Dynamic Programming

T. M. Murali

October 19, 21, 26, 28, 2021
Algorithm Design Techniques

1. Goal: design efficient (polynomial-time) algorithms.
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2. Greedy
   - Pro: natural approach to algorithm design.
   - Con: many greedy approaches to a problem. Only some may work.
   - Con: many problems for which no greedy approach is known.

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   - Pro: simple to develop algorithm skeleton.
   - Con: conquer step can be very hard to implement efficiently.
   - Con: usually reduces time for a problem known to be solvable in polynomial time.
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   - Pro: simple to develop algorithm skeleton.
   - Con: conquer step can be very hard to implement efficiently.
   - Con: usually reduces time for a problem known to be solvable in polynomial time.

4. Dynamic programming
   - More powerful than greedy and divide-and-conquer strategies.
   - Implicitly explore space of all possible solutions.
   - Solve multiple sub-problems and build up correct solutions to larger and larger sub-problems.
   - Careful analysis needed to ensure number of sub-problems solved is polynomial in the size of the input.
History of Dynamic Programming

- Bellman pioneered the systematic study of dynamic programming in the 1950s.

The Secretary of Defense at that time was hostile to mathematical research. Bellman sought an impressive name to avoid confrontation. ▶ “it’s impossible to use dynamic in a pejorative sense” ▶ “something not even a Congressman could object to” (Bellman, R. E., Eye of the Hurricane, An Autobiography).
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▶ “it’s impossible to use dynamic in a pejorative sense”
▶ “something not even a Congressman could object to” (Bellman, R. E., Eye of the Hurricane, An Autobiography).
Applications of Dynamic Programming

- Computational biology: Smith-Waterman algorithm for sequence alignment.
- Operations research: Bellman-Ford algorithm for shortest path routing in networks.
- Control theory: Viterbi algorithm for hidden Markov models.
- Computer science (theory, graphics, AI, . . .): Unix `diff` command for comparing two files.
Weighted Interval Scheduling
Segmented Least Squares
RNA Secondary Structure
Shortest Paths

Input: Start and end time of each ride.
Constraint: Cannot be in two places at one time.
Goal: Compute the largest number of rides you can be on in one day.

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**Review: Interval Scheduling**

**Interval Scheduling**

**INSTANCE:** Set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of $n$ jobs.

**SOLUTION:** The largest subset of mutually compatible jobs.

- Two jobs are *compatible* if they do not overlap.
- For any input set of jobs, algorithm must provably compute the largest set of compatible jobs.
**Interval Scheduling**

**Instance:** Set \((s(i), f(i)), 1 \leq i \leq n\) of start and finish times of \(n\) jobs.

**Solution:** The largest subset of mutually compatible jobs.

- Two jobs are *compatible* if they do not overlap.
- For any input set of jobs, algorithm must provably compute the largest set of compatible jobs.
- Greedy algorithm: sort jobs in increasing order of finish times. Add next job to current subset only if it is compatible with previously-selected jobs.
Weighted Interval Scheduling

Weighted Interval Scheduling

**INSTANCE:** Nonempty set \{((s_i, f_i), 1 \leq i \leq n}\) of start and finish times of \(n\) jobs and a weight \(v_i \geq 0\) associated with each job.

**SOLUTION:** A set \(S\) of mutually compatible jobs such that the value \(\sum_{i \in S} v_i\) is maximised.

Index

1

\[\text{Value} = 1\]

2

\[\text{Value} = 3\]

3

\[\text{Value} = 1\]

Figure 6.1 A simple instance of weighted interval scheduling.
Weighted Interval Scheduling

**INSTANCE:** Nonempty set \(\{(s_i, f_i), 1 \leq i \leq n\}\) of start and finish times of \(n\) jobs and a weight \(v_i \geq 0\) associated with each job.

**SOLUTION:** A set \(S\) of mutually compatible jobs such that the value \(\sum_{i \in S} v_i\) is maximised.

---

**Figure 6.1** A simple instance of weighted interval scheduling.

Greedy algorithm can produce arbitrarily bad results for this problem.
Detour: a Binomial Identity

Pascal's triangle:
Each element is a binomial coefficient.
Each element is the sum of the two elements above it.
\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}
\]

Proof: either we include the \(n\)th element in a subset or not...
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\]

- Proof: either we include the \( n \)th element in a subset or not . . .
**Approach**

- Sort jobs in increasing order of finish time and relabel: $f_1 \leq f_2 \leq \ldots \leq f_n$.
- Job $i$ comes before job $j$ if $i < j$.
- $p(j)$ is the largest index $i < j$ such that job $i$ is compatible with job $j$. $p(j) = 0$ if there is no such job $i$.

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**Index**

<table>
<thead>
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$p(1) = 0$, $p(2) = 0$, $p(3) = 1$, $p(4) = 0$, $p(5) = 3$, $p(6) = 3$.
**Approach**

- Sort jobs in increasing order of finish time and relabel: \( f_1 \leq f_2 \leq \ldots \leq f_n \).
- Job \( i \) comes before job \( j \) if \( i < j \).
- \( p(j) \) is the largest index \( i < j \) such that job \( i \) is compatible with job \( j \).
  - \( p(j) = 0 \) if there is no such job \( i \).
- All jobs that come before job \( p(j) \) are also compatible with job \( j \).

We will develop optimal algorithm from obvious statements about the problem.
Let $O$ be the optimal solution: it contains a subset of the input jobs. Two cases to consider. **One of these cases must be true.**

**Case 1:** job $n$ is not in $O$.

**Case 2:** job $n$ is in $O$. 

$O$ must be the best of these two choices!
Let $O$ be the optimal solution: it contains a subset of the input jobs. Two cases to consider. **One of these cases must be true.**

**Case 1:** job $n$ is not in $O$. $O$ must be the optimal solution for jobs \( \{1, 2, \ldots, n-1\} \).

**Case 2:** job $n$ is in $O$. 

---

\( v_1 = 2 \)

\( p(1) = 0 \)

\( v_2 = 4 \)

\( p(2) = 0 \)

\( v_3 = 4 \)

\( p(3) = 1 \)

\( v_4 = 7 \)

\( p(4) = 0 \)

\( v_5 = 2 \)

\( p(5) = 3 \)

\( v_6 = 1 \)

\( p(6) = 3 \)
Let $O$ be the optimal solution: it contains a subset of the input jobs. Two cases to consider. One of these cases must be true.

**Case 1:** job $n$ is not in $O$. $O$ must be the optimal solution for jobs $\{1, 2, \ldots, n - 1\}$.

**Case 2:** job $n$ is in $O$.

- $O$ cannot use incompatible jobs $\{p(n) + 1, p(n) + 2, \ldots, n - 1\}$.
- Remaining jobs in $O$ must be the optimal solution for jobs $\{1, 2, \ldots, p(n)\}$.
Let \( O \) be the optimal solution: it contains a subset of the input jobs. Two cases to consider. One of these cases must be true.

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\( O \) must be the best of these two choices!
Sub-problems

Let $O$ be the optimal solution: it contains a subset of the input jobs. Two cases to consider. One of these cases must be true.

Case 1: job $n$ is not in $O$. $O$ must be the optimal solution for jobs $\{1, 2, \ldots, n-1\}$.

Case 2: job $n$ is in $O$.

- $O$ cannot use incompatible jobs $\{p(n) + 1, p(n) + 2, \ldots, n-1\}$.
- Remaining jobs in $O$ must be the optimal solution for jobs $\{1, 2, \ldots, p(n)\}$.

$O$ must be the best of these two choices!

Suggests finding optimal solution for sub-problems consisting of jobs $\{1, 2, \ldots, j-1, j\}$, for all values of $j$. 
Let $O_j$ be the optimal solution for jobs $\{1, 2, \ldots, j\}$ and $OPT(j)$ be the value of this solution ($OPT(0) = 0$).
Let \( O_j \) be the optimal solution for jobs \( \{1, 2, \ldots, j\} \) and \( OPT(j) \) be the value of this solution (\( OPT(0) = 0 \)).

We are seeking \( O_n \) with a value of \( OPT(n) \).
Let $O_j$ be the optimal solution for jobs $\{1, 2, \ldots, j\}$ and $OPT(j)$ be the value of this solution ($OPT(0) = 0$).

We are seeking $O_n$ with a value of $OPT(n)$.

To compute $OPT(j)$:

Case 1 $j \notin O_j$:
Let $O_j$ be the optimal solution for jobs $\{1, 2, \ldots, j\}$ and $OPT(j)$ be the value of this solution ($OPT(0) = 0$).

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To compute $OPT(j)$:

- **Case 1** $j \notin O_j$: $OPT(j) = OPT(j - 1)$.
Let $O_j$ be the optimal solution for jobs $\{1, 2, \ldots, j\}$ and $OPT(j)$ be the value of this solution ($OPT(0) = 0$).

We are seeking $O_n$ with a value of $OPT(n)$.

To compute $OPT(j)$:

Case 1 $j \not\in O_j$: $OPT(j) = OPT(j - 1)$.

Case 2 $j \in O_j$: 

\[ OPT(j) = \max (v_j + OPT(p(j)), OPT(j - 1)) \]
Let \( O_j \) be the optimal solution for jobs \( \{1, 2, \ldots, j\} \) and \( \text{OPT}(j) \) be the value of this solution (\( \text{OPT}(0) = 0 \)).

We are seeking \( O_n \) with a value of \( \text{OPT}(n) \).

To compute \( \text{OPT}(j) \):

Case 1 \( j \notin O_j \): \( \text{OPT}(j) = \text{OPT}(j - 1) \).

Case 2 \( j \in O_j \): \( \text{OPT}(j) = v_j + \text{OPT}(p(j)) \).
Let $O_j$ be the optimal solution for jobs $\{1, 2, \ldots, j\}$ and $OPT(j)$ be the value of this solution ($OPT(0) = 0$).

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To compute \( OPT(j) \):

- **Case 1** \( j \notin O_j \): \( OPT(j) = OPT(j - 1) \).
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OPT(j) = \max \left( v_j + OPT(p(j)), OPT(j - 1) \right)
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When does job \( j \) belong to \( O_j \)?

Poll
Let $O_j$ be the optimal solution for jobs $\{1, 2, \ldots, j\}$ and $OPT(j)$ be the value of this solution ($OPT(0) = 0$).

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To compute $OPT(j)$:

Case 1 $j \notin O_j$: $OPT(j) = OPT(j - 1)$.

Case 2 $j \in O_j$: $OPT(j) = v_j + OPT(p(j))$

$$OPT(j) = \max (v_j + OPT(p(j)), OPT(j - 1))$$

When does job $j$ belong to $O_j$? If and only if $v_j + OPT(p(j)) \geq OPT(j - 1)$. 

- Let $O_j$ be the optimal solution for jobs $\{1, 2, \ldots, j\}$ and $OPT(j)$ be the value of this solution ($OPT(0) = 0$).
- We are seeking $O_n$ with a value of $OPT(n)$.
- To compute $OPT(j)$:
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T. M. Murali October 19, 21, 26, 28, 2021 Dynamic Programming
Recursive Algorithm

\[ \text{OPT}(j) = \max(v_j + \text{OPT}(p(j)), \text{OPT}(j - 1)) \]

Compute-Opt(j)

If \( j = 0 \) then

Return 0

Else

Return \( \max(v_j + \text{Compute-Opt}(p(j)), \text{Compute-Opt}(j - 1)) \)

Endif

Correctness of algorithm follows by induction (see textbook for proof).
**Recursive Algorithm**

\[ \text{OPT}(j) = \max(\nu_j + \text{OPT}(p(j)), \text{OPT}(j - 1)) \]

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**Compute-Opt(j)**

If \( j = 0 \) then

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Endif

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- Correctness of algorithm follows by induction (see textbook for proof).
Example of Recursive Algorithm

Index

1  \( v_1 = 2 \)  \( p(1) = 0 \)
2  \( v_2 = 4 \)  \( p(2) = 0 \)
3  \( v_3 = 4 \)  \( p(3) = 1 \)
4  \( v_4 = 7 \)  \( p(4) = 0 \)
5  \( v_5 = 2 \)  \( p(5) = 3 \)
6  \( v_6 = 1 \)  \( p(6) = 3 \)

**Figure 6.2** An instance of weighted interval scheduling with the functions \( p(j) \) defined for each interval \( j \).

\[
\begin{align*}
\text{OPT}(6) &= \text{Poll} \\
\text{OPT}(5) &= \\
\text{OPT}(4) &= \\
\text{OPT}(3) &= \\
\text{OPT}(2) &= \\
\text{OPT}(1) &= \\
\text{OPT}(0) &= 0
\end{align*}
\]
Example of Recursive Algorithm

OPT(6) = \max(v_6 + OPT(p(6)), OPT(5)) = \max(1 + OPT(3), OPT(5))

OPT(5) =

OPT(4) =

OPT(3) =

OPT(2) =

OPT(1) =

OPT(0) = 0
**Example of Recursive Algorithm**

OPT(6) = max( \( v_6 + \text{OPT}(p(6)) \), OPT(5) )
OPT(5) = max( \( v_5 + \text{OPT}(p(5)) \), OPT(4) )
OPT(4) =
OPT(3) =
OPT(2) =
OPT(1) =
OPT(0) = 0

*Figure 6.2* An instance of weighted interval scheduling with the functions \( p(j) \) defined for each interval \( j \).

\( \text{OPT}(6) = \max(v_6 + \text{OPT}(p(6)), \text{OPT}(5)) = \max(1 + \text{OPT}(3), \text{OPT}(5)) \)
\( \text{OPT}(5) = \max(v_5 + \text{OPT}(p(5)), \text{OPT}(4)) = \max(2 + \text{OPT}(3), \text{OPT}(4)) \)
Example of Recursive Algorithm

\[
\text{OPT}(6) = \max(v_6 + \text{OPT}(p(6)), \text{OPT}(5)) = \max(1 + \text{OPT}(3), \text{OPT}(5))
\]
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\text{OPT}(5) = \max(v_5 + \text{OPT}(p(5)), \text{OPT}(4)) = \max(2 + \text{OPT}(3), \text{OPT}(4))
\]
\[
\text{OPT}(4) = \max(v_4 + \text{OPT}(p(4)), \text{OPT}(3)) = \max(7 + \text{OPT}(0), \text{OPT}(3))
\]
\[
\text{OPT}(3) =
\]
\[
\text{OPT}(2) =
\]
\[
\text{OPT}(1) =
\]
\[
\text{OPT}(0) = 0
\]
Example of Recursive Algorithm

\[ \text{OPT}(6) = \max(v_6 + \text{OPT}(p(6)), \text{OPT}(5)) = \max(1 + \text{OPT}(3), \text{OPT}(5)) \]
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\[ \text{OPT}(4) = \max(v_4 + \text{OPT}(p(4)), \text{OPT}(3)) = \max(7 + \text{OPT}(0), \text{OPT}(3)) \]
\[ \text{OPT}(3) = \max(v_3 + \text{OPT}(p(3)), \text{OPT}(2)) = \max(4 + \text{OPT}(1), \text{OPT}(2)) \]
\[ \text{OPT}(2) = \]
\[ \text{OPT}(1) = 2 \]
\[ \text{OPT}(0) = 0 \]

**Figure 6.2** An instance of weighted interval scheduling with the functions \( p(j) \) defined for each interval \( j \).
Optimal solution is job 5, job 3, and job 1.

Figure 6.2 An instance of weighted interval scheduling with the functions $p(j)$ defined for each interval $j$.

OPT(6) = max($v_6 + OPT(p(6))$, OPT(5)) = max(1 + OPT(3), OPT(5))
OPT(5) = max($v_5 + OPT(p(5))$, OPT(4)) = max(2 + OPT(3), OPT(4))
OPT(4) = max($v_4 + OPT(p(4))$, OPT(3)) = max(7 + OPT(0), OPT(3))
OPT(3) = max($v_3 + OPT(p(3))$, OPT(2)) = max(4 + OPT(1), OPT(2))
OPT(2) = max($v_2 + OPT(p(2))$, OPT(1)) = max(4 + OPT(0), OPT(1))
OPT(1) =
OPT(0) = 0
Example of Recursive Algorithm

**Figure 6.2** An instance of weighted interval scheduling with the functions $p(j)$ defined for each interval $j$.

\[
\begin{align*}
\text{OPT}(6) &= \max(v_6 + \text{OPT}(p(6)), \text{OPT}(5)) = \max(1 + \text{OPT}(3), \text{OPT}(5)) \\
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\text{OPT}(4) &= \max(v_4 + \text{OPT}(p(4)), \text{OPT}(3)) = \max(7 + \text{OPT}(0), \text{OPT}(3)) \\
\text{OPT}(3) &= \max(v_3 + \text{OPT}(p(3)), \text{OPT}(2)) = \max(4 + \text{OPT}(1), \text{OPT}(2)) \\
\text{OPT}(2) &= \max(v_2 + \text{OPT}(p(2)), \text{OPT}(1)) = \max(4 + \text{OPT}(0), \text{OPT}(1)) \\
\text{OPT}(1) &= v_1 = 2 \\
\text{OPT}(0) &= 0
\end{align*}
\]
Example of Recursive Algorithm

\[ \text{OPT}(6) = \max(v_6 + \text{OPT}(p(6)), \text{OPT}(5)) = \max(1 + \text{OPT}(3), \text{OPT}(5)) \]
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\[ \text{OPT}(2) = \max(v_2 + \text{OPT}(p(2)), \text{OPT}(1)) = \max(4 + \text{OPT}(0), \text{OPT}(1)) = 4 \]
\[ \text{OPT}(1) = v_1 = 2 \]
\[ \text{OPT}(0) = 0 \]

Figure 6.2: An instance of weighted interval scheduling with the functions \( p(j) \) defined for each interval \( j \).
Example of Recursive Algorithm

<table>
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</table>

\( p(1) = 0 \)
\( p(2) = 0 \)
\( p(3) = 1 \)
\( p(4) = 0 \)
\( p(5) = 3 \)
\( p(6) = 3 \)

\( 
\begin{align*}
\text{OPT}(6) &= \max(v_6 + \text{OPT}(p(6)), \text{OPT}(5)) = \max(1 + \text{OPT}(3), \text{OPT}(5)) \\
\text{OPT}(5) &= \max(v_5 + \text{OPT}(p(5)), \text{OPT}(4)) = \max(2 + \text{OPT}(3), \text{OPT}(4)) \\
\text{OPT}(4) &= \max(v_4 + \text{OPT}(p(4)), \text{OPT}(3)) = \max(7 + \text{OPT}(0), \text{OPT}(3)) \\
\text{OPT}(3) &= \max(v_3 + \text{OPT}(p(3)), \text{OPT}(2)) = \max(4 + \text{OPT}(1), \text{OPT}(2)) = 6 \\
\text{OPT}(2) &= \max(v_2 + \text{OPT}(p(2)), \text{OPT}(1)) = \max(4 + \text{OPT}(0), \text{OPT}(1)) = 4 \\
\text{OPT}(1) &= v_1 = 2 \\
\text{OPT}(0) &= 0 \\
\end{align*} 
\)

Figure 6.2  An instance of weighted interval scheduling with the functions \( p(j) \) defined for each interval \( j \).
Example of Recursive Algorithm

OPT(6) = max(v6 + OPT(p(6)), OPT(5)) = max(1 + OPT(3), OPT(5))
OPT(5) = max(v5 + OPT(p(5)), OPT(4)) = max(2 + OPT(3), OPT(4))
OPT(4) = max(v4 + OPT(p(4)), OPT(3)) = max(7 + OPT(0), OPT(3)) = 7
OPT(3) = max(v3 + OPT(p(3)), OPT(2)) = max(4 + OPT(1), OPT(2)) = 6
OPT(2) = max(v2 + OPT(p(2)), OPT(1)) = max(4 + OPT(0), OPT(1)) = 4
OPT(1) = v1 = 2
OPT(0) = 0

Figure 6.2 An instance of weighted interval scheduling with the functions p(j) defined for each interval j.
Example of Recursive Algorithm

Index

1  $v_1 = 2$  $p(1) = 0$
2  $v_2 = 4$  $p(2) = 0$
3  $v_3 = 4$  $p(3) = 1$
4  $v_4 = 7$  $p(4) = 0$
5  $v_5 = 2$  $p(5) = 3$
6  $v_6 = 1$  $p(6) = 3$

Figure 6.2 An instance of weighted interval scheduling with the functions $p(j)$ defined for each interval $j$.

OPT(6) = $\max(v_6 + \text{OPT}(p(6)), \text{OPT}(5)) = \max(1 + \text{OPT}(3), \text{OPT}(5))$

OPT(5) = $\max(v_5 + \text{OPT}(p(5)), \text{OPT}(4)) = \max(2 + \text{OPT}(3), \text{OPT}(4)) = 8$

OPT(4) = $\max(v_4 + \text{OPT}(p(4)), \text{OPT}(3)) = \max(7 + \text{OPT}(0), \text{OPT}(3)) = 7$

OPT(3) = $\max(v_3 + \text{OPT}(p(3)), \text{OPT}(2)) = \max(4 + \text{OPT}(1), \text{OPT}(2)) = 6$

OPT(2) = $\max(v_2 + \text{OPT}(p(2)), \text{OPT}(1)) = \max(4 + \text{OPT}(0), \text{OPT}(1)) = 4$

OPT(1) = $v_1 = 2$

OPT(0) = 0

Optimal solution is job 5, job 3, and job 1.
Example of Recursive Algorithm

\[ \text{OPT}(6) = \max(v_6 + \text{OPT}(p(6)), \text{OPT}(5)) = \max(1 + \text{OPT}(3), \text{OPT}(5)) = 8 \]
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Figure 6.2  An instance of weighted interval scheduling with the functions \(p(j)\) defined for each interval \(j\).
Example of Recursive Algorithm

\[
\begin{align*}
\text{OPT}(6) &= \max(v_6 + \text{OPT}(p(6)), \text{OPT}(5)) = \max(1 + \text{OPT}(3), \text{OPT}(5)) = 8 \\
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\text{OPT}(2) &= \max(v_2 + \text{OPT}(p(2)), \text{OPT}(1)) = \max(4 + \text{OPT}(0), \text{OPT}(1)) = 4 \\
\text{OPT}(1) &= v_1 = 2 \\
\text{OPT}(0) &= 0
\end{align*}
\]

Optimal solution is \text{Poll}.

\begin{figure}
\centering
\begin{tikzpicture}
\draw [->] (0,0) -- (7,0);
\draw (0,0) -- (1,1) node [above] {$v_1 = 2$};
\draw (1,1) -- (2,2) node [above] {$v_2 = 4$};
\draw (2,2) -- (3,3) node [above] {$v_3 = 4$};
\draw (3,3) -- (4,4) node [above] {$v_4 = 7$};
\draw (4,4) -- (5,5) node [above] {$v_5 = 2$};
\draw (5,5) -- (6,6) node [above] {$v_6 = 1$};
\draw (6,6) -- (7,7) node [above] {$v_6 = 1$};
\end{tikzpicture}
\caption{An instance of weighted interval scheduling with the functions $p(j)$ defined for each interval $j$.}
\end{figure}
Example of Recursive Algorithm

\[ \text{OPT}(6) = \max(v_6 + \text{OPT}(p(6)), \text{OPT}(5)) = \max(1 + \text{OPT}(3), \text{OPT}(5)) = 8 \]
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\[ \text{OPT}(1) = v_1 = 2 \]
\[ \text{OPT}(0) = 0 \]

Optimal solution is job 5, job 3, and job 1.
Running Time of Recursive Algorithm

Compute-Opt(j)
    If j = 0 then
        Return 0
    Else
        Return max(v_j + Compute-Opt(p(j)), Compute-Opt(j - 1))
    Endif
Running Time of Recursive Algorithm

What is the running time of the algorithm?

Compute-Opt($j$)
   If $j = 0$ then
      Return 0
   Else
      Return max($v_j$ + Compute-Opt($p(j)$), Compute-Opt($j - 1$))
   Endif
What is the running time of the algorithm? Can be exponential in $n$. 

```
ComputOpt(j)
    If j = 0 then
        Return 0
    Else
        Return max($v_j$ + ComputOpt($p(j))$, ComputOpt($j - 1))
    Endif
```
Running Time of Recursive Algorithm

- What is the running time of the algorithm? Can be exponential in $n$.
- When $p(j) = j - 2$, for all $j \geq 2$: recursive calls are for $j - 1$ and $j - 2$.

```plaintext
Compute-Opt(j)
  If j = 0 then
    Return 0
  Else
    Return max($v_j$ + Compute-Opt(p(j)), Compute-Opt(j - 1))
  Endif
```

*Figure 6.4* An instance of weighted interval scheduling on which the simple Compute-Opt recursion will take exponential time. The values of all intervals in this instance are 1.

*Figure 6.3* The tree of subproblems called by Compute-Opt on the problem instance of Figure 6.2.
Memoisation

- Store OPT(j) values in a cache and reuse them rather than recompute them.
Memoisation

- Store $OPT(j)$ values in a cache and reuse them rather than recompute them.

---

M-Compute-Opt($j$)

If $j = 0$ then

Return 0

Else if $M[j]$ is not empty then

Return $M[j]$

Else

Define $M[j] = \max(v_j + M-Compute-Opt(p(j)), M-Compute-Opt(j - 1))$

Return $M[j]$

Endif
Running Time of Memoisation

\[
M\text{-Compute-Opt}(j)
\]
\[
\text{If } j = 0 \text{ then}
\]
\[
\quad \text{Return } 0
\]
\[
\text{Else if } M[j] \text{ is not empty then}
\]
\[
\quad \text{Return } M[j]
\]
\[
\text{Else}
\]
\[
\quad \text{Define } M[j] = \max(v_j + M\text{-Compute-Opt}(p(j)), M\text{-Compute-Opt}(j - 1))
\]
\[
\quad \text{Return } M[j]
\]
\[
\text{Endif}
\]

**Claim:** running time of this algorithm is \(O(n)\) (after sorting).
Running Time of Memoisation

M-Compute-Opt(j)
    If $j = 0$ then
        Return 0
    Else if $M[j]$ is not empty then
        Return $M[j]$
    Else
        Define $M[j] = \max(v_j + M-Compute-Opt(p(j)), M-Compute-Opt(j - 1))$
        Return $M[j]$
    Endif

Claim: running time of this algorithm is $O(n)$ (after sorting).
Time spent in a single call to M-Compute-Opt is $O(1)$ apart from time spent in recursive calls.
Total time spent is the order of the number of recursive calls to M-Compute-Opt.
How many such recursive calls are there in total?
Claim: running time of this algorithm is $O(n)$ (after sorting).

Time spent in a single call to $M$-Compute-$Opt$ is $O(1)$ apart from time spent in recursive calls.

Total time spent is the order of the number of recursive calls to $M$-Compute-$Opt$.

How many such recursive calls are there in total?

Use number of filled entries in $M$ as a measure of progress.

Each time $M$-Compute-$Opt$ issues two recursive calls, it fills in a new entry in $M$.

Therefore, total number of recursive calls is $O(n)$.
Computing $O$ in Addition to $\text{OPT}(n)$

Explicitly store $O_j$ in addition to $\text{OPT}(j)$. Running time becomes $O(n^2)$.

Recall: request $j$ belongs to $O_j$ if and only if $v_j + \text{OPT}(p(j)) \geq \text{OPT}(j-1)$.

Can recover $O_j$ from values of the optimal solutions in $O(j)$ time.
Computing $O$ in Addition to $\text{OPT}(n)$

- Explicitly store $O_j$ in addition to $\text{OPT}(j)$. 

Recall: request $j$ belongs to $O_j$ if and only if $v_j + \text{OPT}(p(j)) \geq \text{OPT}(j - 1)$.

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Recall: request $j$ belong to $O_j$ if and only if $v_j + OPT(p(j)) \geq OPT(j - 1)$.

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Computing $O$ in Addition to $\text{OPT}(n)$

- Explicitly store $O_j$ in addition to $\text{OPT}(j)$. Running time becomes $O(n^2)$.
- Recall: request $j$ belong to $O_j$ if and only if $v_j + \text{OPT}(p(j)) \geq \text{OPT}(j-1)$.
- Can recover $O_j$ from values of the optimal solutions in $O(j)$ time.

---

Find-Solution($j$)

If $j = 0$ then
  Output nothing
Else
  If $v_j + M[p(j)] \geq M[j-1]$ then
    Output $j$ together with the result of Find-Solution($p(j)$)
  Else
    Output the result of Find-Solution($j-1$)
  Endif
Endif
From Recursion to Iteration

- Unwind the recursion and convert it into iteration.
- Can compute values in $M$ iteratively in $O(n)$ time.
- Find-Solution works as before.

\[
\text{Iterative-Compute-Opt} \\
M[0] = 0 \\
\text{For } j = 1, 2, \ldots, n \\
M[j] = \max(v_j + M[p(j)], M[j - 1]) \\
\text{Endfor}
\]
Basic Outline of Dynamic Programming

To solve a problem, we need a collection of sub-problems that satisfy a few properties:

1. There are a polynomial number of sub-problems.
2. The solution to the problem can be computed easily from the solutions to the sub-problems.
3. There is a natural ordering of the sub-problems from “smallest” to “largest”.
4. There is an easy-to-compute recurrence that allows us to compute the solution to a sub-problem from the solutions to some smaller sub-problems.
Basic Outline of Dynamic Programming

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4. There is an easy-to-compute recurrence that allows us to compute the solution to a sub-problem from the solutions to some smaller sub-problems.

Difficulties in designing dynamic programming algorithms:

1. Which sub-problems to define?
2. How can we tie together sub-problems using a recurrence?
3. How do we order the sub-problems (to allow iterative computation of optimal solutions to sub-problems)?
Imagery from street view vehicles is accompanied by laser range data, which is aggregated and simplified by robustly fitting it in a coarse mesh that models the dominant scene surfaces.
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Fitting Lines
Fitting Lines
Fitting Lines
Fitting Lines
Fitting Lines
Fitting Lines

\[ y = ax + b \]

Slope = \( a \)

(\( x_1, y_1 \) )

\( b \)
Fitting Lines

\[ y = ax + b \]

Slope = \( a \)

\[ |y_1 - ax_1 - b| \]

\( (x_1, y_1) \)

\( b \)
Least Squares Problem

- Given scientific or statistical data plotted on two axes.
- Find the “best” line that “passes” through these points.

**Least Squares Regression**

**INSTANCE:**

Set $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of $n$ points.

**SOLUTION:**

Line $L: y = ax + b$ that minimises $\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2$.

How many unknown parameters must we find values for?

Two: $a$ and $b$.

Solution is achieved by

\[
\begin{align*}
a &= \frac{\sum_{i=1}^{n} x_i y_i - (\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \\
b &= \frac{\sum_{i=1}^{n} y_i - ax_i}{n}
\end{align*}
\]
Least Squares Problem

- Given scientific or statistical data plotted on two axes.
- Find the “best” line that “passes” through these points.

**Least Squares Regression**

**INSTANCE:** Set $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of $n$ points.

**SOLUTION:** Line $L : y = ax + b$ that minimises

$$\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$
**Least Squares Problem**

- Given scientific or statistical data plotted on two axes.
- Find the “best” line that “passes” through these points.

**Least Squares Regression**

**INSTANCE:** Set \( P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \) of \( n \) points.

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**Least Squares Problem**

- Given scientific or statistical data plotted on two axes.
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**Least Squares Regression**

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**SOLUTION:** Line \( L : y = ax + b \) that minimises

\[
\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.
\]

- How many unknown parameters must we find values for? Two: \( a \) and \( b \).
Least Squares Problem

• Given scientific or statistical data plotted on two axes.
• Find the “best” line that “passes” through these points.

**Least Squares Regression**

**INSTANCE:** Set $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of $n$ points.

**SOLUTION:** Line $L : y = ax + b$ that minimises

$$\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$ 

• How many unknown parameters must we find values for? Two: $a$ and $b$.
• Solution is achieved by

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2} \text{ and } b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$
Segmented Least Squares
Segmented Least Squares

*Figure 6.7* A set of points that lie approximately on two lines.  
*Figure 6.8* A set of points that lie approximately on three lines.

- Want to fit multiple lines through $P$.
- Each line must fit contiguous set of $x$-coordinates.
- Lines must minimise total error.
Example of Segmented Least Squares

Input contains a set of two-dimensional points.
Consider the sorted $x$-coordinates of the points in the input.
Example of Segmented Least Squares

Divide the points into segments; each *segment* contains consecutive points in the sorted order by $x$-coordinate. Here we are defining a meaning for “segment” that is specific to this problem.
Example of Segmented Least Squares

Fit the best line for each segment.
Example of Segmented Least Squares

Illegal solution: black point is not in any segment.
Example of Segmented Least Squares

Illegal solution: leftmost purple point has $x$-coordinate between last two points in green segment.
Formulating Segmented Least Squares Problem

**Segmented Least Squares**

**INSTANCE:** Set $P = \{p_i = (x_i, y_i), 1 \leq i \leq n\}$ of $n$ points, $x_1 < x_2 < \cdots < x_n$.

**SOLUTION:**

How many unknown parameters must we find? 2$k$, and we must find $k$ too!
**Segmented Least Squares**

**INSTANCE:** Set \( P = \{ p_i = (x_i, y_i), 1 \leq i \leq n \} \) of \( n \) points, \( x_1 < x_2 < \cdots < x_n \).

**SOLUTION:**
1. An integer \( k \),
2. a partition of \( P \) into \( k \) segments \( \{ P_1, P_2, \ldots, P_k \} \), and
3. for each segment \( P_j \), the best-fit line \( L_j : y = a_j x + b_j, 1 \leq j \leq k \) that minimise the total error

\[
\sum_{j=1}^{k} \text{Error}(L_j, P_j)
\]
Segmented Least Squares

**INSTANCE:** Set $P = \{ p_i = (x_i, y_i), 1 \leq i \leq n \}$ of $n$ points, $x_1 < x_2 < \cdots < x_n$ and a parameter $C > 0$.

**SOLUTION:**
1. An integer $k$,
2. a partition of $P$ into $k$ segments $\{ P_1, P_2, \ldots, P_k \}$, and
3. for each segment $P_j$, the best-fit line $L_j : y = a_j x + b_j$, $1 \leq j \leq k$

that minimise the total error

$$ \sum_{j=1}^{k} \text{Error}(L_j, P_j) + Ck $$
Formulating Segmented Least Squares Problem

**Segmented Least Squares**

**INSTANCE:** Set \( P = \{ p_i = (x_i, y_i), 1 \leq i \leq n \} \) of \( n \) points, \( x_1 < x_2 < \cdots < x_n \) and a parameter \( C > 0 \).

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Formulating Segmented Least Squares Problem

**Segmented Least Squares**

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1. An integer $k$,
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3. for each segment $P_j$, the best-fit line $L_j : y = a_j x + b_j, 1 \leq j \leq k$ that minimise the total error

$$\sum_{j=1}^{k} \text{Error}(L_j, P_j) + Ck$$

- How many unknown parameters must we find? $2k$, and we must find $k$ too!
- Assume points in $P$ are sorted in increasing order of $x$-coordinate.
Formulating the Recursion: Getting Intuition

Observation: Where does the last segment in the optimal solution end?

Let $OPT(i)$ be the optimal total error for the points $\{p_1, p_2, \ldots, p_i\}$. We want to compute $OPT(n)$. Let $e_{i,j}$ denote the minimum error of a (single) line that fits $\{p_i, p_{i+1}, \ldots, p_j\}$. If the last segment in the optimal partition is $\{p_i, p_{i+1}, \ldots, p_n\}$, then...
Formulating the Recursion: Getting Intuition

Observation: Where does the last segment in the optimal solution end? \( p_n \), and this segment starts at some point \( p_i \). We don’t know \( i \) yet!
Formulating the Recursion: Getting Intuition

Observation: Where does the last segment in the optimal solution end? $p_n$, and this segment starts at some point $p_i$. We don’t know $i$ yet!

If the last segment in the optimal partition is $\{p_i, p_{i+1}, \ldots, p_n\}$, then

optimal total error for $n$ points = Error of the best line fitting $\{p_i, p_{i+1}, \ldots, p_n\} + C +$ optimal total error for the first $i - 1$ points.
Formulating the Recursion: Getting Intuition

- Observation: Where does the last segment in the optimal solution end? $p_n$, and this segment starts at some point $p_i$. We don’t know $i$ yet!
- Let $OPT(i)$ be the optimal total error for the points $\{p_1, p_2, \ldots, p_i\}$.
- We want to compute $OPT(n)$.

- If the last segment in the optimal partition is $\{p_i, p_{i+1}, \ldots, p_n\}$, then optimal total error for $n$ points = Error of the best line fitting $\{p_i, p_{i+1}, \ldots, p_n\} + C +$ optimal total error for the first $i - 1$ points.
Formulating the Recursion: Getting Intuition

- Observation: Where does the last segment in the optimal solution end? \( p_n \), and this segment starts at some point \( p_i \). We don’t know \( i \) yet!

- Let \( OPT(i) \) be the optimal total error for the points \( \{p_1, p_2, \ldots, p_i\} \).
- We want to compute \( OPT(n) \).
- Let \( e_{i,j} \) denote the minimum error of a (single) line that fits \( \{p_i, p_{i+1}, \ldots, p_j\} \).
- If the last segment in the optimal partition is \( \{p_i, p_{i+1}, \ldots, p_n\} \), then optimal total error for \( n \) points = Error of the best line fitting \( \{p_i, p_{i+1}, \ldots, p_n\} \) + \( C \) + optimal total error for the first \( i - 1 \) points.
Formulating the Recursion: Getting Intuition

- Observation: Where does the last segment in the optimal solution end? $p_n$, and this segment starts at some point $p_i$. **We don’t know $i$ yet!**
- Let $OPT(i)$ be the optimal total error for the points $\{p_1, p_2, \ldots, p_i\}$.
- We want to compute $OPT(n)$.
- Let $e_{i,j}$ denote the minimum error of a (single) line that fits $\{p_i, p_2, \ldots, p_j\}$.
- If the last segment in the optimal partition is $\{p_i, p_{i+1}, \ldots, p_n\}$, then

$$OPT(n) = e_{i,n} + C + OPT(i - 1)$$
In general, we want to solve sub-problem on the points \( \{p_1, p_2, \ldots, p_j\} \), i.e., we want to compute \( \text{OPT}(j) \), where \( j \) lies between 1 and \( n \).
In general, we want to solve sub-problem on the points \( \{p_1, p_2, \ldots, p_j\} \), i.e., we want to compute \( \text{OPT}(j) \), where \( j \) lies between 1 and \( n \).

If the last segment in the optimal partition is \( \{p_i, p_{i+1}, \ldots, p_j\} \), then optimal total error for first \( j \) points = Error of the best line fitting \( \{p_i, p_{i+1}, \ldots, p_j\} \) + \( C \) + optimal total error for the first \( i - 1 \) points.
In general, we want to solve sub-problem on the points \( \{p_1, p_2, \ldots, p_j\} \), i.e., we want to compute \( \text{OPT}(j) \), where \( j \) lies between 1 and \( n \).

If the last segment in the optimal partition is \( \{p_i, p_{i+1}, \ldots, p_j\} \), then

\[
\text{OPT}(j) = e_{i,j} + C + \text{OPT}(i - 1)
\]
In general, we want to solve sub-problem on the points \( \{p_1, p_2, \ldots, p_j\} \), i.e., we want to compute \( \text{OPT}(j) \), where \( j \) lies between 1 and \( n \).

If the last segment in the optimal partition is \( \{p_i, p_{i+1}, \ldots, p_j\} \), then

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We don’t know \( i \)!
In general, we want to solve sub-problem on the points \( \{p_1, p_2, \ldots, p_j\} \), i.e., we want to compute \( \text{OPT}(j) \), where \( j \) lies between 1 and \( n \).

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We don’t know \( i \)! But \( i \) can take only \( j \) distinct values: 1, 2, \ldots, \( j - 1, j \).

Therefore,

\[
\text{OPT}(j) = \min_{1 \leq i \leq j} (e_{i,j} + C + \text{OPT}(i - 1))
\]
In general, we want to solve sub-problem on the points \( \{ p_1, p_2, \ldots p_j \} \), i.e., we want to compute \( \text{OPT}(j) \), where \( j \) lies between 1 and \( n \).

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\[
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\]

Segment \( \{ p_i, p_{i+1}, \ldots p_j \} \) is part of the optimal solution for this sub-problem if and only if the minimum value of \( \text{OPT}(j) \) is obtained using index \( i \).
Dynamic Programming Algorithm

\[
\text{OPT}(j) = \min_{1 \leq i \leq j} (e_{i,j} + C + \text{OPT}(i - 1))
\]

Segmented-Least-Squares(n)

Array \( M[0 \ldots n] \)
Set \( M[0] = 0 \)
For all pairs \( i \leq j \)
   Compute the least squares error \( e_{i,j} \) for the segment \( p_i, \ldots, p_j \)
Endfor
For \( j = 1, 2, \ldots, n \)
   Use the recurrence (6.7) to compute \( M[j] \)
Endfor
Return \( M[n] \)
Dynamic Programming Algorithm

\[
\text{OPT}(j) = \min_{1 \leq i \leq j} \left( e_{i,j} + C + \text{OPT}(i - 1) \right)
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Segmented-Least-Squares(n)

Array \( M[0 \ldots n] \)
Set \( M[0] = 0 \)
For all pairs \( i \leq j \)
\begin{itemize}
  \item Compute the least squares error \( e_{i,j} \) for the segment \( p_i, \ldots, p_j \)
\end{itemize} 
Endfor
For \( j = 1, 2, \ldots, n \)
\begin{itemize}
  \item Use the recurrence (6.7) to compute \( M[j] \)
\end{itemize}
Endfor
Return \( M[n] \)

- We can find the segments in the optimal solution by backtracking.
Running Time

$$OPT(j) = \min_{1 \leq i \leq j} \left( e_{i,j} + C + OPT(i - 1) \right)$$

---

Segmented-Least-Squares(n)

Array $M[0...n]$

Set $M[0] = 0$

For all pairs $i \leq j$

- Compute the least squares error $e_{i,j}$ for the segment $p_i, \ldots, p_j$

Endfor

For $j = 1, 2, \ldots, n$

- Use the recurrence (6.7) to compute $M[j]$

Endfor

Return $M[n]$

---

Let $T(n)$ be the running time of this algorithm.

$$T(n) = \sum_{1 \leq j \leq n} \sum_{1 \leq i \leq j} O(j - i) =$$
Running Time

\[ \text{OPT}(j) = \min_{1 \leq i \leq j} (e_{i,j} + C + \text{OPT}(i - 1)) \]

Segmented-Least-Squares(n)

Array \( M[0\ldots n] \)
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For all pairs \( i \leq j \)
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For \( j = 1, 2, \ldots, n \)
    Use the recurrence (6.7) to compute \( M[j] \)
Endfor
Return \( M[n] \)

Let \( T(n) \) be the running time of this algorithm.

\[ T(n) = \sum_{1 \leq j \leq n} \sum_{1 \leq i \leq j} O(j - i) = ? \]
Running Time

\[ \text{OPT}(j) = \min_{1 \leq i \leq j} \left( e_{i,j} + C + \text{OPT}(i-1) \right) \]

---

Segmented-Least-Squares(n)

Array \( M[0...n] \)

Set \( M[0] = 0 \)

For all pairs \( i \leq j \)

- Compute the least squares error \( e_{i,j} \) for the segment \( p_i, \ldots, p_j \)

Endfor

For \( j = 1, 2, \ldots, n \)

- Use the recurrence (6.7) to compute \( M[j] \)

Endfor

Return \( M[n] \)

---

- Let \( T(n) \) be the running time of this algorithm.
- Running time is \( O(n^3) \); can be improved to \( O(n^2) \).

\[ T(n) = \sum_{1 \leq j \leq n} \sum_{1 \leq i \leq j} O(j - i) = O(n^3) \]
RNA Molecules

- RNA is a basic biological molecule. It is single stranded.
- RNA molecules fold into complex “secondary structures.”
- Secondary structure often governs the behaviour of an RNA molecule.
- Various rules govern secondary structure formation:

  1. Pairs of bases match up; each base matches with \( \leq 1 \) other base.
  2. Adenine always matches with Uracil.
  3. Cytosine always matches with Guanine.
  4. There are no kinks in the folded molecule.
  5. Structures are “knot-free.”

Problem: given an RNA molecule, predict its secondary structure.

Hypothesis: In the cell, RNA molecules form the secondary structure with the lowest total free energy.
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Figure 6.13 An RNA secondary structure. Thick lines connect adjacent elements of the sequence; thin lines indicate pairs of elements that are matched.
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*Figure 6.13* An RNA secondary structure. Thick lines connect adjacent elements of the sequence; thin lines indicate pairs of elements that are matched.
Formulating the Problem

- An RNA molecule is a string $B = b_1 b_2 \ldots b_n$; each $b_i \in \{A, C, G, U\}$.
- A secondary structure on $B$ is a set of pairs $S = \{(i, j)\}$, where $1 \leq i, j \leq n$ and

![Figure 6.14](image_url)

Figure 6.14 Two views of an RNA secondary structure. In the second view, (b), the string has been "stretched" lengthwise, and edges connecting matched pairs appear as noncrossing "bubbles" over the string.
Formulating the Problem

An **RNA molecule** is a string $B = b_1 b_2 \ldots b_n$; each $b_i \in \{A, C, G, U\}$.

A **secondary structure on $B$** is a set of pairs $S = \{(i, j)\}$, where $1 \leq i, j \leq n$ and

1. *(No kinks.)* If $(i, j) \in S$, then $i < j - 4$.
2. *(Watson-Crick)* The elements in each pair in $S$ consist of either $\{A, U\}$ or $\{C, G\}$ (in either order).
3. $S$ is a **matching**: no index appears in more than one pair.
4. *(No knots)* If $(i, j)$ and $(k, l)$ are two pairs in $S$, then we cannot have $i < k < j < l$.

The **energy** of a secondary structure $\propto$ the number of base pairs in it.

Problem: Compute the largest secondary structure, i.e., with the largest number of base pairs.
Illegal Secondary Structures

A C A U G G C C A U G U

Watson-Crick

A C A U G G C C A U G U

Kink Matching

Knot

A C A U G G C C A U G U

T. M. Murali  October 19, 21, 26, 28, 2021  Dynamic Programming
Legal Secondary Structures

A C A U G G C C A U G U
Dynamic Programming Approach

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$.  

**Poll**
Dynamic Programming Approach

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$. $OPT(j) = 0$, if $j \leq 5$. 

In the optimal secondary structure on $b_1 b_2 \ldots b_j$ if $j$ is not a member of any pair, use $OPT(j-1)$. 

If $j$ pairs with some $t < j - 4$, the knot condition yields two independent sub-problems: $OPT(t-1)$ and ???
Dynamic Programming Approach

- **OPT(j)** is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$. OPT($j$) = 0, if $j \leq 5$.
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Dynamic Programming Approach

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$. $OPT(j) = 0$, if $j \leq 5$.
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**Dynamic Programming Approach**

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$. $OPT(j) = 0$, if $j \leq 5$.

- In the optimal secondary structure on $b_1 b_2 \ldots b_j$
  1. if $j$ is not a member of any pair, use $OPT(j - 1)$.
  2. if $j$ pairs with some $t < j - 4$,

---

**Figure 6.15** Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Dynamic Programming Approach

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$. $OPT(j) = 0$, if $j \leq 5$.

- In the optimal secondary structure on $b_1 b_2 \ldots b_j$
  1. if $j$ is not a member of any pair, use $OPT(j - 1)$.
  2. if $j$ pairs with some $t < j - 4$, knot condition yields two independent sub-problems!

\[ \text{Figure 6.15 Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.} \]
**Dynamic Programming Approach**

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1 b_2 \ldots b_j$. $OPT(j) = 0$, if $j \leq 5$.

- In the optimal secondary structure on $b_1 b_2 \ldots b_j$
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  2. if $j$ pairs with some $t < j - 4$, knot condition yields two independent sub-problems! $OPT(t - 1)$ and ??

![Diagram](attachment:image.png)

*Figure 6.15* Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Dynamic Programming Approach

- $OPT(j)$ is the maximum number of base pairs in a secondary structure for $b_1b_2\ldots b_j$. $OPT(j) = 0$, if $j \leq 5$.

- In the optimal secondary structure on $b_1b_2\ldots b_j$
  1. if $j$ is not a member of any pair, use $OPT(j - 1)$.
  2. if $j$ pairs with some $t < j - 4$, knot condition yields two independent sub-problems! $OPT(t - 1)$ and $OPT(t - 1)$.

- Insight: need sub-problems indexed both by start and by end.

**Figure 6.15** Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Correct Dynamic Programming Approach

OPT\((i, j)\) is the maximum number of base pairs in a secondary structure for \(b_i b_{i+1} \ldots b_j\).

**Figure 6.15** Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Correct Dynamic Programming Approach

**OPT**\((i, j)\) is the maximum number of base pairs in a secondary structure for \(b_i b_{i+1} \ldots b_j\). **OPT**\((i, j) = 0\), if \(i \geq j - 4\).

---

**Figure 6.15** Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Correct Dynamic Programming Approach

OPT(i, j) is the maximum number of base pairs in a secondary structure for $b_i b_{i+1} \ldots b_j$. OPT(i, j) = 0, if $i \geq j - 4$.

In the optimal secondary structure on $b_i b_{i+1} \ldots b_j$

$$\text{OPT}(i, j) = \max \left( \right)$$
Correct Dynamic Programming Approach

OPT(i, j) is the maximum number of base pairs in a secondary structure for $b_i b_{i+1} \ldots b_j$. OPT(i, j) = 0, if $i \geq j - 4$.

In the optimal secondary structure on $b_i b_{i+1} \ldots b_j$

1. if $j$ is not a member of any pair, compute OPT(i, j − 1).

OPT(i, j) = \max \left( OPT(i, j − 1), \right)
Correct Dynamic Programming Approach

OPT(i, j) is the maximum number of base pairs in a secondary structure for $b_i b_{i+1} \ldots b_j$. OPT(i, j) = 0, if $i \geq j - 4$.

In the optimal secondary structure on $b_i b_{i+1} \ldots b_j$

1. if $j$ is not a member of any pair, compute OPT(i, j − 1).
2. if $j$ pairs with some $t < j - 4$, compute OPT(i, t) and OPT(t + 1, j − 1).

$$OPT(i, j) = \max \left( OPT(i, j - 1), \right)$$
Correct Dynamic Programming Approach

OPT(i, j) is the maximum number of base pairs in a secondary structure for $b_i b_{i+1} \ldots b_j$. OPT(i, j) = 0, if $i \geq j - 4$.

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Correct Dynamic Programming Approach

- \( OPT(i, j) \) is the maximum number of base pairs in a secondary structure for \( b_i b_{i+1} \ldots b_j \). \( OPT(i, j) = 0 \), if \( i \geq j - 4 \).

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  2. If \( j \) pairs with some \( t < j - 4 \), compute \( OPT(i, t - 1) \) and \( OPT(t + 1, j - 1) \).

- Since \( t \) can range from \( i \) to \( j - 5 \),

\[
OPT(i, j) = \max \left( OPT(i, j - 1), \max_t \left( 1 + OPT(i, t - 1) + OPT(t + 1, j - 1) \right) \right)
\]
**Correct Dynamic Programming Approach**

- $OPT(i, j)$ is the maximum number of base pairs in a secondary structure for $b_ib_{i+1} \ldots b_j$. $OPT(i, j) = 0$, if $i \geq j - 4$.
- In the optimal secondary structure on $b_ib_{i+1} \ldots b_j$,
  1. if $j$ is not a member of any pair, compute $OPT(i, j - 1)$.
  2. if $j$ pairs with some $t < j - 4$, compute $OPT(i, t - 1)$ and $OPT(t + 1, j - 1)$.
- Since $t$ can range from $i$ to $j - 5$,
  $$OPT(i, j) = \max \left( OPT(i, j - 1), \max_t (1 + OPT(i, t - 1) + OPT(t + 1, j - 1)) \right)$$
- In the “inner” maximisation, $t$ runs over all indices between $i$ and $j - 5$ that are allowed to pair with $j$.

---

**Figure 6.15** Schematic views of the dynamic programming recurrence using (a) one variable, and (b) two variables.
Example of Dynamic Programming Algorithm
### Dynamic Programming Algorithm

\[
\text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \max_t \left( 1 + \text{OPT}(i, t - 1) + \text{OPT}(t + 1, j - 1) \right) \right)
\]

- There are \( n \) sub-problems.
**Dynamic Programming Algorithm**

\[
\text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \max_t (1 + \text{OPT}(i, t - 1) + \text{OPT}(t + 1, j - 1)) \right)
\]

- There are \( O(n^2) \) sub-problems.
- How do we order them from “smallest” to “largest”? [proceeding with further explanation]
Dynamic Programming Algorithm

\[ \text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \max_t (1 + \text{OPT}(i, t - 1) + \text{OPT}(t + 1, j - 1)) \right) \]

- There are \( O(n^2) \) sub-problems.
- How do we order them from “smallest” to “largest”?
- Computing \( \text{OPT}(i, j) \) involves sub-problems of the form \( \text{OPT}(l, j - 1) \).
- We should compute \( \text{OPT}() \) values in increasing order of the second argument.
Dynamic Programming Algorithm

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OPT(i, j) = \max \left( \OPT(i, j - 1), \max_t (1 + \OPT(i, t - 1) + \OPT(t + 1, j - 1)) \right)
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- There are \(O(n^2)\) sub-problems.
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- Computing \(\OPT(i, j)\) involves sub-problems of the form \(\OPT(l, j - 1)\).
- We should compute \(\OPT()\) values in increasing order of the second argument.

Initialise \(\OPT(i, j) = 0\) for every \(l, j\) such that \(i \geq j - 4\)

for \(j = 1, 2, \ldots, n - 1, n\)

for \(i = 1, 2, \ldots, j - 6, j - 5\)

Compute \(\OPT(i, j)\) using the recurrence above.

- How long does it take to compute \(\OPT(i, j)\)\
- What is the running time of the algorithm? \(O(n^3)\).
**Dynamic Programming Algorithm**

\[
\text{OPT}(i, j) = \max \left( \text{OPT}(i, j - 1), \max_{t} \left( 1 + \text{OPT}(i, t - 1) + \text{OPT}(t + 1, j - 1) \right) \right)
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- There are \(O(n^2)\) sub-problems.
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Initialise \(\text{OPT}(i, j) = 0\) for every \(l, j\) such that \(i \geq j - 4\)
for \(j = 1, 2, \ldots, n - 1, n\)
for \(i = 1, 2, \ldots, j - 6, j - 5\)
Compute \(\text{OPT}(i, j)\) using the recurrence above.

- How long does it take to compute \(\text{OPT}(i, j)\)? \(O(j - i)\)
- What is the running time of the algorithm? \(O(n^3)\).
Motivation

- Computational finance:
  - Each node is a financial agent.
  - The cost $c_{uv}$ of an edge $(u, v)$ is the cost of a transaction in which we buy from agent $u$ and sell to agent $v$.
  - Negative cost corresponds to a profit.

- Internet routing protocols
  - Dijkstra’s algorithm needs knowledge of the entire network.
  - Routers only know which other routers they are connected to.
  - Algorithm for shortest paths with negative edges is decentralised.
  - We will not study this algorithm in the class. See Chapter 6.9.
Problem Statement

- **Input:** a directed graph $G = (V, E)$ with a cost function $c : E \rightarrow \mathbb{R}$, i.e., $c_{uv}$ is the cost of the edge $(u, v) \in E$.

- A *negative cycle* is a directed cycle whose edges have a total cost that is negative.

- **Two related problems:**
  1. If $G$ has no negative cycles, find the *shortest s-t path*: a path from source $s$ to destination $t$ with minimum total cost.
  2. Does $G$ have a *negative cycle*? Application is to arbitrage opportunities.
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![Diagram](image)

**Figure 6.20** In this graph, one can find \( s-t \) paths of arbitrarily negative cost (by going around the cycle \( C \) many times).
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![Graph Diagram](image)

**Figure 6.20** In this graph, one can find $s$-$t$ paths of arbitrarily negative cost (by going around the cycle $C$ many times).
Run Dijkstra’s algorithm.

Figure 6.21 (a) With negative edge costs, Dijkstra’s Algorithm can give the wrong answer for the Shortest-Path Problem. (b) Adding 3 to the cost of each edge will make all edges nonnegative, but it will change the identity of the shortest s-t path.
Run Dijkstra’s algorithm. Computes incorrect answers because it is greedy.

**Figure 6.21** (a) With negative edge costs, Dijkstra’s Algorithm can give the wrong answer for the Shortest-Path Problem. (b) Adding 3 to the cost of each edge will make all edges nonnegative, but it will change the identity of the shortest s-t path.
Approaches for Shortest Path Algorithm

1. Run Dijkstra’s algorithm. Computes incorrect answers because it is greedy.

2. Add some large constant to each edge.

Figure 6.21 (a) With negative edge costs, Dijkstra’s Algorithm can give the wrong answer for the Shortest-Path Problem. (b) Adding 3 to the cost of each edge will make all edges nonnegative, but it will change the identity of the shortest s-t path.
Approaches for Shortest Path Algorithm

1. Run Dijkstra’s algorithm. Computes incorrect answers because it is greedy.

2. Add some large constant to each edge. Computes incorrect answers because the minimum cost path changes.

Figure 6.21 (a) With negative edge costs, Dijkstra’s Algorithm can give the wrong answer for the Shortest-Path Problem. (b) Adding 3 to the cost of each edge will make all edges nonnegative, but it will change the identity of the shortest s-t path.
Dynamic Programming Approach

- Assume $G$ has no negative cycles.
- Claim: There is a shortest path from $s$ to $t$ that is simple (does not repeat a node).
Dynamic Programming Approach

- Assume $G$ has no negative cycles.
- Claim: There is a shortest path from $s$ to $t$ that is *simple* (does not repeat a node) and hence has at most $n - 1$ edges.
Dynamic Programming Approach

- Assume $G$ has no negative cycles.
- Claim: There is a shortest path from $s$ to $t$ that is *simple* (does not repeat a node) and hence has at most $n - 1$ edges.
- How do we define sub-problems?
**Dynamic Programming Approach**

- Assume $G$ has no negative cycles.
- Claim: There is a shortest path from $s$ to $t$ that is *simple* (does not repeat a node) and hence has at most $n - 1$ edges.

**How do we define sub-problems?**

- Shortest $s$-$t$ path has $\leq n - 1$ edges: how we can reach $t$ using $i$ edges, for different values of $i$?
- We do not know which nodes will be in shortest $s$-$t$ path: how we can reach $t$ from each node in $V$?
Dynamic Programming Approach

- Assume $G$ has no negative cycles.
- Claim: There is a shortest path from $s$ to $t$ that is simple (does not repeat a node) and hence has at most $n - 1$ edges.

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- We do not know which nodes will be in shortest $s$-$t$ path: how we can reach $t$ from each node in $V$?

Sub-problems defined by varying the number of edges in the shortest path and by varying the starting node in the shortest path.
Dynamic Programming Recursion

- $OPT(i, v)$: minimum cost of a $v$-$t$ path that uses at most $i$ edges.
- $t$ is not explicitly mentioned in the sub-problems.
- Goal is to compute $OPT(n - 1, s)$. 

Let $P$ be the optimal path whose cost is $OPT(i, v)$.

1. If $P$ actually uses $i - 1$ edges, then $OPT(i, v) = OPT(i - 1, v)$.
2. If first node on $P$ is $w$, then $OPT(i, v) = c_{vw} + OPT(i - 1, w)$.

$OPT(i, v) = \min(\text{OPT}(i - 1, v), \\min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)))$.
**Dynamic Programming Recursion**

- **$OPT(i, v)$**: minimum cost of a $v$-$t$ path that uses at most $i$ edges.
- $t$ is not explicitly mentioned in the sub-problems.
- Goal is to compute $OPT(n - 1, s)$.

Let $P$ be the optimal path whose cost is $OPT(i, v)$.

![Diagram](image)

**Figure 6.22** The minimum-cost path $P$ from $v$ to $t$ using at most $i$ edges.
Dynamic Programming Recursion

- **OPT(i, v)**: minimum cost of a v-t path that uses at most i edges.
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![Diagram of minimum-cost path P from v to t using at most i edges.](image)

**Figure 6.22** The minimum-cost path P from v to t using at most i edges.

- Let P be the optimal path whose cost is OPT(i, v).
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Example of Dynamic Programming Recursion

\[ \text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right) \]
Example of Dynamic Programming Recursion

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Alternate Dynamic Programming Formulation

- $OPT_{\pi}(i, v)$: minimum cost of a $v$-$t$ path that uses exactly $i$ edges. Goal is to compute
Alternate Dynamic Programming Formulation

- \( OPT_{\equiv}(i, v) \): minimum cost of a \( v-t \) path that uses exactly \( i \) edges. Goal is to compute

\[
\min_{i=1}^{n-1} OPT_{\equiv}(i, s).
\]
Alternate Dynamic Programming Formulation

- \( OPT_{\Xi}(i, v) \): minimum cost of a \( v-t \) path that uses exactly \( i \) edges. Goal is to compute

\[
\min_{i=1}^{n-1} OPT_{\Xi}(i, s).
\]

- Let \( P \) be the optimal path whose cost is \( OPT_{\Xi}(i, v) \).
Alternate Dynamic Programming Formulation

- $OPT_{\equiv}(i, v)$: minimum cost of a $v$-$t$ path that uses exactly $i$ edges. Goal is to compute

$$\min_{i=1}^{n-1} OPT_{\equiv}(i, s).$$

- Let $P$ be the optimal path whose cost is $OPT_{\equiv}(i, v)$.
  - If first node on $P$ is $w$, then $OPT_{\equiv}(i, v) = c_{vw} + OPT_{\equiv}(i - 1, w)$. 
Alternate Dynamic Programming Formulation

- \(OPT_{\leq}(i, v)\): minimum cost of a \(v\)-\(t\) path that uses exactly \(i\) edges. Goal is to compute

\[
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\]

- Let \(P\) be the optimal path whose cost is \(OPT_{\leq}(i, v)\).
  - If first node on \(P\) is \(w\), then \(OPT_{\leq}(i, v) = c_{vw} + OPT_{\leq}(i-1, w)\).

\[
OPT_{\leq}(i, v) = \min_{w \in V} \left( c_{vw} + OPT_{\leq}(i-1, w) \right)
\]
Alternate Dynamic Programming Formulation

- **$OPT_{=}(i, v)$**: minimum cost of a $v$-$t$ path that uses exactly $i$ edges. Goal is to compute

$$\min_{i=1}^{n-1} OPT_{=}(i, s).$$

- Let $P$ be the optimal path whose cost is $OPT_{=}(i, v)$.
  - If first node on $P$ is $w$, then $OPT_{=}(i, v) = c_{vw} + OPT_{=}(i - 1, w)$.

$$OPT_{=}(i, v) = \min_{w \in V} (c_{vw} + OPT_{=}(i - 1, w))$$

- Compare the two desired solutions:

$$\min_{i=1}^{n-1} OPT_{=}(i, s) = \min_{i=1}^{n-1} \left( \min_{w \in V} (c_{sw} + OPT_{=}(i - 1, w)) \right)$$

$$OPT(n - 1, s) = \min \left( OPT(n - 2, s), \min_{w \in V} (c_{sw} + OPT(n - 2, w)) \right)$$
Bellman-Ford Algorithm

\[ \text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} \left( c_{vw} + \text{OPT}(i - 1, w) \right) \right) \]

---

Shortest-Path(G, s, t)

1. \( n = \text{number of nodes in } G \)
2. \( \text{Array } M[0 \ldots n-1, V] \)
3. Define \( M[0, t] = 0 \) and \( M[0, v] = \infty \) for all other \( v \in V \)
4. For \( i = 1, \ldots, n-1 \)
   1. For \( v \in V \) in any order
      1. Compute \( M[i, v] \) using the recurrence (6.23)
   2. Endfor
5. Endfor
6. Return \( M[n - 1, s] \)
**Bellman-Ford Algorithm**

\[ \text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right) \]

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  - Endfor
- Endfor
- Return \( M[n - 1, s] \)

- Space used is \( \text{Poll} \). Running time is \( \text{Poll} \).
### Bellman-Ford Algorithm

\[
\text{OPT}(i, v) = \min \left( \text{OPT}(i - 1, v), \min_{w \in V} (c_{vw} + \text{OPT}(i - 1, w)) \right)
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**Shortest-Path(G, s, t)**

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For \[ i = 1, \ldots, n - 1 \]

For \[ v \in V \text{ in any order} \]

Compute \[ M[i, v] \text{ using the recurrence (6.23)} \]

Endfor

Endfor

Return \[ M[n - 1, s] \]

---

- Space used is \( O(n^2) \). Running time is \( O(n^3) \).
- If shortest path uses \( k \) edges, we can recover it in \( O(kn) \) time by tracing back through smaller sub-problems.
An Improved Bound on the Running Time

- Suppose $G$ has $n$ nodes and $m \ll \binom{n}{2}$ edges. Can we demonstrate a better upper bound on the running time?
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$$M[i, v] = \min \left( M[i - 1, v], \min_{w \in V} (c_{vw} + M[i - 1, w]) \right)$$
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\[
M[i, v] = \min \left( M[i - 1, v], \min_{w \in N_v} (c_{vw} + M[i - 1, w]) \right)
\]

- $w$ only needs to range over outgoing neighbours $N_v$ of $v$.
- If $n_v = |N_v|$ is the number of outgoing neighbours of $v$, then in each round, we spend time equal to

\[
\sum_{v \in V} n_v = \ldots
\]
An Improved Bound on the Running Time

Suppose $G$ has $n$ nodes and $m \ll \binom{n}{2}$ edges. Can we demonstrate a better upper bound on the running time?

$$M[i, \nu] = \min \left( M[i - 1, \nu], \min_{w \in \mathcal{N}_\nu} (c_{\nu w} + M[i - 1, w]) \right)$$

- $w$ only needs to range over outgoing neighbours $\mathcal{N}_\nu$ of $\nu$.
- If $n_\nu = |\mathcal{N}_\nu|$ is the number of outgoing neighbours of $\nu$, then in each round, we spend time equal to

$$\sum_{\nu \in V} n_\nu = m.$$

- The total running time is $O(mn)$. 
Improving the Memory Requirements

\[ M[i, \nu] = \min \left( M[i - 1, \nu], \min_{w \in N_\nu} (c_{\nu w} + M[i - 1, w]) \right) \]

- The algorithm uses \( O(n^2) \) space to store the array \( M \).
Improving the Memory Requirements

\[ M[i, v] = \min \left( M[i - 1, v], \min_{w \in \mathcal{N}_v} (c_{vw} + M[i - 1, w]) \right) \]

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- Observe that \( M[i, v] \) depends only on \( M[i - 1, *] \) and no other indices.
Improving the Memory Requirements

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- Modified algorithm:
  1. Maintain two arrays \( M \) and \( M' \) indexed over \( V \).
  2. At the beginning of each iteration, copy \( M \) into \( M' \).
  3. To update \( M \), use

\[ M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right) \]
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\[ M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right) \]

- Claim: at the beginning of iteration \( i \), \( M \) stores values of \( \text{OPT}(i - 1, v) \) for all nodes \( v \in V \).
- Space used is \( O(n) \).
Computing the Shortest Path: Algorithm

\[ M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right) \]

- How can we recover the shortest path that has cost \( M[v] \)?
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- For each node \( v \), compute and update \( f(v) \), the first node after \( v \) in the current shortest path from \( v \) to \( t \).
- Updating \( f(v) \):
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  \[ \min_{w \in N_v} (c_{vw} + M'[w]) \] and \( M'[v] > c_{vx} + M'[x] \), then
  - set \( M[v] = c_{vx} + M'[x] \) and
  - set \( f(v) = x \).

- At the end, follow \( f(v) \) pointers from \( s \) to \( t \) (and hope for the best).
Example of Maintaining Pointers

\[ M[v] = \min \left( M'[v], \min_{w \in N_v} \left( c_{vw} + M'[w] \right) \right) \]

\[
\begin{array}{c|ccccccc}
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\hline
\hline
t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
a & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\hline
b & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\hline
c & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\hline
d & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\hline
e & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\hline
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Computing the Shortest Path: Correctness

- **Pointer graph** \( P(V, F) \): each edge in \( F \) is \((v, f(v))\).
  - Can \( P \) have cycles?
  - Is there a path from \( s \) to \( t \) in \( P \)?
  - Can there be multiple paths \( s \) to \( t \) in \( P \)?
  - Which of these is the shortest path?

![Diagram](image)

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T. M. Murali October 19, 21, 26, 28, 2021 Dynamic Programming
Computing the Shortest Path: Cycles in \( P \)

\[
M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right)
\]

- Claim: If \( P \) has a cycle \( C \), then \( C \) has negative cost.
Computing the Shortest Path: Cycles in $P$

$M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right)$

- **Claim:** If $P$ has a cycle $C$, then $C$ has negative cost.
  - Suppose we set $f(v) = w$. At this instant, $M[v] = c_{vw} + M'[w]$. 

```
M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right)
```

```
\begin{itemize}
  \item Claim: If $P$ has a cycle $C$, then $C$ has negative cost.
  \begin{itemize}
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  \end{itemize}
\end{itemize}
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- Claim: If $P$ has a cycle $C$, then $C$ has negative cost.
  - Suppose we set $f(v) = w$. At this instant, $M[v] = c_{vw} + M'[w]$.
  - Comparing $M[w]$ and $M'[w]$. 

Corollary: If $G$ has no negative cycles that $P$ does not either.
Computing the Shortest Path: Cycles in $P$

$M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right)$

- **Claim:** If $P$ has a cycle $C$, then $C$ has negative cost.
  
  ▶ Suppose we set $f(v) = w$. At this instant, $M[v] = c_{vw} + M'[w]$.
  
  ▶ Comparing $M[w]$ and $M'[w]$, we know that $M[w] \leq M'[w]$.
  
  ▶ Between this assignment and the assignment of $f(v)$ to some other node, $M[w]$ may itself further decrease. Hence, $M[v] \geq c_{vw} + M[w]$, in general.
Computing the Shortest Path: Cycles in $P$

$M[v] = \min \left( M'[v], \min_{w \in \mathcal{N}_v} (c_{vw} + M'[w]) \right)$

- **Claim**: If $P$ has a cycle $C$, then $C$ has negative cost.
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  - Between this assignment and the assignment of $f(v)$ to some other node, $M[w]$ may itself further decrease. Hence, $M[v] \geq c_{vw} + M[w]$, in general.
  - Let $v_1, v_2, \ldots v_k$ be the nodes in $C$ and assume that $(v_k, v_1)$ is the last edge to have been added.
  - What is the situation just before this addition?
Computing the Shortest Path: Cycles in $P$

$$M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right)$$

Claim: If $P$ has a cycle $C$, then $C$ has negative cost.

- Suppose we set $f(v) = w$. At this instant, $M[v] = c_{vw} + M'[w]$.
- Comparing $M[w]$ and $M'[w]$, we know that $M[w] \leq M'[w]$.
- Between this assignment and the assignment of $f(v)$ to some other node, $M[w]$ may itself further decrease. Hence, $M[v] \geq c_{vw} + M[w]$, in general.
- Let $v_1, v_2, \ldots v_k$ be the nodes in $C$ and assume that $(v_k, v_1)$ is the last edge to have been added.
- What is the situation just before this addition?
- $M[v_i] - M[v_{i+1}] \geq c_{v_i,v_{i+1}}$, for all $1 \leq i < k - 1$. 

Corollary: if $G$ has no negative cycles that $P$ does not either.
Computing the Shortest Path: Cycles in \( P \)

\[
M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right)
\]

Claim: If \( P \) has a cycle \( C \), then \( C \) has negative cost.

- Suppose we set \( f(v) = w \). At this instant, \( M[v] = c_{vw} + M'[w] \).
- Comparing \( M[w] \) and \( M'[w] \), we know that \( M[w] \leq M'[w] \).
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- Let \( v_1, v_2, \ldots, v_k \) be the nodes in \( C \) and assume that \((v_k, v_1)\) is the last edge to have been added.
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- \( M[v_i] - M[v_{i+1}] \geq c_{v_i,v_{i+1}}, \text{ for all } 1 \leq i < k - 1. \)
- \( \blacktriangleright \) Poll
Computing the Shortest Path: Cycles in $P$

$M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right)$

*Claim*: If $P$ has a cycle $C$, then $C$ has negative cost.

- Suppose we set $f(v) = w$. At this instant, $M[v] = c_{vw} + M'[w]$.
- Comparing $M[w]$ and $M'[w]$, we know that $M[w] \leq M'[w]$.
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- Let $v_1, v_2, \ldots v_k$ be the nodes in $C$ and assume that $(v_k, v_1)$ is the last edge to have been added.
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  - $M[v_i] - M[v_{i+1}] \geq c_{v_i,v_{i+1}}$, for all $1 \leq i < k - 1$.
  - $M[v_k] - M[v_1] > c_{v_k,v_1}$.

Corollary: if $G$ has no negative cycles that $P$ does not either.
Weighted Interval Scheduling  
Segmented Least Squares  
RNA Secondary Structure  
Shortest Paths

**Computing the Shortest Path: Cycles in \( P \)**

\[ M[v] = \min \left( M'[v], \min_{w \in N_v} (c_{vw} + M'[w]) \right) \]

- **Claim:** If \( P \) has a cycle \( C \), then \( C \) has negative cost.
  - Suppose we set \( f(v) = w \). At this instant, \( M[v] = c_{vw} + M'[w] \).
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  - Let \( v_1, v_2, \ldots, v_k \) be the nodes in \( C \) and assume that \((v_k, v_1)\) is the last edge to have been added.
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    - \( M[v_i] - M[v_{i+1}] \geq c_{v_i,v_{i+1}} \), for all \( 1 \leq i < k - 1 \).
    - \( M[v_k] - M[v_1] > c_{v_k,v_1} \).
    - Adding all these inequalities, \( 0 > \sum_{i=1}^{k-1} c_{v_i,v_{i+1}} + c_{v_k,v_1} = \text{cost of } C \).

**Corollary:** if \( G \) has no negative cycles that \( P \) does not either.
\[ M[v] = \min \left( M'[v], \min_{w \in \mathcal{N}_v} (c_{vw} + M'[w]) \right) \]

- **Claim:** If \( P \) has a cycle \( C \), then \( C \) has negative cost.
  
  - Suppose we set \( f(v) = w \). At this instant, \( M[v] = c_{vw} + M'[w] \).
  
  - Comparing \( M[w] \) and \( M'[w] \), we know that \( M[w] \leq M'[w] \).
  
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- **Corollary:** if \( G \) has no negative cycles that \( P \) does not either.
Computing the Shortest Path: Paths in $P$

- Let $P$ be the pointer graph upon termination of the algorithm.
- Consider the path $P_v$ in $P$ obtained by following the pointers from $v$ to $f(v) = v_1$, to $f(v_1) = v_2$, and so on.

Claim: $P_v$ terminates at $t$.

Claim: $P_v$ is the shortest path in $G$ from $v$ to $t$. 

T. M. Murali October 19, 21, 26, 28, 2021 Dynamic Programming
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Poll
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Claim: $P_v$ is the shortest path in $G$ from $v$ to $t$. 
Bellman-Ford Algorithm: One Array

\[ M[v] = \min \left( M[v], \min_{w \in N_v} (c_{vw} + M[w]) \right) \]

- We can prove algorithm’s correctness in this case as well.
Bellman-Ford Algorithm: Early Termination

\[ M[v] = \min \left( M[v], \min_{w \in N_v} (c_{vw} + M[w]) \right) \]

- In general, after \( i \) iterations, the path whose length is \( M[v] \) may have many more than \( i \) edges.
Bellman-Ford Algorithm: Early Termination

\[ M[v] = \min \left( M[v], \min_{w \in N_v} (c_{vw} + M[w]) \right) \]

- In general, after \( i \) iterations, the path whose length is \( M[v] \) may have many more than \( i \) edges.
- Early termination: If \( M \) does not change after processing all the nodes, we have computed all the shortest paths to \( t \).