NP and Computational Intractability

T. M. Murali

November 16, 18, 30, 2021
Algorithm Design

Patterns
- Greed. \(O(n \log n)\) interval scheduling.
- Divide-and-conquer. \(O(n \log n)\) counting inversions.
- Dynamic programming. \(O(n^3)\) RNA folding.
- Duality. \(O(n^2 m)\) maximum flow and minimum cuts.

Image segmentation \(\leq P\) Minimum s-t cut

"Anti-patterns"
- NP-completeness. \(O(n^k)\) algorithm unlikely.
- PSPACE-completeness. \(O(n^k)\) certification algorithm unlikely.
- Undecidability. No algorithm possible.
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- Local search.
- Randomization. Image segmentation \(\leq_P\) Minimum \(s-t\) cut
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NP and Computational Intractability
Computational Tractability

- When is an algorithm an efficient solution to a problem?
Computational Tractability

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Computational Tractability

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**Polynomial time**
- Shortest path
- Matching
- Minimum cut
- 2-SAT
- Planar four-colour
- Bipartite vertex cover
- Primality testing

**Probably not**
- Longest path
- 3-D matching
- Maximum cut
- 3-SAT
- Planar three-colour
- Vertex cover
- Factoring
Problem Classification

- Classify problems based on whether they admit efficient solutions or not.
- Some extremely hard problems cannot be solved efficiently (e.g., chess on an $n$-by-$n$ board).
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- Some extremely hard problems cannot be solved efficiently (e.g., chess on an $n$-by-$n$ board).
- However, classification is unclear for a very large number of discrete computational problems.
Problem Classification

- Classify problems based on whether they admit efficient solutions or not.
- Some extremely hard problems cannot be solved efficiently (e.g., chess on an \(n\)-by-\(n\) board).
- However, classification is unclear for a very large number of discrete computational problems.
- We can prove that these problems are fundamentally equivalent and are manifestations of the same problem!
Polynomial-Time Reduction

- Goal is to express statements of the type “Problem \( X \) is at least as hard as problem \( Y \).”
- Use the notion of *reductions*.
- \( Y \) is polynomial-time reducible to \( X \) (\( Y \leq_P X \))
Polynomial-Time Reduction

- Goal is to express statements of the type “Problem $X$ is at least as hard as problem $Y$.”
- Use the notion of reductions.
- $Y$ is polynomial-time reducible to $X$ ($Y \leq_P X$) if any arbitrary input to $Y$ can be solved using a polynomial number of standard operations, plus one call to a black box that solves problem $X$. 
Polynomial-Time Reduction

Maximum Bipartite Matching $\leq_P$ Maximum $s-t$ Flow
Polynomial-Time Reduction

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Maximum Bipartite Matching $\leq_P$ Maximum $s$-$t$ Flow

Image Segmentation $\leq_P$ Minimum $s$-$t$ Cut

$Y \leq_P X$ implies that “$X$ is at least as hard as $Y$.”

It is possible to solve $Y$ using (potentially unknown) algorithm that solves $X$.

Not the reverse: we can solve $X$ using an algorithm for $Y$.

Such reductions are Karp reductions. Cook reductions allow a polynomial number of calls to the black box that solves $X$. 

T. M. Murali November 16, 18, 30, 2021 NP and Computational Intractability
Polynomial-Time Reduction

- Goal is to express statements of the type “Problem X is at least as hard as problem Y.”
- Use the notion of reductions.
- Y is polynomial-time reducible to X (Y ≤_P X) if any arbitrary input to Y can be solved using a polynomial number of standard operations, plus one call to a black box that solves problem X.
  - Maximum Bipartite Matching ≤_P Maximum s-t Flow
  - Image Segmentation ≤_P Minimum s-t Cut
Polynomial-Time Reduction

- Goal is to express statements of the type “Problem X is at least as hard as problem Y.”
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  - Not the reverse: we can solve $X$ using an algorithm for $Y$.
- Such reductions are Karp reductions. Cook reductions allow a polynomial number of calls to the black box that solves $X$. 
Usefulness of Reductions

- Claim: If $Y \leq_P X$ and $X$ can be solved in polynomial time, then $Y$ can be solved in polynomial time.
Usefulness of Reductions

- Claim: If $Y \leq_P X$ and $X$ can be solved in polynomial time, then $Y$ can be solved in polynomial time.

- Contrapositive: If $Y \leq_P X$ and $Y$ cannot be solved in polynomial time, then $X$ cannot be solved in polynomial time.

- Informally: If $Y$ is hard, and we can show that $Y$ reduces to $X$, then the hardness “spreads” to $X$. 
Reduction Strategies

- Simple equivalence.
- Special case to general case.
- Encoding with gadgets.
Optimisation versus Decision Problems

So far, we have developed algorithms that solve optimisation problems.

- Compute the *largest* flow.
- Find the *closest* pair of points.
- Find the schedule with the *least* completion time.
Optimisation versus Decision Problems

So far, we have developed algorithms that solve optimisation problems.

- Compute the *largest* flow.
- Find the *closest* pair of points.
- Find the schedule with the *least* completion time.

Now, we will focus on *decision versions* of problems, e.g., is there a flow with value at least $k$, for a given value of $k$?

**Decision problem**: answer to every input is yes or no.

**Primes**

**INSTANCE**: A natural number $n$

**QUESTION**: Is $n$ prime?
**Independent Set and Vertex Cover**

- Given an undirected graph $G(V, E)$, a subset $S \subseteq V$ is an **independent set** if no two vertices in $S$ are connected by an edge.
- Given an undirected graph $G(V, E)$, a subset $S \subseteq V$ is a **vertex cover** if every edge in $E$ is incident on at least one vertex in $S$. 

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NP and Computational Intractability
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**Independent Set**

**INSTANCE:** Undirected graph $G$ and an integer $k$

**QUESTION:** Does $G$ contain an independent set of size $\geq k$?

**Vertex Cover**

**INSTANCE:** Undirected graph $G$ and an integer $l$

**QUESTION:** Does $G$ contain a vertex cover of size $\leq l$?
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**INDEPENDENT SET**

**INSTANCE:** Undirected graph $G$ and an integer $k$

**QUESTION:** Does $G$ contain an independent set of size $\geq k$?

**Demonstrate simple equivalence between these two problems.**

**VERTEX COVER**

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Independent Set and Vertex Cover

- Given an undirected graph \( G(V, E) \), a subset \( S \subseteq V \) is an **independent set** if no two vertices in \( S \) are connected by an edge.
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**Independent Set**

**INSTANCE:** Undirected graph \( G \) and an integer \( k \)

**QUESTION:** Does \( G \) contain an independent set of size \( \geq k \)?

**Claim:** \( \text{Independent Set} \leq_p \text{Vertex Cover} \) and \( \text{Vertex Cover} \leq_p \text{Independent Set} \).
Strategy for Proving Indep. Set $\leq_P$ Vertex Cover

Yes, there is an independent set of size at least 3

No, every independent set is of size 3 or less

Black box algorithm for solving vertex cover

Input graph for the vertex cover problem

$k = 3$
Strategy for Proving Indep. Set $\leq_P$ Vertex Cover

1. Start with an arbitrary input to INDEPENDENT SET: an undirected graph $G(V, E)$ and an integer $k$.

2. From $G(V, E)$ and $k$, create an input to VERTEX COVER: an undirected graph $G'(V', E')$ and an integer $l$.
   - $G'$ related to $G$ in some way.
   - $l$ can depend upon $k$ and size of $G$.

3. Prove that $G(V, E)$ has an independent set of size $\geq k$ if and only if $G'(V', E')$ has a vertex cover of size $\leq l$. 
Strategy for Proving Indep. Set $\leq_P$ Vertex Cover

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3. Prove that $G(V, E)$ has an independent set of size $\geq k$ if and only if $G'(V', E')$ has a vertex cover of size $\leq l$.

- Transformation and proof must be correct for all possible graphs $G(V, E)$ and all possible values of $k$.
- Why is the proof an iff statement?
Reason for Two-Way Proof

$k = 3$

Yes, there is an independent set of size at least 3

No, every independent set is of size 3 or less

Why is the proof an iff statement?

\[
l = ?
\]

Input graph for the vertex cover problem

Yes

Black box algorithm for solving vertex cover

No
**Reason for Two-Way Proof**

- **Why is the proof an iff statement?** In the reduction, we are using black box for *Vertex Cover* to solve *Independent Set*.

  1. If there is an independent set size $\geq k$, we must be sure that there is a vertex cover of size $\leq l$, so that we know that the black box will find this vertex cover.
  2. If the black box finds a vertex cover of size $\leq l$, we must be sure we can construct an independent set of size $\geq k$ from this vertex cover.
Proof that Independent Set $\leq_P$ Vertex Cover

1. Arbitrary input to **INDEPENDENT SET**: an undirected graph $G(V, E)$ and an integer $k$.

2. Let $|V| = n$.

3. Create an input to **VERTEX COVER**: same undirected graph $G(V, E)$ and integer $l = n - k$. 
Proof that Independent Set $\leq_P$ Vertex Cover

1. Arbitrary input to **Independent Set**: an undirected graph $G(V, E)$ and an integer $k$.

2. Let $|V| = n$.

3. Create an input to **Vertex Cover**: same undirected graph $G(V, E)$ and integer $l = n - k$.

4. **Claim**: $G(V, E)$ has an independent set of size $\geq k$ iff $G(V, E)$ has a vertex cover of size $\leq n - k$.

   **Proof**: $S$ is an independent set in $G$ iff $V - S$ is a vertex cover in $G$. 
Proof that Independent Set $\leq_P$ Vertex Cover

1. Arbitrary input to **Independent Set**: an undirected graph $G(V, E)$ and an integer $k$.

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   Proof: $S$ is an independent set in $G$ iff $V - S$ is a vertex cover in $G$.

   * Same idea proves that **Vertex Cover $\leq_P$ Independent Set**
**Vertex Cover and Set Cover**

- **Independent Set** is a “packing” problem: pack as many vertices as possible, subject to constraints (the edges).
- **Vertex Cover** is a “covering” problem: cover all edges in the graph with as few vertices as possible.
- There are more general covering problems.
Vertex Cover and Set Cover

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**Microbe Cover**

**INSTANCE:** A set $U$ of $n$ compounds, a collection $M_1, M_2, \ldots, M_l$ of microbes, where each microbe can make a subset of compounds in $U$, and an integer $k$.

**QUESTION:** Is there a subset of $\leq k$ microbes that can together make all the compounds in $U$?

- Define a “microbe” to be the set of compounds it can make, e.g., $M_1 = \{c_1, c_2, c_4, c_7\}$.

**Instance:**

- $n = 10, l = 6, k = 3$

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$n = 10, \ l = 6, \ k = 3$
**Vertex Cover \( \leq_p \) Microbe Cover**

- **Input to** $\text{Vertex Cover}$: an undirected graph $G(V, E)$ and an integer $k$.
- Let $|V| = l$.
- Create an input $\{U, \{M_1, M_2, \ldots, M_l\}\}$ to $\text{Microbe Cover}$ where
**Vertex Cover \( \leq_P \) Microbe Cover**

Input to **Vertex Cover**: an undirected graph \( G(V, E) \) and an integer \( k \).

- Let \( |V| = l \).
- Create an input \( \{U, \{M_1, M_2, \ldots, M_l\}\} \) to **Microbe Cover** where
  - \( U = E \), i.e., each element of \( U \) is an edge of \( G \), and
  - for each node \( i \in V \), create a microbe \( M_i \) whose compounds are the set of edges incident on \( i \).
Input to Vertex Cover: an undirected graph $G(V, E)$ and an integer $k$.

Let $|V| = l$.

Create an input $\{U, \{M_1, M_2, \ldots M_l\}\}$ to Microbe Cover where

- $U = E$, i.e., each element of $U$ is an edge of $G$, and
- for each node $i \in V$, create a microbe $M_i$ whose compounds are the set of edges incident on $i$.

Claim: $U$ can be covered with $\leq k$ microbes iff $G$ has a vertex cover with at $\leq k$ nodes.

Proof strategy:

1. If $G$ has a vertex cover of size $\leq k$, then $U$ can be covered with $\leq k$ microbes.
2. If $U$ can be covered with $\leq k$ microbes, then $G$ has a vertex cover of size $\leq k$. 
Microbe Cover and Set Cover

**Microbe Cover**

**INSTANCE:** A set \( U \) of \( n \) compounds, a collection \( M_1, M_2, \ldots, M_l \) of microbes, where each microbe can make a subset of compounds in \( U \), and an integer \( k \).

**QUESTION:** Is there a subset of \( \leq k \) microbes that can together make all the compounds in \( U \)?

Purely combinatorial problem: a “microbe” is just a set of “compounds.”

\[ n = 10, l = 6 \]
Microbe Cover and Set Cover

**Microbe Cover**

**INSTANCE:** A set $U$ of $n$ compounds, a collection $M_1, M_2, \ldots, M_l$ of microbes, where each microbe can make a subset of compounds in $U$, and an integer $k$.

**QUESTION:** Is there a subset of $\leq k$ microbes that can together make all the compounds in $U$?

- Purely combinatorial problem: a “microbe” is just a set of “compounds.”

**Set Cover**

**INSTANCE:** A set $U$ of $n$ elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of $U$, and an integer $k$.

**QUESTION:** Is there a collection of $\leq k$ sets in the collection whose union is $U$?

$n = 10, l = 6$
Boolean Satisfiability

- Abstract problems formulated in Boolean notation.
Abstract problems formulated in Boolean notation.

Given a set $X = \{x_1, x_2, \ldots, x_n\}$ of $n$ Boolean variables.

Each variable can take the value 0 or 1.

Term: a variable $x_i$ or its negation $\overline{x_i}$.

Clause of length $l$: ($\lor$) of $l$ distinct terms $t_1 \lor t_2 \lor \cdots t_l$.

Truth assignment for $X$: is a function $\nu : X \rightarrow \{0, 1\}$.

An assignment $\nu$ satisfies a clause $C$ if it causes at least one term in $C$ to evaluate to 1 (since $C$ is an $\lor$ of terms).

An assignment satisfies a collection of clauses $C_1, C_2, \ldots, C_k$ if it causes all clauses to evaluate to 1, i.e., $C_1 \land C_2 \land \cdots C_k = 1$.

$\nu$ is a satisfying assignment with respect to $C_1, C_2, \ldots, C_k$.

set of clauses $C_1, C_2, \ldots, C_k$ is satisfiable.
Example

- $X = \{x_1, x_2, x_3, x_4\}$
- Terms: $x_1, \overline{x_1}, x_2, \overline{x_2}, x_3, \overline{x_3}, x_4, \overline{x_4}$
Example

- $X = \{x_1, x_2, x_3, x_4\}$
- Terms: $x_1, \overline{x_1}, x_2, \overline{x_2}, x_3, \overline{x_3}, x_4, \overline{x_4}$
- Clauses:

  - $x_1 \lor \overline{x_2} \lor \overline{x_3}$
  - $x_2 \lor \overline{x_3} \lor x_4$
  - $x_3 \lor \overline{x_4}$
Example

- $X = \{x_1, x_2, x_3, x_4\}$
- Terms: $x_1, \overline{x_1}, x_2, \overline{x_2}, x_3, \overline{x_3}, x_4, \overline{x_4}$
- Clauses:
  
  \[
  \begin{align*}
  x_1 \lor \overline{x_2} \lor \overline{x_3} \\
  x_2 \lor \overline{x_3} \lor x_4 \\
  x_3 \lor \overline{x_4}
  \end{align*}
  \]

- Assignment: $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$
  
  \[
  \begin{align*}
  x_1 \lor \overline{x_2} \lor \overline{x_3} \\
  x_2 \lor \overline{x_3} \lor x_4 \\
  x_3 \lor \overline{x_4}
  \end{align*}
  \]
  
  ▶ Not a satisfying assignment
Example

- \( X = \{x_1, x_2, x_3, x_4\} \)
- Terms: \( x_1, \overline{x_1}, x_2, \overline{x_2}, x_3, \overline{x_3}, x_4, \overline{x_4} \)
- Clauses:
  \[
  x_1 \lor \overline{x_2} \lor \overline{x_3} \\
  x_2 \lor \overline{x_3} \lor x_4 \\
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Example

- \( X = \{x_1, x_2, x_3, x_4\} \)
- Terms: \( x_1, \overline{x_1}, x_2, \overline{x_2}, x_3, \overline{x_3}, x_4, \overline{x_4} \)
- Clauses:
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  x_1 \vee \overline{x_2} \vee \overline{x_3} \\
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- Assignment: \( x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 1 \)
  - Is not a satisfying assignment
SAT and 3-SAT

Satisfiability Problem (SAT)

INSTANCE: A set of clauses $C_1, C_2, \ldots C_k$ over a set $X = \{x_1, x_2, \ldots x_n\}$ of $n$ variables.

QUESTION: Is there a satisfying truth assignment for $X$ with respect to $C$?
SAT and 3-SAT

3-Satisfiability Problem (SAT)

INSTANCE: A set of clauses $C_1, C_2, \ldots C_k$, each of length three, over a set $X = \{x_1, x_2, \ldots x_n\}$ of $n$ variables.

QUESTION: Is there a satisfying truth assignment for $X$ with respect to $C$?
SAT and 3-SAT

3-Satisfiability Problem (SAT)

INSTANCE: A set of clauses \( C_1, C_2, \ldots C_k \), each of length three, over a set \( X = \{x_1, x_2, \ldots x_n\} \) of \( n \) variables.

QUESTION: Is there a satisfying truth assignment for \( X \) with respect to \( C \)?

- SAT and 3-SAT are fundamental combinatorial search problems.
- We have to make \( n \) independent decisions (the assignments for each variable) while satisfying a set of constraints.
- Satisfying each constraint in isolation is easy, but we have to make our decisions so that all constraints are satisfied simultaneously.
Examples of 3-SAT

Example:

- $C_1 = x_1 \lor 0 \lor 0$
- $C_2 = x_2 \lor 0 \lor 0$
- $C_3 = \overline{x_1} \lor \overline{x_2} \lor 0$

Poll

1. Is $C_1 \land C_2$ satisfiable? Yes, by $x_1 = 1$, $x_2 = 1$.
2. Is $C_1 \land C_3$ satisfiable? Yes, by $x_1 = 1$, $x_2 = 0$.
3. Is $C_2 \land C_3$ satisfiable? Yes, by $x_1 = 0$, $x_2 = 1$.
4. Is $C_1 \land C_2 \land C_3$ satisfiable? No.
Examples of 3-SAT

Example:

- $C_1 = x_1 \lor 0 \lor 0$
- $C_2 = x_2 \lor 0 \lor 0$
- $C_3 = \overline{x_1} \lor \overline{x_2} \lor 0$

Is $C_1 \land C_2$ satisfiable?
Examples of 3-SAT

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- $C_3 = \overline{x_1} \lor \overline{x_2} \lor 0$

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2. Is $C_1 \land C_3$ satisfiable? Yes, by $x_1 = 1, x_2 = 0$.

3. Is $C_2 \land C_3$ satisfiable? Yes, by $x_1 = 0, x_2 = 1$.

4. Is $C_1 \land C_2 \land C_3$ satisfiable? No.
Examples of 3-SAT

Example:

- $C_1 = x_1 \lor 0 \lor 0$
- $C_2 = x_2 \lor 0 \lor 0$
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Examples of 3-SAT

Example:

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- $C_3 = \overline{x_1} \lor \overline{x_2} \lor 0$

1. Is $C_1 \land C_2$ satisfiable? Yes, by $x_1 = 1, x_2 = 1$.
2. Is $C_1 \land C_3$ satisfiable? Yes, by $x_1 = 1, x_2 = 0$. 
Examples of 3-SAT

Example:
- \( C_1 = x_1 \lor 0 \lor 0 \)
- \( C_2 = x_2 \lor 0 \lor 0 \)
- \( C_3 = \overline{x_1} \lor \overline{x_2} \lor 0 \)

1. Is \( C_1 \land C_2 \) satisfiable? Yes, by \( x_1 = 1, x_2 = 1 \).
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3. Is \( C_2 \land C_3 \) satisfiable?
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3. Is $C_2 \land C_3$ satisfiable? Yes, by $x_1 = 0, x_2 = 1$. 
Examples of 3-SAT

Example:
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2. Is $C_1 \land C_3$ satisfiable? Yes, by $x_1 = 1$, $x_2 = 0$.
3. Is $C_2 \land C_3$ satisfiable? Yes, by $x_1 = 0$, $x_2 = 1$.
4. Is $C_1 \land C_2 \land C_3$ satisfiable?
Examples of 3-SAT

Example:

1. $C_1 = x_1 \lor 0 \lor 0$
2. $C_2 = x_2 \lor 0 \lor 0$
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2. Is $C_1 \land C_3$ satisfiable? Yes, by $x_1 = 1, x_2 = 0$.
3. Is $C_2 \land C_3$ satisfiable? Yes, by $x_1 = 0, x_2 = 1$.
4. Is $C_1 \land C_2 \land C_3$ satisfiable? No.
3-SAT and Independent Set

\[ C_1 = x_1 \lor \overline{x_2} \lor \overline{x_3} \]
\[ C_2 = \overline{x_1} \lor x_2 \lor x_4 \]
\[ C_3 = \overline{x_1} \lor x_3 \lor \overline{x_4} \]

- We want to prove \(3\text{-SAT} \leq_P \text{INDEPENDENT SET}.\)
3-SAT and Independent Set

\[ C_1 = x_1 \lor \overline{x_2} \lor \overline{x_3} \]
\[ C_2 = \overline{x_1} \lor x_2 \lor x_4 \]
\[ C_3 = \overline{x_1} \lor x_3 \lor \overline{x_4} \]

1. Select \( x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1 \).

We want to prove \( 3\text{-SAT} \leq_P \text{INDEPENDENT SET} \).

Two ways to think about 3-SAT:

1. Make an independent 0/1 decision on each variable and succeed if we achieve one of three ways in which to satisfy each clause.
3-SAT and Independent Set

\[ C_1 = x_1 \lor \overline{x_2} \lor \overline{x_3} \]
\[ C_2 = \overline{x_1} \lor x_2 \lor x_4 \]
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1. Select \( x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1 \).
2. Choose one literal from each clause to evaluate to true.

- We want to prove 3-SAT \( \leq_P \) Independent Set.

Two ways to think about 3-SAT:

1. Make an independent 0/1 decision on each variable and succeed if we achieve one of three ways in which to satisfy each clause.
2. Choose (at least) one term from each clause. Find a truth assignment that causes each chosen term to evaluate to 1. Ensure that no two terms selected conflict, e.g., select \( \overline{x_2} \) in \( C_1 \) and \( x_2 \) in \( C_2 \).
3-SAT and Independent Set

\[ C_1 = x_1 \lor \overline{x_2} \lor \overline{x_3} \]
\[ C_2 = \overline{x_1} \lor x_2 \lor x_4 \]
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1. Select \( x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1 \).

2. Choose one literal from each clause to evaluate to true.
   - Choices of selected literals imply \( x_1 = 0, x_2 = 0, x_4 = 1 \).

We want to prove \( 3\text{-SAT} \leq_P \text{INDEPENDENT SET} \).

Two ways to think about 3-SAT:

1. Make an independent 0/1 decision on each variable and succeed if we achieve one of three ways in which to satisfy each clause.

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Proving $3$-SAT $\leq_p$ Independent Set

$C_1 = x_1 \lor \overline{x_2} \lor \overline{x_3}$
$C_2 = \overline{x_1} \lor x_2 \lor x_4$
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- We are given an input to 3-SAT with $k$ clauses of length three over $n$ variables.
- Construct an input to independent set: graph $G(V, E)$ with $3k$ nodes.
Proving $\text{3-SAT} \leq_P \text{Independent Set}$

$C_1 = x_1 \lor \overline{x_2} \lor \overline{x_3}$
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- We are given an input to 3-SAT with $k$ clauses of length three over $n$ variables.
- Construct an input to independent set: graph $G(V, E)$ with $3k$ nodes.
  - For each clause $C_i$, $1 \leq i \leq k$, add a triangle of three nodes $v_{i1}, v_{i2}, v_{i3}$ and three edges to $G$.
  - Label each node $v_{ij}, 1 \leq j \leq 3$ with the $j$th term in $C_i$.  

$\text{Poll}$
We are given an input to 3-SAT with \( k \) clauses of length three over \( n \) variables.

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- Label each node \( v_{ij}, 1 \leq j \leq 3 \) with the \( j \)th term in \( C_i \).
- Add an edge between each pair of nodes whose labels correspond to terms that conflict.
Claim: Input to 3-SAT is satisfiable iff $G$ has an independent set of size $k$. 

\[
C_1 = x_1 \lor \neg x_2 \lor \neg x_3 \\
C_2 = \neg x_1 \lor x_2 \lor x_4 \\
C_3 = \neg x_1 \lor x_3 \lor \neg x_4
\]
Proving $3$-SAT $\leq_P$ Independent Set

Claim: Input to 3-SAT is satisfiable iff $G$ has an independent set of size $k$.

Satisfiable assignment $\rightarrow$ independent set of size $k$: 

$C_1 = x_1 \lor \overline{x_2} \lor \overline{x_3}$
$C_2 = \overline{x_1} \lor x_2 \lor x_4$
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Proving $3$-SAT $\leq_P \text{Independent Set}$

Claim: Input to $3$-SAT is satisfiable iff $G$ has an independent set of size $k$.

Satisfiable assignment $\rightarrow$ independent set of size $k$: Each triangle in $G$ has at least one node whose label evaluates to 1. Set $S$ of nodes consisting of one such node from each triangle forms an independent set of size $= k$. Why?
Proving $3\text{-SAT} \leq_P \text{Independent Set}$

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Proving $3$-SAT $\leq_P$ Independent Set

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Independent set $S$ of size $k$ $\rightarrow$ satisfiable assignment: the size of this set is $k$. How do we construct a satisfying truth assignment from the nodes in the independent set?
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- For each variable $x_i$, only $x_i$ or $\overline{x_i}$ is the label of a node in $S$. Why?
Proving $3$-SAT $\leq_P$ Independent Set

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    ▶ For each variable $x_i$, only $x_i$ or $\overline{x_i}$ is the label of a node in $S$. Why?
    ▶ If $x_i$ is the label of a node in $S$, set $x_i = 1$; else set $x_i = 0$.
    ▶ Why is each clause satisfied?
Transitivity of Reductions

Claim: If $Z \leq_P Y$ and $Y \leq_P X$, then $Z \leq_P X$. 
Transitivity of Reductions

- Claim: If $Z \leq_P Y$ and $Y \leq_P X$, then $Z \leq_P X$.
- We have shown
  
  $3\text{-SAT} \leq_P \text{INDEPENDENT SET} \leq_P \text{VERTEX COVER} \leq_P \text{SET COVER}$
Finding vs. Certifying

- Is it easy to check if a given set of vertices in an undirected graph forms an independent set of size at least $k$?
- Is it easy to check if a particular truth assignment satisfies a set of clauses?
Finding vs. Certifying

- Is it easy to check if a given set of vertices in an undirected graph forms an independent set of size at least $k$?
- Is it easy to check if a particular truth assignment satisfies a set of clauses?
- We draw a contrast between finding a solution and checking a solution (in polynomial time).
- Since we have not been able to develop efficient algorithms to solve many decision problems, let us turn our attention to whether we can check if a proposed solution is correct.
Problems and Algorithms

PRIMES

INSTANCE: A natural number \( n \)

QUESTION: Is \( n \) prime?

- Decision problem \( X \): for every input \( s \), answer \( X(s) \) is yes or no.
Problems and Algorithms

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- An algorithm $A$ for a decision problem receives an input $s$ and returns $A(s) \in \{\text{yes}, \text{no}\}$.
- An algorithm $A$ solves the problem $X$ if for every input $s$,
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  - if \( X(s) = \text{no} \) then \( A(s) = \text{no} \)
- \( A \) has a polynomial running time if there is a polynomial function \( p(\cdot) \) such that for every input \( s \), \( A \) terminates on \( s \) in at most \( O(p(|s|)) \) steps.
  - There is an algorithm such that \( p(|s|) = |s|^{12} \) for PRIMES (Agarwal, Kayal, Saxena, 2002, improved to \( |s|^6 \) by Pomerance and Lenstra, 2005).
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- $A$ has a **polynomial running time** if there is a polynomial function $p(\cdot)$ such that for every input $s$, $A$ terminates on $s$ in at most $O(p(|s|))$ steps.
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- $\mathcal{P}$: set of problems $X$ for which there is a polynomial time algorithm.
Introduction Reductions

Problems and Algorithms

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- $\mathcal{P}$: set of problems $X$ for which there is a polynomial time algorithm.

A decision problem $X$ is in $\mathcal{P}$ iff there is an algorithm $A$ with polynomial running time that solves $X$. 
Efficient Certification

- A “checking” algorithm for a decision problem $X$ has a different structure from an algorithm that solves $X$.

- Checking algorithm needs input $s$ as well as a separate “certificate” $t$ that contains evidence that $X(s) = \text{yes}$.
Efficient Certification

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- An algorithm $B$ is an efficient certifier for a problem $X$ if
  1. $B$ is a polynomial time algorithm that takes two inputs $s$ and $t$ and
  2. for all inputs $s$
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- Certifier’s job is to take a candidate certificate ($t$) that $s \in X$ and check in polynomial time whether $t$ is a correct certificate.

- Certificate $t$ must be “short” so that certifier can run in polynomial time.
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Certifier’s job is to take a candidate certificate ($t$) that $s \in X$ and check in polynomial time whether $t$ is a correct certificate.

Certificate $t$ must be “short” so that certifier can run in polynomial time.

Certifier does not care about how to find these certificates.
\( \mathcal{NP} \)

- \( \mathcal{P} \): set of problems \( X \) for which there is a polynomial time algorithm.
NP

- $\mathcal{P}$: set of problems $X$ for which there is a polynomial time algorithm.
- $\mathcal{NP}$ is the set of all problems for which there exists an efficient certifier.
- 3-SAT $\in \mathcal{NP}$:

  - Certificate $t$: a truth assignment to the variables.
  - Certifier $B$: checks whether assignment causes each clause to evaluate to true.

  Independent Set $\in \mathcal{NP}$:

  - Certificate $t$: a set of at least $k$ vertices.
  - Certifier $B$: checks that no pair of these vertices are connected by an edge.

  Set Cover $\in \mathcal{NP}$:

  - Certificate $t$: a list of $k$ sets from the collection.
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Claim: $\mathcal{P} \subseteq \mathcal{NP}$. 
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- Let $X$ be any problem in $\mathcal{P}$.
- There is a polynomial time algorithm $A$ that solves $X$. 

One of the major unsolved problems in computer science.

$1$M prize offered by Clay Mathematics Institute.
Claim: \( P \subseteq NP \).

- Let \( X \) be any problem in \( P \).
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- \( B \) ignores \( t \) and simply returns \( A(s) \). Why is \( B \) an efficient certifier?

Is \( P = NP \) or is \( \text{NP} - \text{P} \neq \emptyset \)?

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**P vs. NP**

- **Claim:** $\mathcal{P} \subseteq \mathcal{NP}$.
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Is $\mathcal{P} = \mathcal{NP}$ or is $\mathcal{NP} - \mathcal{P} \neq \emptyset$? One of the major unsolved problems in computer science. $1\text{M}$ prize offered by Clay Mathematics Institute.

P vs NP Problem

Suppose that you are organizing housing accommodations for a group of four hundred university students. Space is limited and only one hundred of the students will receive places in the dormitory. To complicate matters, the Dean has provided you with a list of pairs of incompatible students, and requested that no pair from this list appear in your final choice. This is an example of what computer scientists call an NP-problem, since it is easy to check if a given choice of one hundred students proposed by a coworker is satisfactory (i.e., no pair taken from your coworker's list also appears on the list from the Dean's office), however the task of generating such a list from scratch seems to be so hard as to be completely impractical. Indeed, the total number of ways of choosing one hundred students from the four hundred applicants is greater than the number of atoms in the known universe! Thus no future civilization could ever hope to build a supercomputer capable of solving the problem by brute force; that is, by checking every possible combination of 100 students. However, this apparent difficulty may only reflect the lack of ingenuity of your programmer. In fact, one of the outstanding problems in computer science is determining whether questions exist whose answer can be quickly checked, but which require an impossibly long time to solve by any direct procedure. Problems like the one listed above certainly seem to be of this kind, but so far no one has managed to prove that any of them really are so hard as they appear, i.e., that there really is no feasible way to generate an answer with the help of a computer. Stephen Cook and Leonid Levin formulated the P (i.e., easy to find) versus NP (i.e., easy to check) problem independently in 1971.

Image credit: on the left, Stephen Cook by Jiří Janiček (cropped).  CC-BY-SA 3.0
Dealing with *decision problems*: for every input, the answer is yes or no.

A problem is in \( \mathcal{P} \) if there is a polynomial time algorithm that solves it.
Dealing with *decision problems*: for every input, the answer is yes or no.

A problem is in \( \mathcal{P} \) if there is a polynomial time algorithm that solves it.

A problem is in \( \mathcal{NP} \) if there is a polynomial time certifying algorithm for *yes* inputs:

- Given an input and a “certificate”, the certifier can use the certificate to verify in polynomial time if the answer is *yes* for that input.
- Definition of \( \mathcal{NP} \) does not care about inputs for which the answer is *no*. 
Summary

NP

- Dealing with *decision problems*: for every input, the answer is yes or no.
- A problem is in $P$ if there is a polynomial time algorithm that solves it.
- A problem is in $NP$ if there is a polynomial time certifying algorithm for yes inputs:
  - Given an input and a “certificate”, the certifier can use the certificate to verify in polynomial time if the answer is yes for that input.
  - Definition of $NP$ does not care about inputs for which the answer is no.
- $P \subseteq NP$
- 3-SAT, VertexCover, SetCover, IndependentSet are in $NP$.
- 3-SAT $\leq_P$ Independent Set $\leq_P$ Vertex Cover $\leq_P$ Set Cover
Dealing with *decision problems*: for every input, the answer is yes or no.

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What is the structure of the problems in $\mathcal{NP}$?

1. Is there a sequence of problems $X_1, X_2, X_3, \ldots$ in $\mathcal{NP}$, such that $X_1 \leq_p X_2 \leq_p X_3 \leq_p \ldots$?
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2. Are there two problems $X_1$ and $X_2$ in $\mathcal{NP}$ such that there is no problem $X \in \mathcal{NP}$ where $X_1 \leq_p X$ and $X_2 \leq_p X$?
\( \mathsf{NP} \)-Complete and \( \mathsf{NP} \)-Hard Problems

- What are the hardest problems in \( \mathsf{NP} \)?
\textbf{\NP-Complete and \NP-Hard Problems}

- What are the hardest problems in \NP?

A problem \(X\) is \textit{\NP-Complete} if
- \(X \in \NP\) and
- for every problem \(Y \in \NP\), \(Y \leq_P X\).

A problem \(X\) is \textit{\NP-Hard} if
- for every problem \(Y \in \NP\), \(Y \leq_P X\).

\textbf{Claim:} Suppose \(X\) is \NP-Complete. Then \(X \in \P\) iff \(\P = \NP\).

\textbf{Corollary:} If there is any problem in \NP that cannot be solved in polynomial time, then no \NP-Complete problem can be solved in polynomial time.

Does even one \NP-Complete problem exist?! If it does, how can we prove that every problem in \NP reduces to this problem?
NP-Complete and NP-Hard Problems

- What are the hardest problems in \( \mathcal{NP} \)?

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A problem \( X \) is \( \mathcal{NP} \)-Hard if

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Claim: Suppose \( X \) is \( \mathcal{NP} \)-Complete. Then \( X \in \mathcal{P} \) iff \( \mathcal{P} = \mathcal{NP} \).
**NP-Complete and NP-Hard Problems**

- **What are the hardest problems in NP?**

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  A problem $X$ is **NP-Hard** if
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- **Claim:** Suppose $X$ is NP-Complete. Then $X \in \mathcal{P}$ iff $\mathcal{P} = \mathcal{NP}$.

- **Corollary:** If there is any problem in NP that cannot be solved in polynomial time, then no NP-Complete problem can be solved in polynomial time.
**NP-Complete and NP-Hard Problems**

- What are the hardest problems in NP?

A problem $X$ is **NP-Complete** if

1. $X \in \text{NP}$
2. For every problem $Y \in \text{NP}$, $Y \leq_p X$.

A problem $X$ is **NP-Hard** if

1. For every problem $Y \in \text{NP}$, $Y \leq_p X$.

**Claim:** Suppose $X$ is $\text{NP}$-Complete. Then $X \in \text{P}$ iff $\text{P} = \text{NP}$.

**Corollary:** If there is any problem in $\text{NP}$ that cannot be solved in polynomial time, then no $\text{NP}$-Complete problem can be solved in polynomial time.

**Does even one $\text{NP}$-Complete problem exist?!** If it does, how can we prove that every problem in $\text{NP}$ reduces to this problem?
Circuit Satisfiability

- **Cook-Levin Theorem:** Circuit Satisfiability is $NP$-Complete.
Circuit Satisfiability

- **Cook-Levin Theorem:** \textsc{Circuit Satisfiability} is $\mathcal{NP}$-Complete.

- A circuit $K$ is a labelled, directed acyclic graph such that:
  1. the sources in $K$ are labelled with constants (0 or 1) or the name of a distinct variable (the inputs to the circuit).
  2. every other node is labelled with one Boolean operator $\land$, $\lor$, or $\neg$.
  3. a single node with no outgoing edges represents the output of $K$.

![Diagram of a circuit](image)

**Figure 8.4** A circuit with three inputs, two additional sources that have assigned truth values, and one output.
Circuit Satisfiability

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**Circuit Satisfiability**

**INSTANCE**: A circuit $K$.

**QUESTION**: Is there a truth assignment to the inputs that causes the output to have value 1?

![Diagram of a circuit with three inputs, two additional sources, and one output.](image)

*Figure 8.4* A circuit with three inputs, two additional sources that have assigned truth values, and one output.

> Skip proof; read textbook or Chapter 2.6 of Garey and Johnson.
Proving Circuit Satisfiability is \( \mathcal{NP} \)-Complete
Proving Circuit Satisfiability is \( \mathcal{NP} \)-Complete

- Take an arbitrary problem \( X \in \mathcal{NP} \) and show that \( X \leq_{P} \text{Circuit Satisfiability} \).
Proving Circuit Satisfiability is $\mathcal{NP}$-Complete

- Take an arbitrary problem $X \in \mathcal{NP}$ and show that $X \leq_p \text{Circuit Satisfiability}$.
- Claim we will not prove: any algorithm that takes a fixed number $n$ of bits as input and produces a yes/no answer
  1. can be represented by an equivalent circuit and
  2. if the running time of the algorithm is polynomial in $n$, the size of the circuit is a polynomial in $n$. 
Proving Circuit Satisfiability is $\mathcal{NP}$-Complete

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- To show $X \leq_P \text{Circuit Satisfiability}$, given an input $s$ of length $n$, we want to determine whether $s \in X$ using a black box that solves Circuit Satisfiability.
Proving Circuit Satisfiability is $\mathcal{NP}$-Complete

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- To show $X \leq_p \text{Circuit Satisfiability}$, given an input $s$ of length $n$, we want to determine whether $s \in X$ using a black box that solves $\text{Circuit Satisfiability}$.

- What do we know about $X$?
Proving Circuit Satisfiability is $\mathcal{NP}$-Complete

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- To show $X \leq_p \text{Circuit Satisfiability}$, given an input $s$ of length $n$, we want to determine whether $s \in X$ using a black box that solves Circuit Satisfiability.
- What do we know about $X$? It has an efficient certifier $B(\cdot, \cdot)$. 
Proving Circuit Satisfiability is $\mathcal{NP}$-Complete

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- What do we know about $X$? It has an efficient certifier $B(\cdot, \cdot)$.
- To determine whether $s \in X$, we ask “Is there a certificate $t$ of length $p(n)$ such that $B(s, t) = \text{yes}$?”
Proving Circuit Satisfiability is \( \mathcal{NP} \)-Complete

- To determine whether \( s \in X \), we ask “Is there a certificate \( t \) of length \( p(|s|) \) such that \( B(s, t) = \text{yes} \)”
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To determine whether $s \in X$, we ask “Is there a certificate $t$ of length $p(|s|)$ such that $B(s, t) = \text{yes}$?”

View $B(\cdot, \cdot)$ as an algorithm on $n + p(n)$ bits.

Convert $B$ to a polynomial-sized circuit $K$ with $n + p(n)$ sources.

1. First $n$ sources are hard-coded with the bits of $s$.
2. The remaining $p(n)$ sources labelled with variables representing the bits of $t$. 
Proving Circuit Satisfiability is $\mathcal{NP}$-Complete

- To determine whether $s \in X$, we ask “Is there a certificate $t$ of length $p(|s|)$ such that $B(s, t) = \text{yes}$?”
- View $B(\cdot, \cdot)$ as an algorithm on $n + p(n)$ bits.
- Convert $B$ to a polynomial-sized circuit $K$ with $n + p(n)$ sources.
  1. First $n$ sources are hard-coded with the bits of $s$.
  2. The remaining $p(n)$ sources labelled with variables representing the bits of $t$.
- $s \in X$ iff there is an assignment of the input bits of $K$ that makes $K$ satisfiable.
Example of Transformation to Circuit Satisfiability

- Does a graph $G$ on $n$ nodes have a two-node independent set?
Example of Transformation to Circuit Satisfiability

Does a graph $G$ on $n$ nodes have a two-node independent set?

- $s$ encodes the graph $G$ with $\binom{n}{2}$ bits.
- $t$ encodes the independent set with $n$ bits.
- Certifier needs to check if
  1. at least two bits in $t$ are set to 1 and
  2. no two bits in $t$ are set to 1 if they form the ends of an edge (the corresponding bit in $s$ is set to 1).
Example of Transformation to Circuit Satisfiability

- Suppose $G$ contains three nodes $u$, $v$, and $w$ with $v$ connected to $u$ and $w$. 
Example of Transformation to Circuit Satisfiability

- Suppose $G$ contains three nodes $u$, $v$, and $w$ with $v$ connected to $u$ and $w$.

![Circuit Diagram]

Figure 8.5 A circuit to verify whether a 3-node graph contains a 2-node independent set.
Asymmetry of Certification

Definition of efficient certification and $\mathcal{NP}$ is fundamentally asymmetric:

- An input $s$ is a “yes” instance iff there exists a short certificate $t$ such that $B(s, t) = \text{yes}$.
- An input $s$ is a “no” instance iff for all short certificates $t$, $B(s, t) = \text{no}$.
Asymmetry of Certification

Definition of efficient certification and $\mathcal{NP}$ is fundamentally asymmetric:

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- An input $s$ is a “no” instance iff for all short certificates $t$, $B(s, t) = \text{no}$. The definition of $\mathcal{NP}$ does not guarantee a short proof for “no” instances.
For a decision problem $X$, its complementary problem $\overline{X}$ is the set of inputs $s$ such that $s \in \overline{X}$ iff $s \notin X$. 

Claim: If $\text{NP} \neq \text{co-NP}$ then $\text{P} \neq \text{NP}$. 

T. M. Murali November 16, 18, 30, 2021 NP and Computational Intractability
co-$\mathcal{NP}$

- For a decision problem $X$, its *complementary problem* $\overline{X}$ is the set of inputs $s$ such that $s \in \overline{X}$ iff $s \notin X$.
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For a decision problem \( X \), its *complementary problem* \( \overline{X} \) is the set of inputs \( s \) such that \( s \in \overline{X} \) iff \( s \notin X \).

If \( X \in P \), then \( \overline{X} \in P \).
For a decision problem $X$, its complementary problem $\overline{X}$ is the set of inputs $s$ such that $s \in \overline{X}$ iff $s \not\in X$.

- If $X \in \mathcal{P}$, then $\overline{X} \in \mathcal{P}$.
- If $X \in \mathcal{NP}$, then is $\overline{X} \in \mathcal{NP}$?
**co-\(\mathcal{NP}\)**

- For a decision problem \(X\), its *complementary problem* \(\overline{X}\) is the set of inputs \(s\) such that \(s \in \overline{X}\) iff \(s \notin X\).
- If \(X \in \mathcal{P}\), then \(\overline{X} \in \mathcal{P}\).
- If \(X \in \mathcal{NP}\), then is \(\overline{X} \in \mathcal{NP}\)? Unclear in general.
- A problem \(X\) belongs to the class *co-\(\mathcal{NP}\)* iff \(\overline{X}\) belongs to \(\mathcal{NP}\).
For a decision problem $X$, its *complementary problem* $\overline{X}$ is the set of inputs $s$ such that $s \in \overline{X}$ iff $s \not\in X$.

If $X \in \mathcal{P}$, then $\overline{X} \in \mathcal{P}$.

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Open problem: Is $\mathcal{NP} = \text{co-}\mathcal{NP}$?
For a decision problem \( X \), its \textit{complementary problem} \( \overline{X} \) is the set of inputs \( s \) such that \( s \in \overline{X} \) iff \( s \notin X \).

If \( X \in \mathcal{P} \), then \( \overline{X} \in \mathcal{P} \).

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A problem \( X \) belongs to the class \textit{co-NP} iff \( \overline{X} \) belongs to \( \mathcal{NP} \).

Open problem: Is \( \mathcal{NP} = \text{co-NP} \)?

Claim: If \( \mathcal{NP} \neq \text{co-NP} \) then \( \mathcal{P} \neq \mathcal{NP} \).
Good Characterisations: the Class $\mathcal{NP} \cap \text{co-NP}$

- If a problem belongs to both $\mathcal{NP}$ and co-$\mathcal{NP}$, then
  - When the answer is yes, there is a short proof.
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- Example is the problem of determining if a flow network contains a flow of value at least $\nu$, for some given value of $\nu$.
  - Yes: construct a flow of value at least $\nu$.
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![Venn Diagram]

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