.
- Efficiently combine solutions for sub-problems into final solution.

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- Solve each part recursively.
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- Efficiently combine solutions for sub-problems into final solution.
- Common use:
  - Partition problem into two equal sub-problems of size $n/2$.
  - Solve each part recursively.
  - Combine the two solutions in $O(n)$ time.
  - Resulting running time is $O(n \log n)$.

Mergesort

SORT

INSTANCE: Nonempty list $L = x_1, x_2, \ldots, x_n$ of integers.

SOLUTION: A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1}$, for all $1 \leq i < n$.

- Mergesort is a divide-and-conquer algorithm for sorting.
  1. Partition L into two lists A and B of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ respectively.
  2. Recursively sort A.
  3. Recursively sort B.
  4. Merge the sorted lists A and B into a single sorted list.

Merging Two Sorted Lists

- Merge two sorted lists $A = a_1, a_2, \ldots, a_k$ and $B = b_1, b_2, \ldots b_l$.

  Maintain a *current* pointer for each list.
  Initialise each pointer to the front of the list.
  While both lists are nonempty:

    Let a_i and b_j be the elements pointed to by the *current* pointers.
    Append the smaller of the two to the output list.
    Advance the current pointer in the list that the smaller element belonged to.

  EndWhile
  Append the rest of the non-empty list to the output.

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- Running time of this algorithm is $O(k + l)$.

Analysing Mergesort

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Time to split the input into two lists +
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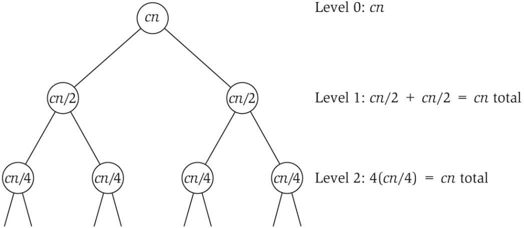
- Three basic ways of solving this recurrence relation:
  1. "Unroll" the recurrence (somewhat informal method).
  2. Guess a solution and substitute into recurrence to check.
  3. Guess solution in $O()$ form and substitute into recurrence to determine the constants. *Read from the textbook.*

Unrolling the recurrence



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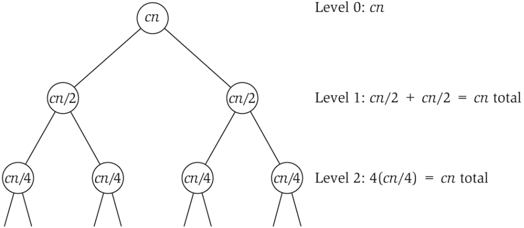

Figure 5.1 Unrolling the recurrence $T(n) \leq 2T(n/2) + O(n)$.

- Input to each sub-problem on level i has size ▸ Poll .
- Recursion tree has ▸ Poll levels.
- Number of sub-problems on level i has size ▸ Poll .

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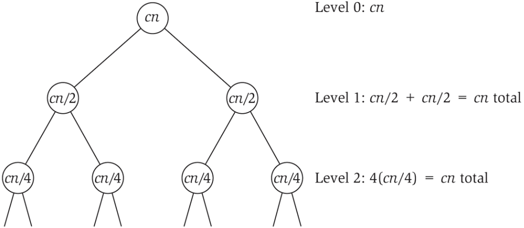

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- Input to each sub-problem on level i has size $n/2^i$.
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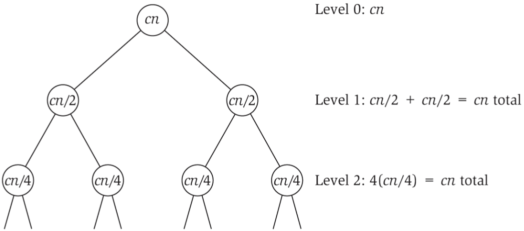

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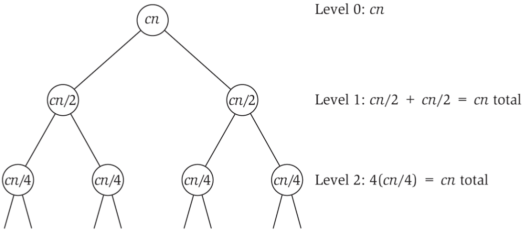

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- Input to each sub-problem on level i has size $n/2^i$.
- Recursion tree has $\log n$ levels.
- Number of sub-problems on level i has size 2^i.
- Total work done at each level is cn.
- Running time of the algorithm is $cn \log n$.
- Use this method only to get an idea of the solution.

Substituting a Solution into the Recurrence

- Guess that the solution is $T(n) \leq cn \log n$ (logarithm to the base 2).
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- Why is $T(n) \leq kn^2$ a "loose" bound?
- Why doesn't an attempt to prove $T(n) \leq kn$, for some $k > 0$ work?

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- Let m be the smallest power of 2 larger than n.
- $T(n) \leq T(m) = O(m \log m) = O(n \log n)$, because $m \leq 2n$.

Other Recurrence Relations

- Divide into q sub-problems of size $n/2$ and merge in $O(n)$ time. Two distinct cases: $q = 1$ and $q > 2$.
- Divide into two sub-problems of size $n/2$ and merge in $O(n^2)$ time.

$T(n) = qT(n/2) + cn, q = 1$

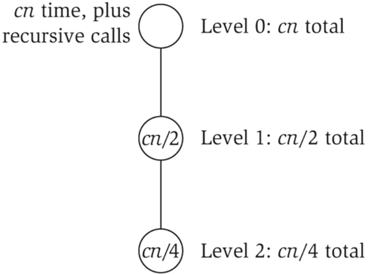

Figure 5.3 Unrolling the recurrence $T(n) \leq T(n/2) + O(n)$.

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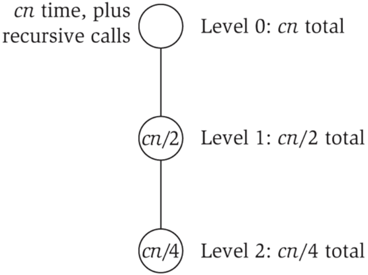

Figure 5.3 Unrolling the recurrence $T(n) \leq T(n/2) + O(n)$.

- Each invocation reduces the problem size by a factor of 2 $\Rightarrow$ there are $\log n$ levels in the recursion tree.
- At level i of the tree, the problem size is $n/2^i$ and the work done is $cn/2^i$.
- Therefore, the total work done is

$$\sum_{i=0}^{i=\log n} \frac{cn}{2^i} = \text{Poll}.$$

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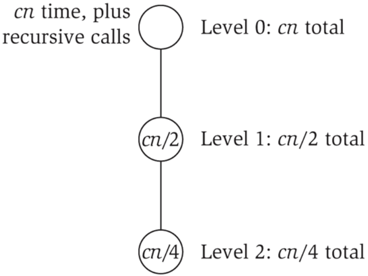

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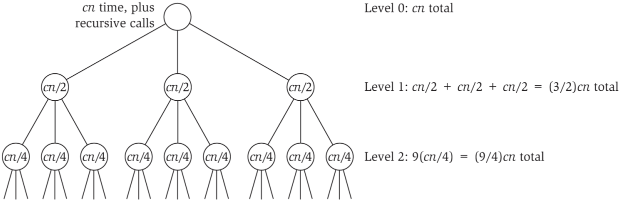

Figure 5.2 Unrolling the recurrence $T(n) \leq 3T(n/2) + O(n)$.

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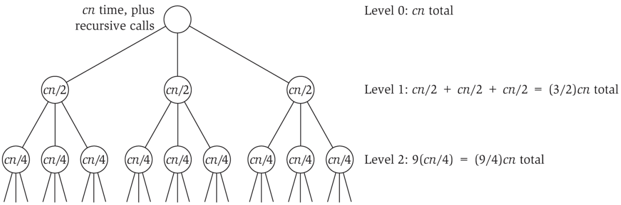

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- There are $\log n$ levels in the recursion tree.
- At level i of the tree, there are q^i sub-problems, each of size $n/2^i$.
- The total work done at level i is $q^i cn/2^i$. Therefore, the total work done is

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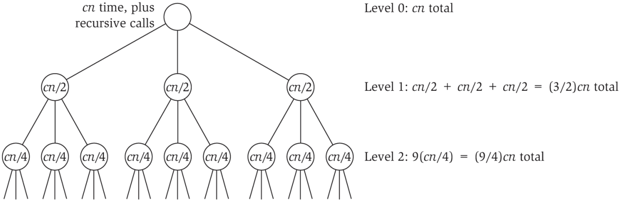

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$$
\begin{aligned}
T(n) &\leq \sum_{i=0}^{i=\log_2 n} q^i \frac{cn}{2^i} \leq cn \sum_{i=0}^{i=\log_2 n} \left(\frac{q}{2}\right)^i \\
&= O\left(cn \left(\frac{q}{2}\right)^{\log_2 n}\right) = O\left(cn \left(\frac{q}{2}\right)^{(\log_{q/2} n)(\log_2 q/2)}\right) \\
&= O(cn\, n^{\log_2 q/2}) = O(n^{\log_2 q}).
\end{aligned}
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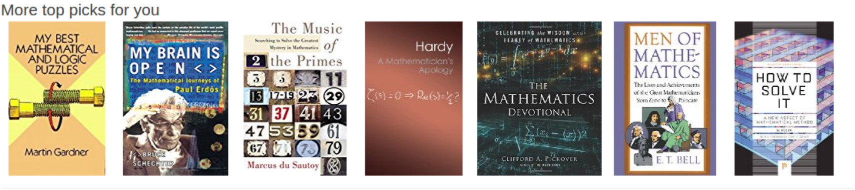
Motivation



- Collaborative filtering: match one user's preferences to those of other users, e.g., purchases, books, music.
- Meta-search engines: merge results of multiple search engines into a better search result.

Fundamental Question

- How do we compare a pair of rankings?
  - My ranking of songs: ordered list of integers from 1 to n.
  - Your ranking of songs: $a_1, a_2, \ldots, a_n$, a permutation of the integers from 1 to n.

1	2	3	4	5	6	7	8	9	10	11	12

4	1	2	6	8	5	3	9	7	11	12	10

Comparing Rankings

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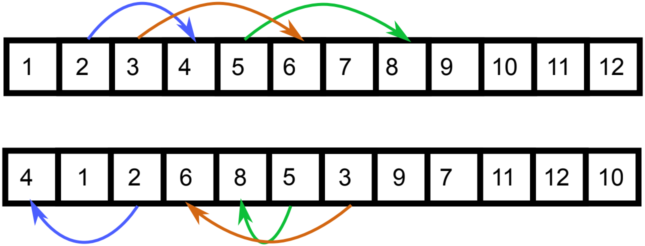

- Suggestion: two rankings of songs are very similar if they have few inversions.
  - The second ranking has an *inversion* if there exist i, j such that $i < j$ but $a_i > a_j$.
  - The number of inversions s is a measure of the difference between the rankings.
- Question also arises in statistics: *Kendall's rank correlation* of two lists of numbers is $1 - 2s/\left(n(n-1)\right)$.

Counting Inversions

Count Inversions

INSTANCE: A list $L = x_1, x_2, \ldots, x_n$ of distinct integers between 1 and n.

SOLUTION: The number of pairs $(i, j), 1 \leq i < j \leq n$ such $x_i > x_j$.

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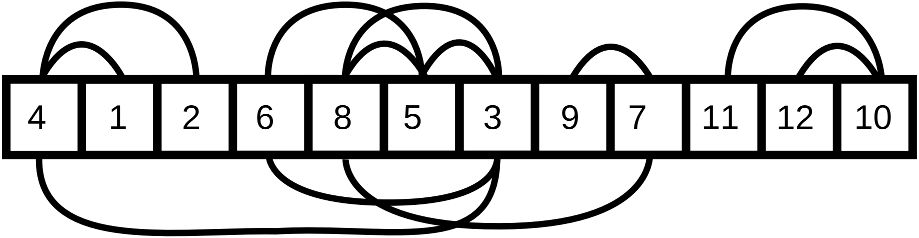
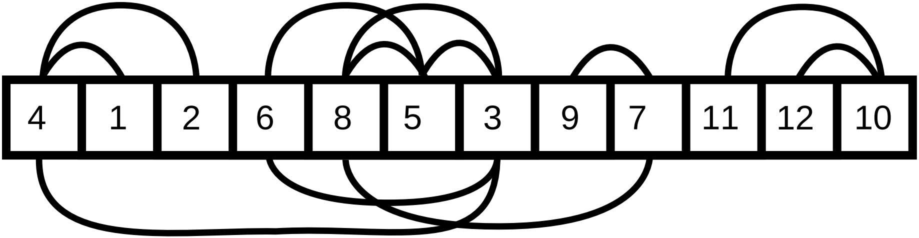


Counting Inversions: Algorithm

- How many inversions can be there in a list of n numbers? Poll



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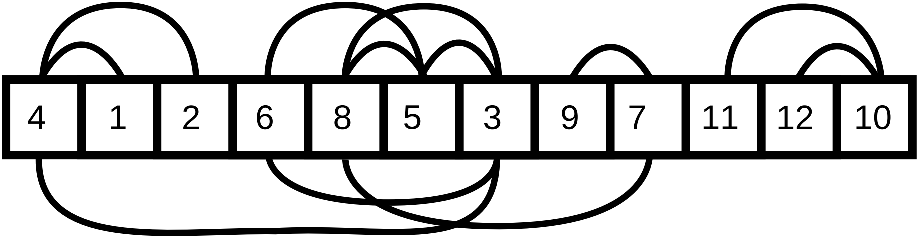

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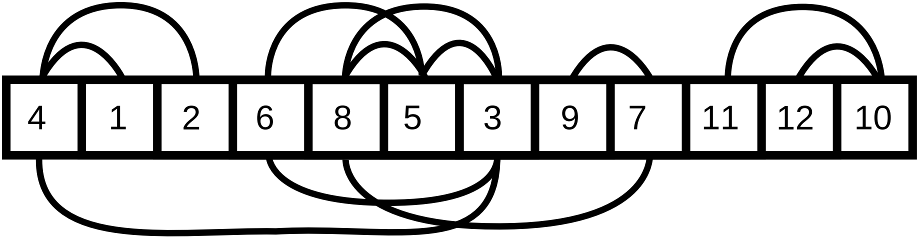
- Sorting removes all inversions in $O(n \log n)$ time. Can we modify the Mergesort algorithm to count inversions?



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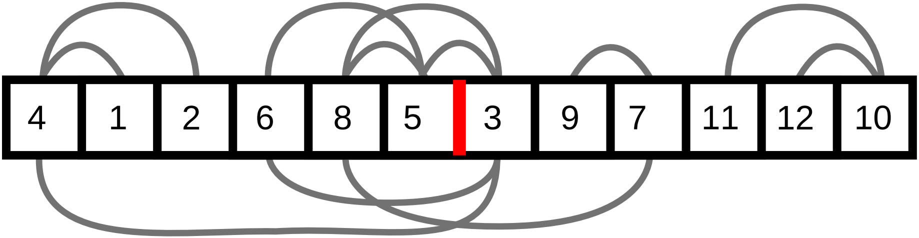

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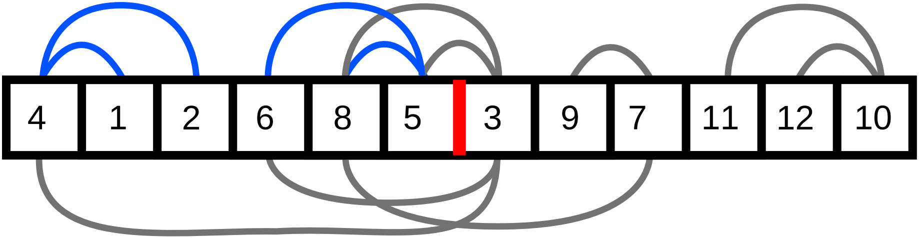

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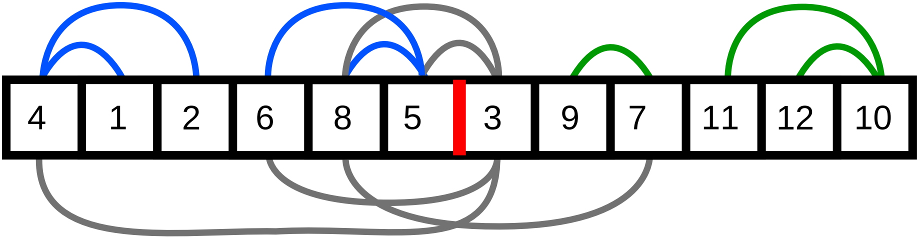

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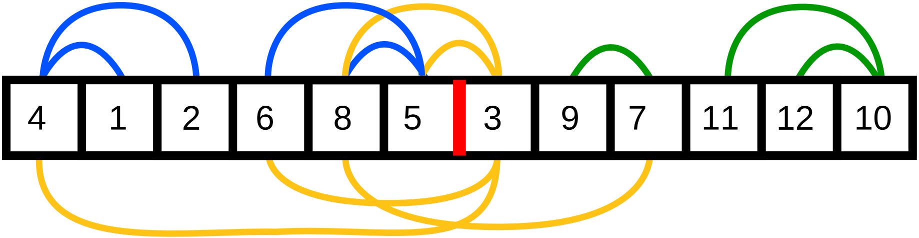

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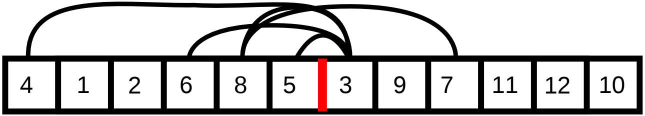

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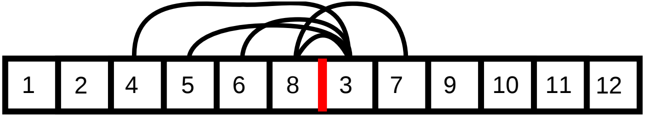

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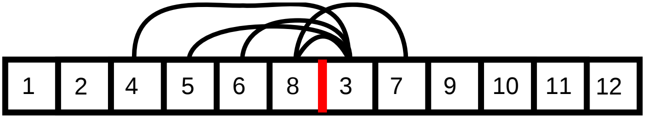

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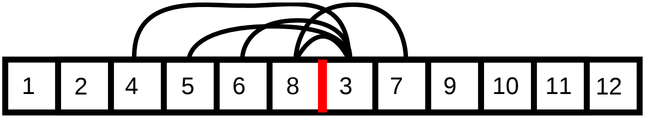

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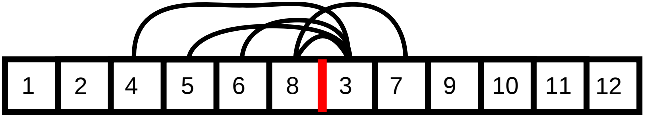

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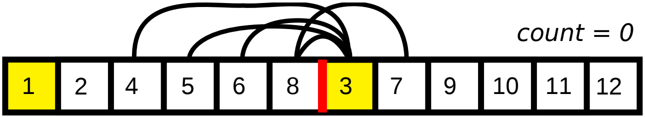

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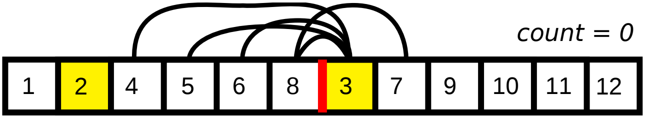

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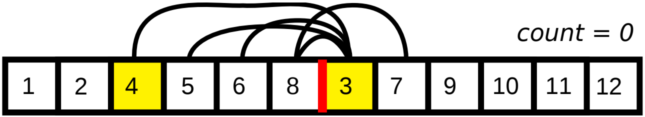

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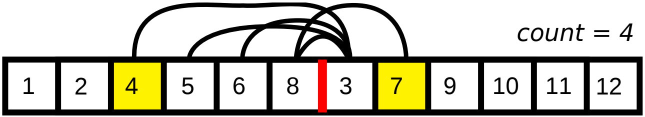

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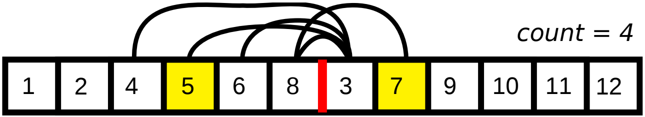

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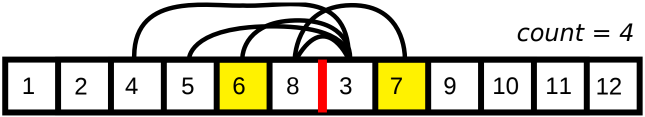

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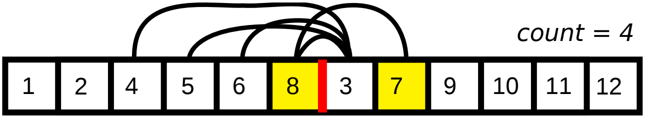

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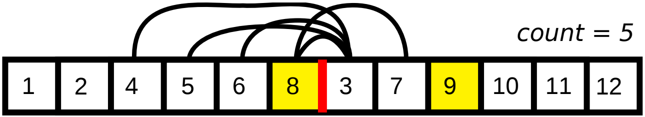

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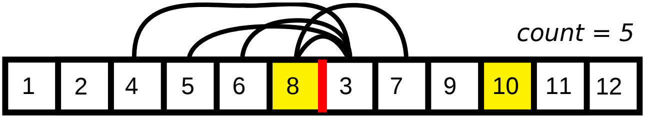

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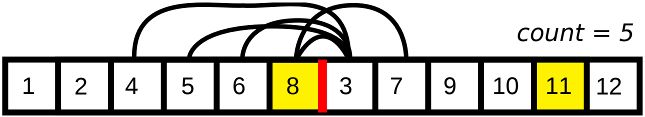

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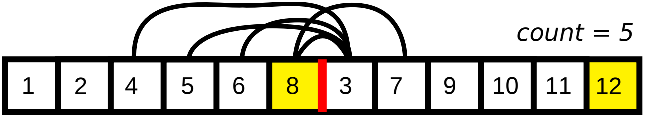

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Counting Inversions: Final Algorithm

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Sort-and-Count(L)
  If the list has one element then
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  Else
      Divide the list into two halves:
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- Running time $T(n)$ of the algorithm is $O(n \log n)$ because $T(n) \leq 2T(n/2) + O(n)$.

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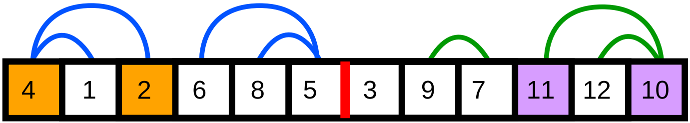

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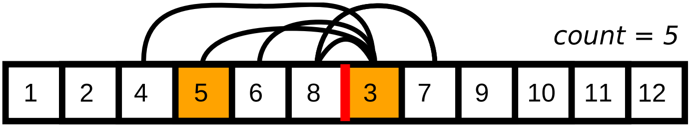

Counting Inversions: Correctness of Sort-and-Count

- Prove by induction. Strategy: (a) every inversion in the data is counted exactly once and (b) No non-inversion is counted.
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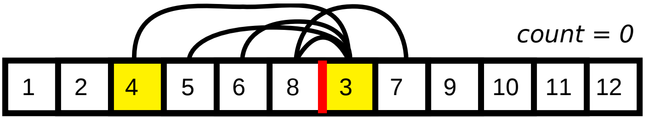

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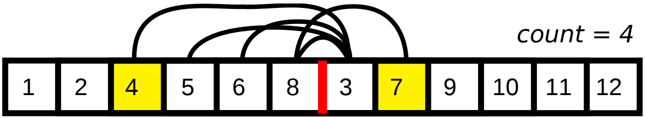

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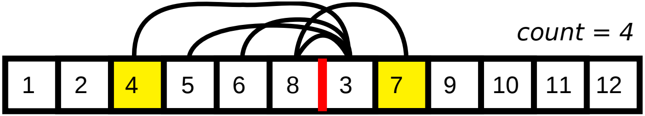
    - Why is no non-inversion counted, i.e., Why does every pair counted correspond to an inversion? When x_l is output, it is smaller than all remaining elements in A, since A is sorted.



Integer Multiplication

MULTIPLY INTEGERS
INSTANCE: Two n-digit binary integers x and y
SOLUTION: The product xy

Integer Multiplication

MULTIPLY INTEGERS
INSTANCE: Two *n*-digit binary integers x and y
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- Multiply two *n*-digit integers.

Integer Multiplication

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INSTANCE: Two n-digit binary integers x and y

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- Multiply two n-digit integers.
- Result has at most $2n$ digits.

Integer Multiplication

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```
                      1100
                    × 1101
                    ------
       12             1100
     × 13            0000
     ----           1100
       36          1100
      12           --------
     ----          10011100
      156

      (a)             (b)
```

Figure 5.8 The elementary-school algorithm for multiplying two integers, in (a) decimal and (b) binary representation.

Integer Multiplication

Multiply Integers

INSTANCE: Two n-digit binary integers x and y

SOLUTION: The product xy

- Multiply two n-digit integers.
- Result has at most $2n$ digits.
- Algorithm we learnt in school takes $O(n^2)$ operations. Size of the input is not 2 but $2n$,

```
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               × 1101
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Divide-and-Conquer Idea

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n bits

$n/2$ bits

Divide-and-Conquer Algorithm

$$xy = x_1 y_1 2^n + (x_1 y_0 + x_0 y_1) 2^{n/2} + x_0 y_0$$

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  1. Compute $x_1 y_1$, $x_1 y_0$, $x_0 y_1$, and $x_0 y_0$ recursively.
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     1. Multiple $x_1 y_1$ by 2^n
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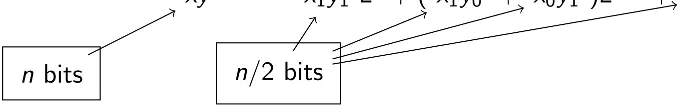

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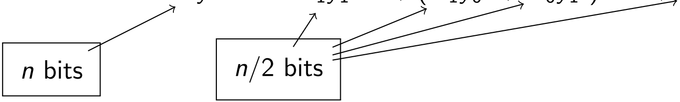

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$$T(n) \leq 4T(n/2) + cn \leq O(n^2)$$

Improving the Algorithm

- Four sub-problems lead to an $O(n^2)$ algorithm.
- How can we reduce the number of sub-problems?

Improving the Algorithm

- Four sub-problems lead to an $O(n^2)$ algorithm.
- How can we reduce the number of sub-problems?
  - No need to compute $x_1 y_0$ and $x_0 y_1$ independently; we just need their sum.

$$(x_0 + x_1)(y_0 + y_1) = x_1 y_1 + (x_1 y_0 + x_0 y_1) + x_0 y_0$$

$$(x_1 y_0 + x_0 y_1) = (x_0 + x_1)(y_0 + y_1) - x_1 y_1 - x_0 y_0$$

Need this sum

$n/2$ bits

- Compute $x_1 y_1$, $x_0 y_0$ and $(x_0 + x_1)(y_0 + y_1)$ recursively and then compute $(x_1 y_0 + x_0 y_1)$ by subtraction.
- Strategy: simple arithmetic manipulations.

Final Algorithm

```
Recursive-Multiply(x,y):
  Write x = x_1 · 2^{n/2} + x_0
        y = y_1 · 2^{n/2} + y_0
  Compute x_1 + x_0 and y_1 + y_0
  p = Recursive-Multiply(x_1 + x_0, y_1 + y_0)
  x_1y_1 = Recursive-Multiply(x_1, y_1)
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  Return x_1y_1 · 2^n + (p − x_1y_1 − x_0y_0) · 2^{n/2} + x_0y_0
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- We have three sub-problems of size $n/2$.
- What is the running time $T(n)$?

$$T(n) \leq 3T(n/2) + cn$$

Final Algorithm

Recursive-Multiply(x,y):

  Write $x = x_1 \cdot 2^{n/2} + x_0$

  $y = y_1 \cdot 2^{n/2} + y_0$

  Compute $x_1 + x_0$ and $y_1 + y_0$

  p = Recursive-Multiply($x_1 + x_0$, $y_1 + y_0$)

  x_1y_1 = Recursive-Multiply(x_1, y_1)

  x_0y_0 = Recursive-Multiply(x_0, y_0)

  Return $x_1y_1 \cdot 2^n + (p - x_1y_1 - x_0y_0) \cdot 2^{n/2} + x_0y_0$

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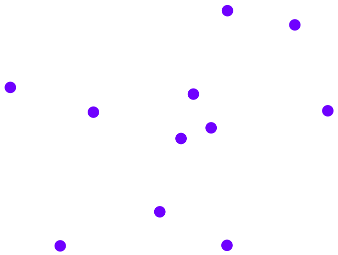
- What is the running time $T(n)$?

$$\begin{aligned} T(n) &\leq 3T(n/2) + cn \\ &\leq O(n^{\log_2 3}) = O(n^{1.59}) \end{aligned}$$

Computational Geometry

- Algorithms for geometric objects: points, lines, segments, triangles, spheres, polyhedra, ldots.
- Started in 1975 by Shamos and Hoey.
- Problems studied have applications in a vast number of fields: ecology, molecular biology, statistics, computational finance, computer graphics, computer vision, . . .

Closest Pair of Points on the Plane

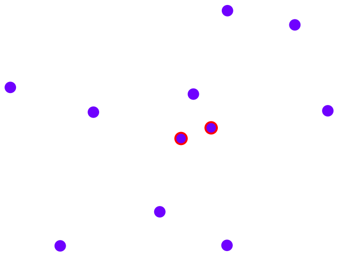


Closest Pair of Points

INSTANCE: A set P of n points in the plane

SOLUTION: The pair of points in P that are the closest to each other.

Closest Pair of Points on the Plane



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Closest Pair of Points on the Plane

CLOSEST PAIR OF POINTS

INSTANCE: A set P of n points in the plane

SOLUTION: The pair of points in P that are the closest to each other.

- At first glance, it seems any algorithm must take $\Omega(n^2)$ time.
- Shamos and Hoey figured out an ingenious $O(n \log n)$ divide and conquer algorithm.

Closest Pair: Set-up

- Let $P = \{p_1, p_2, \ldots, p_n\}$ with $p_i = (x_i, y_i)$.
- Use $d(p_i, p_j)$ to denote the Euclidean distance between p_i and p_j. For a specific pair of points, can compute $d(p_i, p_j)$ in $O(1)$ time.
- Goal: find the pair of points p_i and p_j that minimise $d(p_i, p_j)$.

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  - Sort: closest pair must be adjacent in the sorted order.
  - Divide and conquer after sorting: closest pair must be closest of
    1. closest pair in left half: distance δ_Q.
    2. closest pair in right half: distance δ_R.
    3. closest among pairs that span the left and right halves and are at most $\min(\delta_Q, \delta_R)$ apart. How many such pairs do we need to consider?

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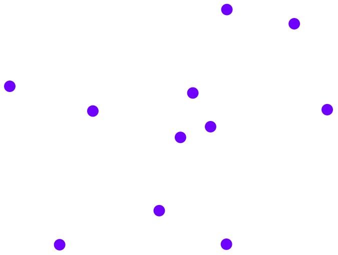
- Generalize the second idea to 2D.

Closest Pair: Algorithm Skeleton

1. Divide P into two sets Q and R of $n/2$ points such that each point in Q has x-coordinate less than any point in R.
2. Recursively compute closest pair in Q and in R, respectively.



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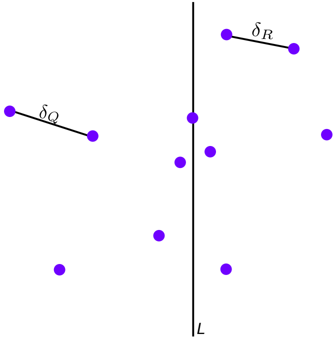

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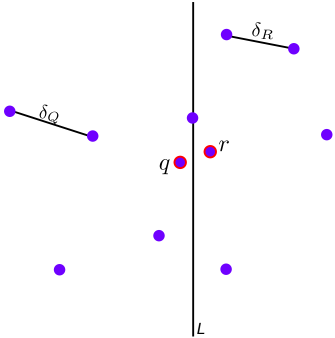
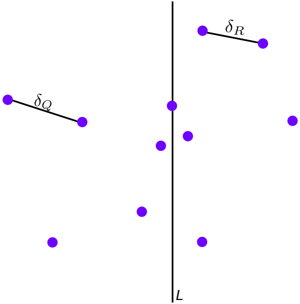
3. Let δ_Q be the distance computed for Q, δ_R be the distance computed for R, and $\delta = \min(\delta_Q, \delta_R)$.
4. Compute pair (q, r) of points such that $q \in Q$, $r \in R$, $d(q, r) < \delta$ and $d(q, r)$ is the smallest possible.



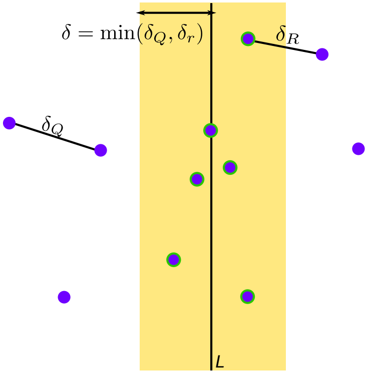
Closest Pair: Proof Sketch

- Prove by induction: Let (s, t) be the closest pair.
  1. (i) both are in Q: computed correctly by recursive call.
  2. (ii) both are in R: computed correctly by recursive call.
  3. (iii) one is in Q and the other is in R: computed correctly in $O(n)$ time by the procedure we will discuss.
- Strategy: Pairs of points for which we do not compute the distance between cannot be the closest pair.
- Overall running time is $O(n \log n)$.



Closest Pair: Conquer Step

- Line L passes through right-most point in Q.
- Let S be the set of points within distance δ of L. (In image, $\delta = \delta_R$.) Poll



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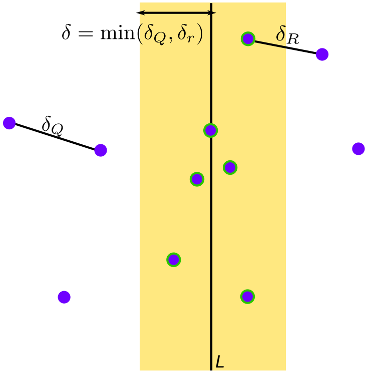
- Claim: There exist $q \in Q$, $r \in R$ such that $d(q, r) < \delta$ if and only if $q, r \in S$.



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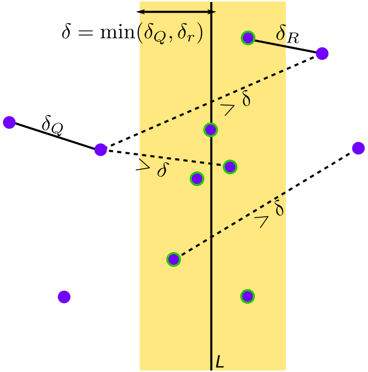
- Claim: There exist $q \in Q$, $r \in R$ such that $d(q, r) < \delta$ if and only if $q, r \in S$.
- Corollary: If $t \in Q - S$ or $u \in R - S$, then (t, u) cannot be the closest pair.



Closest Pair: Packing Argument

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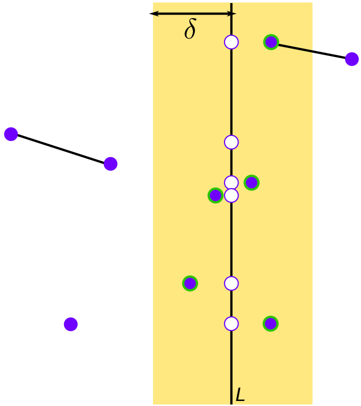

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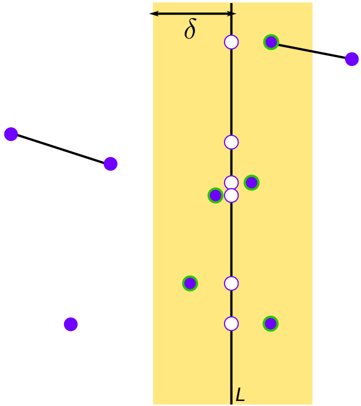
- Let S_y denote the set of points in S sorted by increasing y-coordinate and let s_y denote the y-coordinate of a point $s \in S$.
- Claim: If there exist $s, s' \in S$ such that $d(s, s') < \delta$ then s and s' are at most 15 indices apart in S_y.



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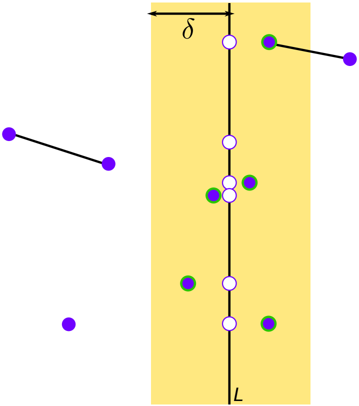

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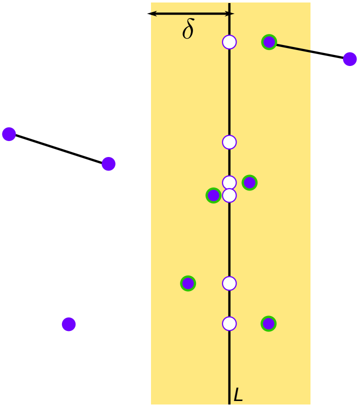
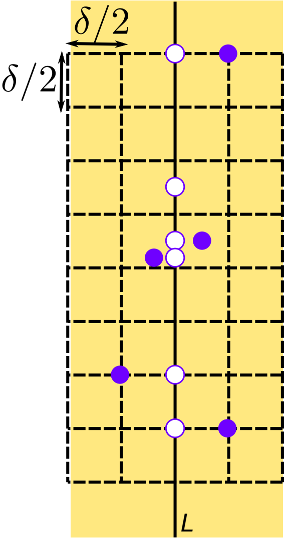
- Use the claim in the algorithm: For every point $s \in S_y$, compute distances only to the next 15 points in S_y.
- Other pairs of points cannot be candidates for the closest pair.



Closest Pair: Proof of Packing Argument

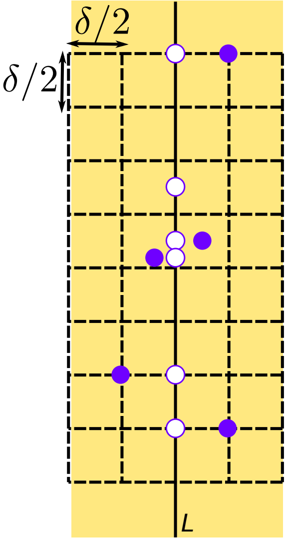
- Claim: If there exist $s, s' \in S$ such that s' appears 16 or more indices after s in S_y, then $s'_y - s_y \geq \delta$.



Closest Pair: Proof of Packing Argument

- Claim: If there exist $s, s' \in S$ such that s' appears 16 or more indices after s in S_y, then $s'_y - s_y \geq \delta$.
- Pack the plane with squares of side $\delta/2$.

Poll



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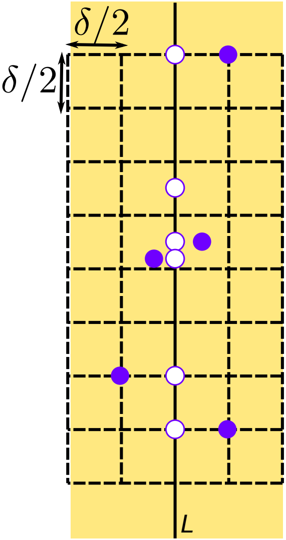
- Each square contains at most one point.



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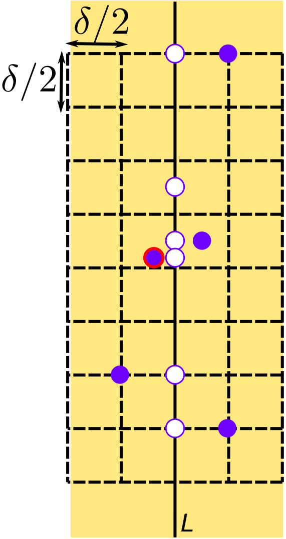
- Each square contains at most one point.
- Let s lie in one of the squares.



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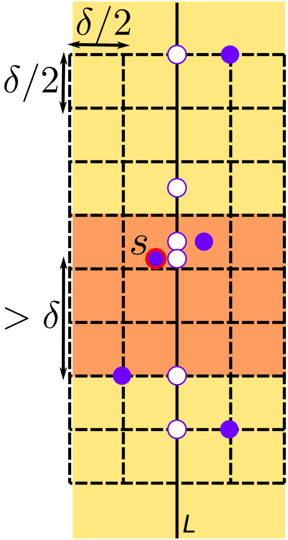
- Each square contains at most one point.
- Let s lie in one of the squares.
- Any point in the third row of the packing below s has a y-coordinate at least δ more than s_y.



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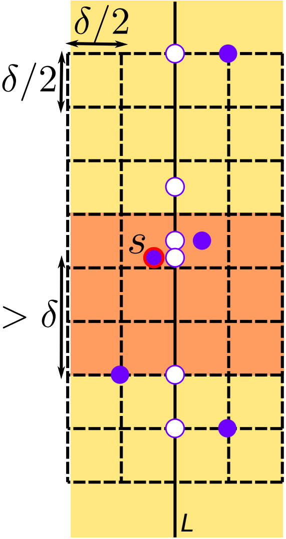
  Poll
- Each square contains at most one point.
- Let s lie in one of the squares.
- Any point in the third row of the packing below s has a y-coordinate at least δ more than s_y.
- We get a count of 12 or more indices (textbook says 16).



Closest Pair: Final Algorithm

```
Closest-Pair(P)
  Construct P_x and P_y  (O(n log n) time)
  (p*_0, p*_1) = Closest-Pair-Rec(P_x,P_y)

Closest-Pair-Rec(P_x, P_y)
  If |P| ≤ 3 then
    find closest pair by measuring all pairwise distances
  Endif

  Construct Q_x, Q_y, R_x, R_y (O(n) time)
  (q*_0,q*_1) = Closest-Pair-Rec(Q_x, Q_y)
  (r*_0,r*_1) = Closest-Pair-Rec(R_x, R_y)

  δ = min(d(q*_0,q*_1), d(r*_0,r*_1))
  x* = maximum x-coordinate of a point in set Q
  L = {(x,y) : x = x*}
  S = points in P within distance δ of L.

  Construct S_y (O(n) time)
  For each point s ∈ S_y, compute distance from s
     to each of next 15 points in S_y
     Let s, s' be pair achieving minimum of these distances
     (O(n) time)

  If d(s,s') < δ then
     Return (s,s')
  Else if d(q*_0,q*_1) < d(r*_0,r*_1) then
     Return (q*_0,q*_1)
  Else
     Return (r*_0,r*_1)
  Endif
```

Closest Pair: Final Algorithm

Closest-Pair(P)

  Construct P_x and P_y ($O(n \log n)$ time)

  (p_0^*, p_1^*) = Closest-Pair-Rec(P_x,P_y)

Closest-Pair-Rec(P_x, P_y)

  If $|P| \leq 3$ then

    find closest pair by measuring all pairwise distances

  Endif

  Construct Q_x, Q_y, R_x, R_y ($O(n)$ time)

  (q_0^*,q_1^*) = Closest-Pair-Rec(Q_x, Q_y)

  (r_0^*,r_1^*) = Closest-Pair-Rec(R_x, R_y)

  $\delta = \min(d(q_0^*,q_1^*),\ d(r_0^*,r_1^*))$

  x^* = maximum x-coordinate of a point in set Q

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If $d(s,s') < \delta$ then

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Else if $d(q_0^*,q_1^*) < d(r_0^*,r_1^*)$ then

  Return (q_0^*,q_1^*)

Else

  Return (r_0^*,r_1^*)

Endif