NP-Complete Problems

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Review: Definitions of $\mathcal{NP}$-Complete and $\mathcal{NP}$-Hard

A problem X is *$\mathcal{NP}$-Complete* if

1. (i) $X \in \mathcal{NP}$ and
2. (ii) for every problem $Y \in \mathcal{NP}$, $Y \leq_P X$.

A problem X is *$\mathcal{NP}$-Hard* if

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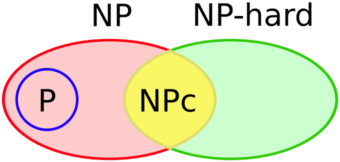
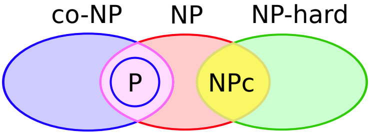


Proving Other Problems $\mathcal{NP}$-Complete



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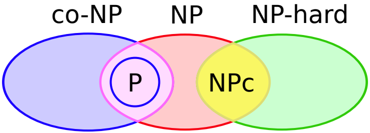

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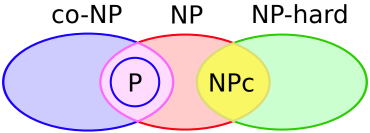

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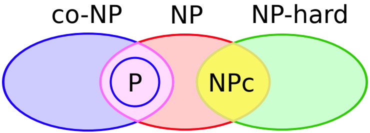

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- To prove X is $\mathcal{NP}$-Complete, reduce a known $\mathcal{NP}$-Complete problem Y to X. Do not prove reduction in the opposite direction, i.e., $X \leq_P Y$.

Proving a Problem $\mathcal{NP}$-Complete with Karp Reduction

1. Prove that $X \in \mathcal{NP}$.
2. Select a problem Y known to be $\mathcal{NP}$-Complete.
3. Consider an arbitrary input s to problem Y. Show how to construct, in polynomial time, an input t to problem X such that
   a. If $Y(s) = \texttt{yes}$, then $X(t) = \texttt{yes}$ and
   b. If $X(t) = \texttt{yes}$, then $Y(s) = \texttt{yes}$ (equivalently, if $Y(s) = \texttt{no}$, then $X(t) = \texttt{no}$).

3-SAT is $\mathcal{NP}$-Complete

- Why is 3-SAT in NP?

3-SAT is $\mathcal{NP}$-Complete

- Why is 3-SAT in NP?
- Circuit Satisfiability $\leq_P$ 3-SAT.
  1. Given an input to Circuit Satisfiability, create an input to SAT, in which each clause has *at most* three variables.
  2. Convert this input to SAT into an input to 3-SAT.

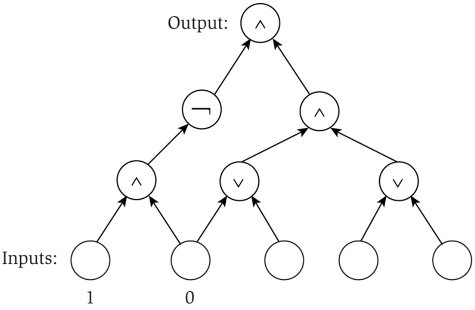

Figure 8.4 A circuit with three inputs, two additional sources that have assigned truth values, and one output.

Skip proof that Circuit Satisfiability $\leq_P$ 3-SAT

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- Output: if o is the output node, use the clause (x_o).

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  - If a clause has a two terms t and t', replace the clause with $t \vee t' \vee z_1$.

More $\mathcal{NP}$-Complete problems

- Circuit Satisfiability is $\mathcal{NP}$-Complete.
- We just showed that Circuit Satisfiability $\leq_P$ 3-SAT.
- We know that

$$\text{3-SAT} \leq_P \text{Independent Set} \leq_P \text{Vertex Cover} \leq_P \text{Set Cover}$$

- All these problems are in $\mathcal{NP}$.
- Therefore, Independent Set, Vertex Cover, and Set Cover are $\mathcal{NP}$-Complete.

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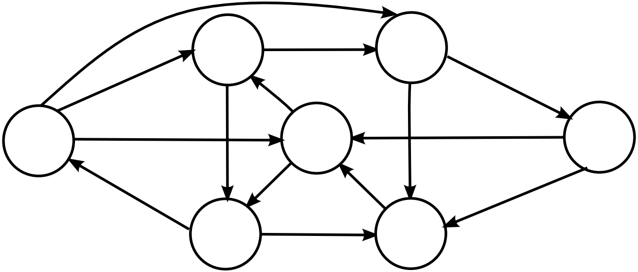

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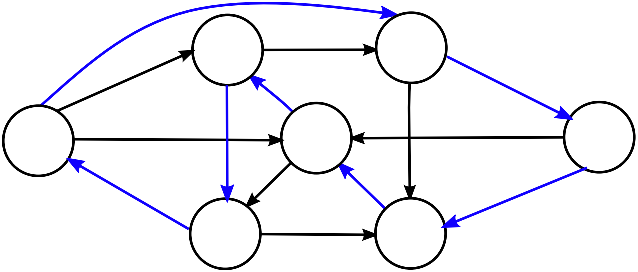

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- Why is the problem in $\mathcal{NP}$?
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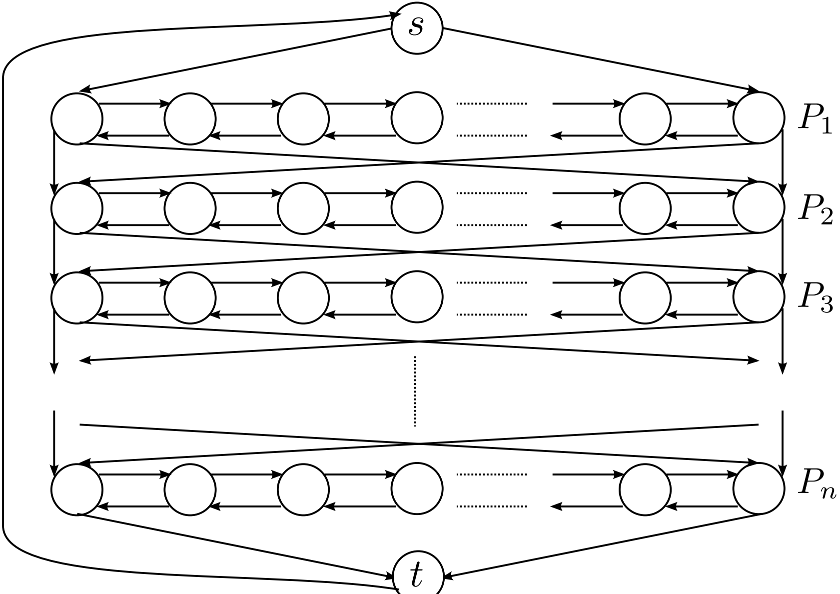
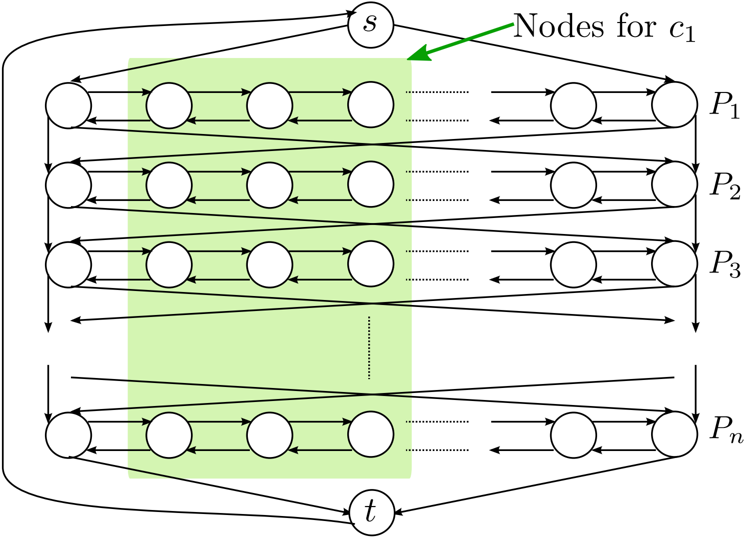
- Consider an arbitrary input to 3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots C_k$.
- Strategy:
  1. Construct a graph G with $O(nk)$ nodes and edges and 2^n Hamiltonian cycles with a one-to-one correspondence with 2^n truth assignments.
  2. Add nodes to impose constraints arising from clauses.
  3. Construction takes $O(nk)$ time.
- G contains n paths $P_1, P_2, \ldots P_n$, one for each variable.
- Each P_i contains $b = 3k + 3$ nodes $v_{i,1}, v_{i,2}, \ldots v_{i,b}$, three for each clause and some extra nodes.

3-SAT $\leq_P$ Hamiltonian Cycle: Constructing G



3-SAT $\leq_P$ Hamiltonian Cycle: Modelling clauses

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.



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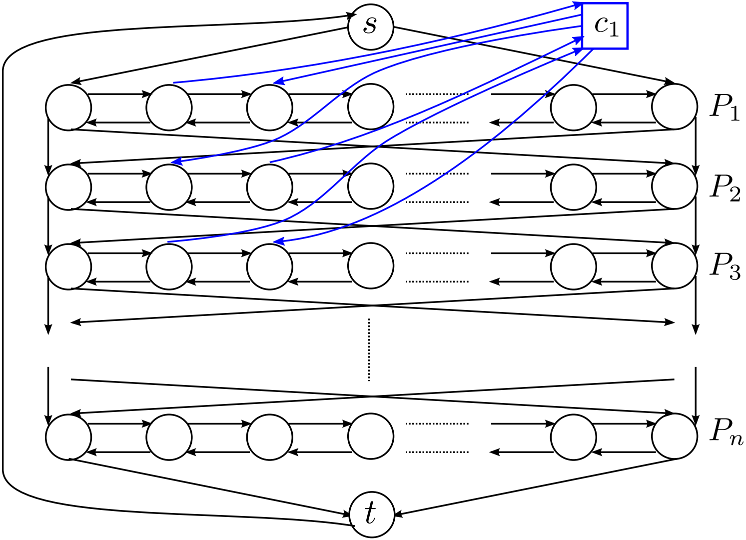

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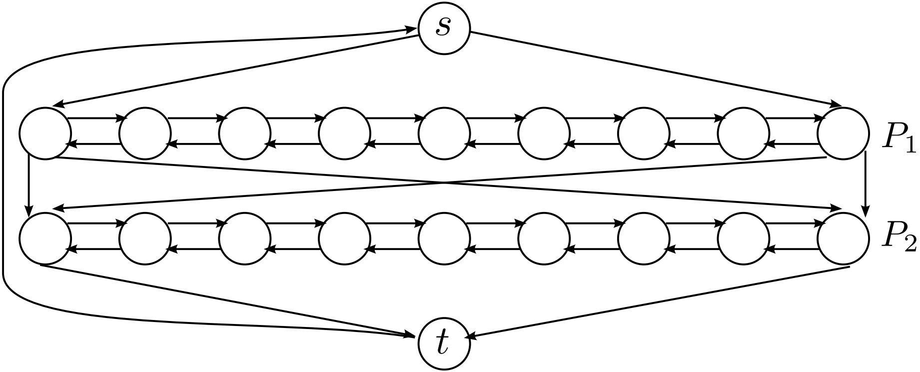

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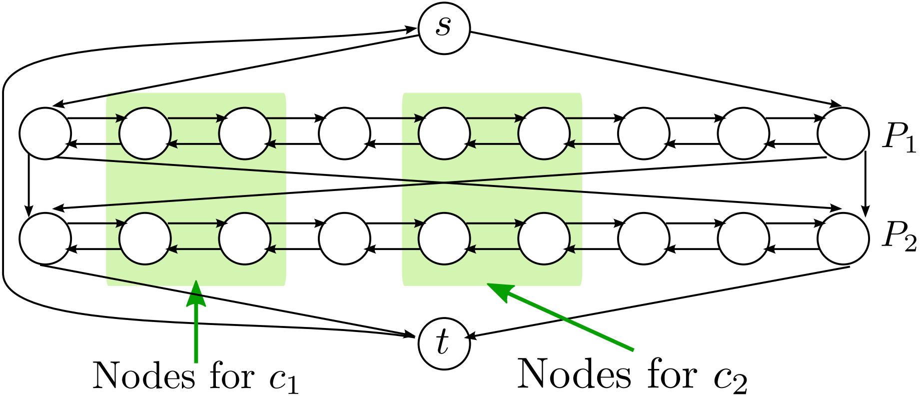

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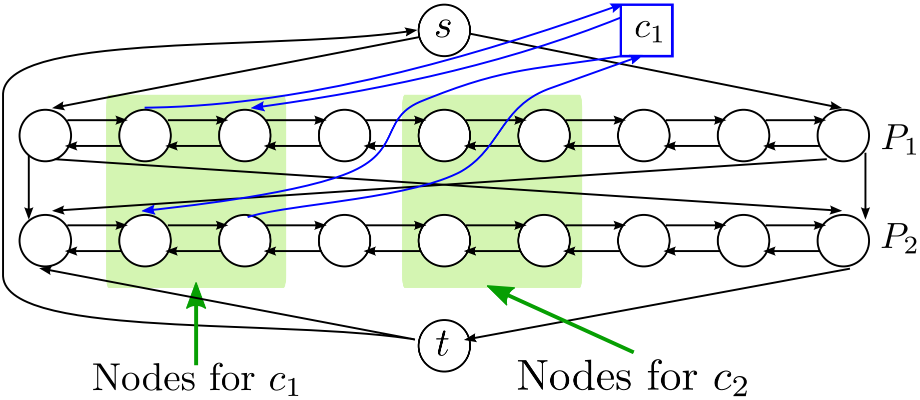

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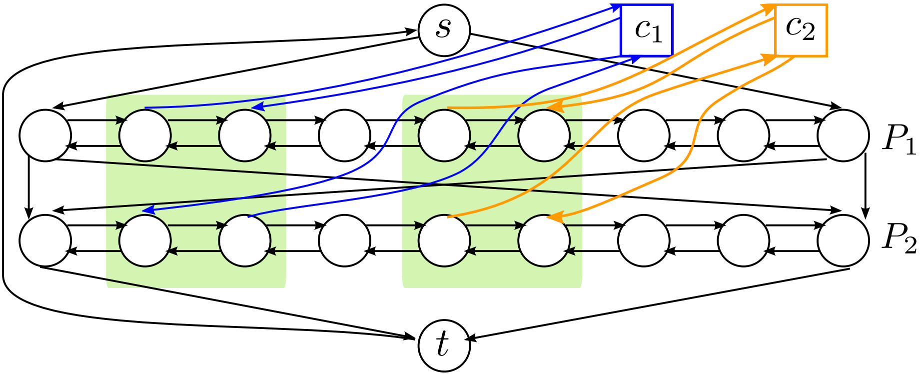

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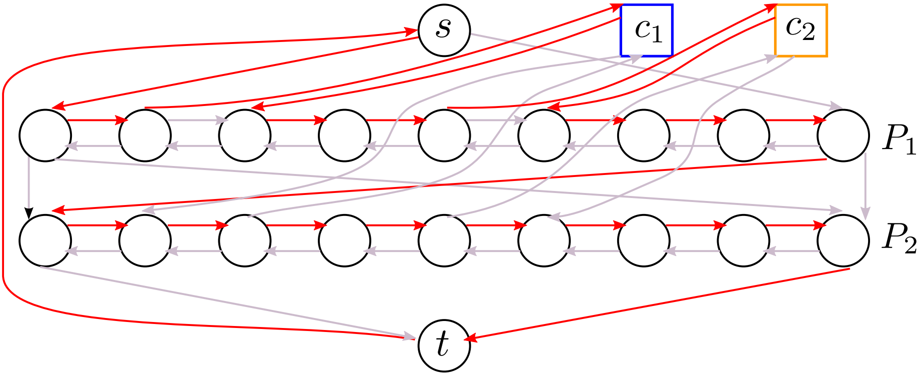

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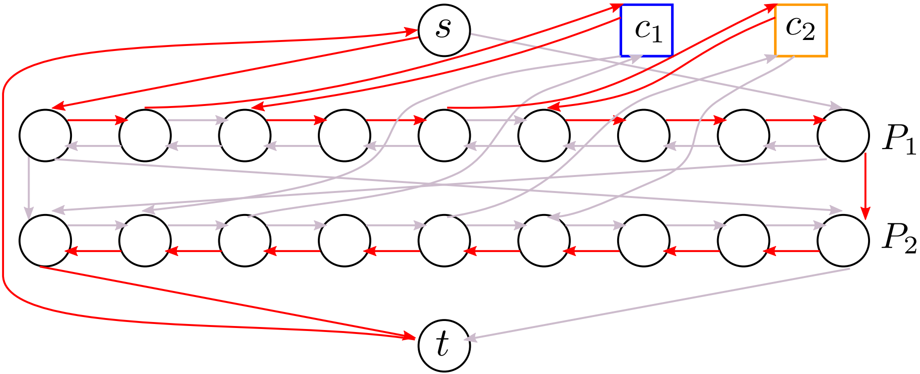
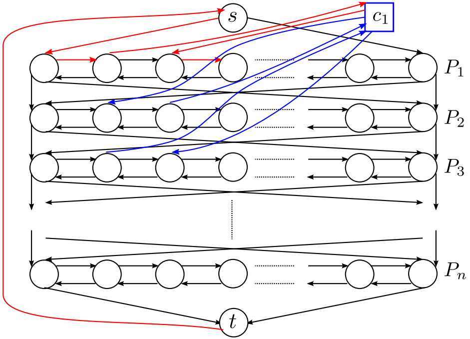


3-SAT $\leq_P$ Hamiltonian Cycle: Proof Part 1



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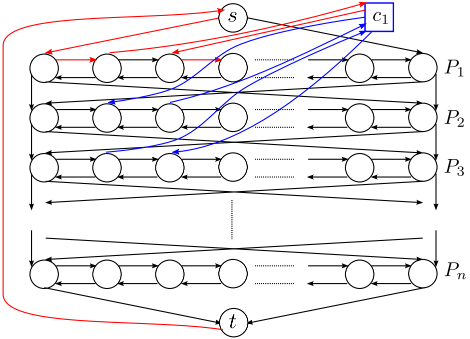
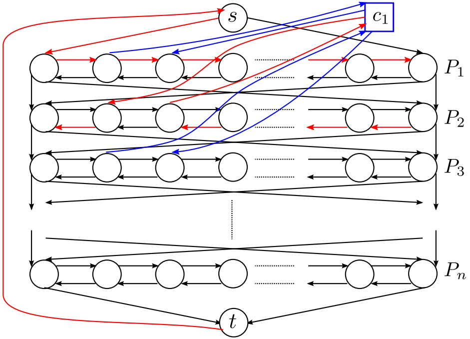


- 3-SAT input is satisfiable $\rightarrow$ G has a Hamiltonian cycle.
  - Construct a Hamiltonian cycle $\mathcal{C}$ as follows:
  - If $x_i = 1$, traverse P_i from left to right in $\mathcal{C}$.
  - Otherwise, traverse P_i from right to left in $\mathcal{C}$.
  - For each clause C_j, there is at least one term set to 1. If the term is x_i, splice c_j into $\mathcal{C}$ using edge from $v_{i,3j}$ and edge to $v_{i,3j+1}$. Analogous construction if term is $\overline{x_i}$.

3-SAT $\leq_P$ Hamiltonian Cycle: Proof Part 2



- G has a Hamiltonian cycle $\mathcal{C} \rightarrow$ Input to 3-SAT is satisfiable.
  - If $\mathcal{C}$ enters c_j on an edge from $v_{i,3j}$, it must leave c_j along the edge to $v_{i,3j+1}$.
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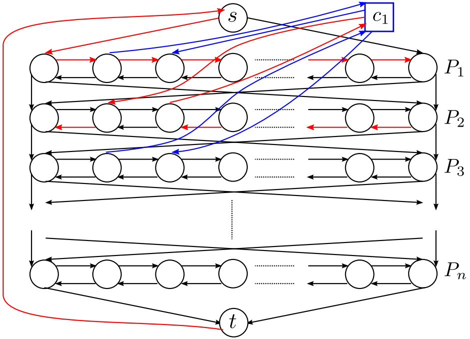

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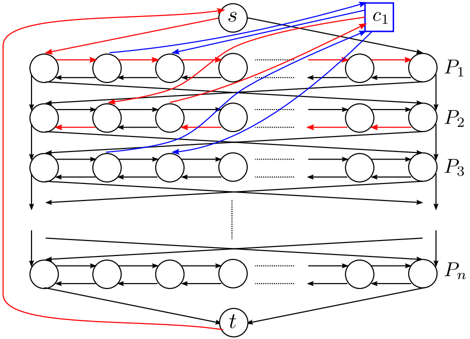

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  - Nodes immediately before and after c_j in $\mathcal{C}$ are themselves connected by an edge e in G.
  - If we remove all such edges e from $\mathcal{C}$, we get a Hamiltonian cycle $\mathcal{C}'$ in $G - \{c_1, c_2, \ldots, c_k\}$.
  - Use $\mathcal{C}'$ to construct truth assignment to variables; prove assignment is satisfying.

The Travelling Salesman Problem

- A salesman must visit n cities $v_1, v_2, \ldots v_n$ starting at home city v_1.
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- Goal is to find as short a tour as possible.
- For every pair of cities v_i and v_j, $d(v_i, v_j) > 0$ is the distance from v_i to v_j.
- A *tour* is a permutation $v_{i_1} = v_1, v_{i_2}, \ldots v_{i_n}$.
- The *length* of the tour is $\sum_{j=1}^{n-1} d(v_{i_j} v_{i_{j+1}}) + d(v_{i_n}, v_{i_1})$.

The Travelling Salesman Problem

- A salesman must visit n cities $v_1, v_2, \ldots v_n$ starting at home city v_1.
- Salesman must find a *tour*, an order in which to visit each city exactly once, and return home.
- Goal is to find as short a tour as possible.
- For every pair of cities v_i and v_j, $d(v_i, v_j) > 0$ is the distance from v_i to v_j.
- A *tour* is a permutation $v_{i_1} = v_1, v_{i_2}, \ldots v_{i_n}$.
- The *length* of the tour is $\sum_{j=1}^{n-1} d(v_{i_j} v_{i_{j+1}}) + d(v_{i_n}, v_{i_1})$.

TRAVELLING SALESMAN

INSTANCE: A set V of n cities, a function $d : V \times V \to \mathbb{R}^+$, and a number $D > 0$.

QUESTION: Is there a tour of length at most D?

Travelling Salesman is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?
- Why is the problem $\mathcal{NP}$-Complete?

Travelling Salesman is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?
- Why is the problem $\mathcal{NP}$-Complete?
- Claim: $\textsc{Hamiltonian Cycle} \leq_P \textsc{Travelling Salesman}$.

Travelling Salesman is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?
- Why is the problem $\mathcal{NP}$-Complete?
- Claim: HAMILTONIAN CYCLE $\leq_P$ TRAVELLING SALESMAN.

HAMILTONIAN CYCLE	TRAVELLING SALESMAN
Directed graph $G(V, E)$	Cities
Edges have identical weights	Distances between cities can vary
Not all pairs of nodes are connected in G	*Every pair* of cities has a distance
(u, v) and (v, u) may both be edges	$d(v_i, v_j) \neq d(v_j, v_i)$, in general
Does a cycle exist?	Does a tour of length $\leq D$ exist?

Travelling Salesman is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?
- Why is the problem $\mathcal{NP}$-Complete?
- Claim: Hamiltonian Cycle $\leq_P$ Travelling Salesman.

Hamiltonian Cycle	Travelling Salesman
Directed graph $G(V, E)$	Cities
Edges have identical weights	Distances between cities can vary
Not all pairs of nodes are connected in G	*Every pair* of cities has a distance
(u, v) and (v, u) may both be edges	$d(v_i, v_j) \neq d(v_j, v_i)$, in general
Does a cycle exist?	Does a tour of length $\leq D$ exist?

- Given a directed graph $G(V, E)$ (input to Hamiltonian Cycle),
  - Create a city v_i for each node $i \in V$.
  - Define $d(v_i, v_j) = 1$ if $(i, j) \in E$.
  - Define $d(v_i, v_j) = 2$ if $(i, j) \notin E$.

Travelling Salesman is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?
- Why is the problem $\mathcal{NP}$-Complete?
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Hamiltonian Cycle	Travelling Salesman
Directed graph $G(V, E)$	Cities
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Not all pairs of nodes are connected in G	*Every pair* of cities has a distance
(u, v) and (v, u) may both be edges	$d(v_i, v_j) \neq d(v_j, v_i)$, in general
Does a cycle exist?	Does a tour of length $\leq D$ exist?

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- Claim: G has a Hamiltonian cycle iff the input to Travelling Salesman has a tour of length at most

Travelling Salesman is $\mathcal{NP}$-Complete

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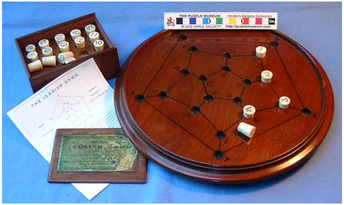
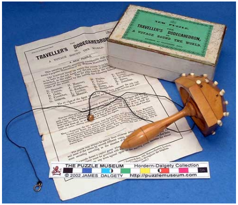
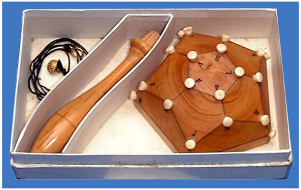
Hamiltonian Cycle	Travelling Salesman
Directed graph $G(V, E)$	Cities
Edges have identical weights	Distances between cities can vary
Not all pairs of nodes are connected in G	*Every pair* of cities has a distance
(u, v) and (v, u) may both be edges	$d(v_i, v_j) \neq d(v_j, v_i)$, in general
Does a cycle exist?	Does a tour of length $\leq D$ exist?

- Given a directed graph $G(V, E)$ (input to Hamiltonian Cycle),
  - Create a city v_i for each node $i \in V$.
  - Define $d(v_i, v_j) = 1$ if $(i, j) \in E$.
  - Define $d(v_i, v_j) = 2$ if $(i, j) \notin E$.
- Claim: G has a Hamiltonian cycle iff the input to Travelling Salesman has a tour of length at most n.

Special Cases and Extensions that are $\mathcal{NP}$-Complete

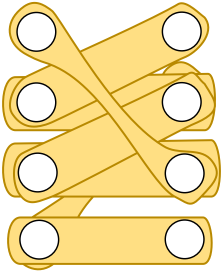






- Hamiltonian Cycle for undirected graphs.
- Hamiltonian Path for directed and undirected graphs.
- Travelling Salesman with symmetric distances (by reducing Hamiltonian Cycle for undirected graphs to it).
- Travelling Salesman with distances defined by points on the plane.

2-Dimensional Matching

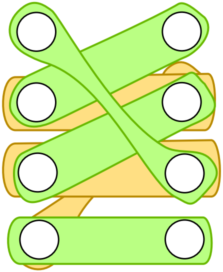


Bipartite Matching

INSTANCE: Disjoint sets X, Y, each of size n, and a set $T \subseteq X \times Y$ of pairs

QUESTION: Is there a set of n pairs in T such that each element of $X \cup Y$ is contained in exactly one of these pairs?

3-Dimensional Matching



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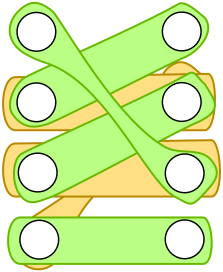
3-Dimensional Matching



- 3-DIMENSIONAL MATCHING is a harder version of BIPARTITE MATCHING.

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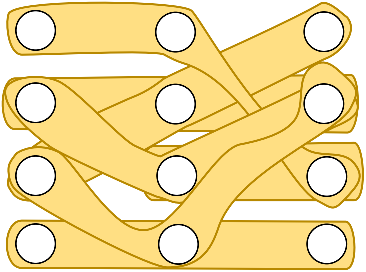


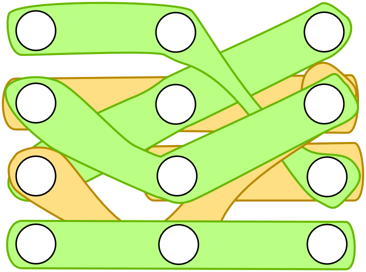
- 3-DIMENSIONAL MATCHING is a harder version of BIPARTITE MATCHING.

  3-DIMENSIONAL MATCHING

  INSTANCE: Disjoint sets X, Y, and Z, each of size n, and a set $T \subseteq X \times Y \times Z$ of triples

  QUESTION: Is there a set of n triples in T such that each element of $X \cup Y \cup Z$ is contained in exactly one of these triples?

3-Dimensional Matching



- 3-Dimensional Matching is a harder version of Bipartite Matching.

  3-Dimensional Matching

  INSTANCE: Disjoint sets X, Y, and Z, each of size n, and a set $T \subseteq X \times Y \times Z$ of triples

  QUESTION: Is there a set of n triples in T such that each element of $X \cup Y \cup Z$ is contained in exactly one of these triples?

- Easy to show 3-Dimensional Matching $\leq_P$ Set Cover and 3-Dimensional Matching $\leq_P$ Set Packing.

3-Dimensional Matching is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?

3-Dimensional Matching is $\mathcal{NP}$-Complete

- Why is the problem in $\mathcal{NP}$?
- Show that 3-SAT $\leq_P$ 3-DIMENSIONAL MATCHING. Jump to Colouring
- Strategy:
  - Start with an input to 3-SAT with n variables and k clauses.
  - Create a gadget for each variable x_i that encodes the choice of truth assignment to x_i.
  - Add gadgets that encode constraints imposed by clauses.

3-SAT $\leq_P$ 3-Dimensional Matching: Variables

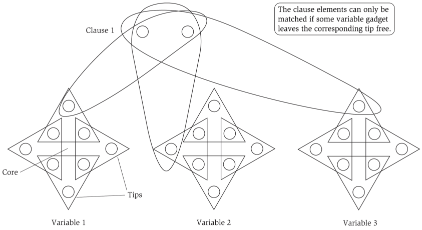
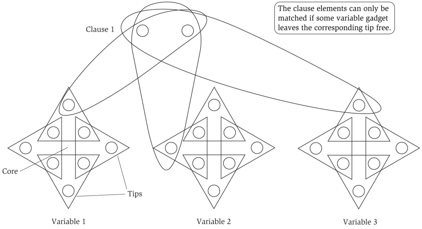


Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

- Each x_i corresponds to a *variable gadget* i with $2k$ *core* elements $A_i = \{a_{i,1}, a_{i,2}, \ldots a_{i,2k}\}$ and $2k$ *tips* $B_i = \{b_{i,1}, b_{i,2}, \ldots b_{i,2k}\}$.
- For each $1 \leq j \leq 2k$, variable gadget i includes a triple $t_{ij} = (a_{i,j}, a_{i,j+1}, b_{i,j})$.
- A triple (tip) is *even* if j is even. Otherwise, the triple (tip) is *odd*.
- Only these triples contain elements in A_i.

3-SAT $\leq_P$ 3-Dimensional Matching: Variables



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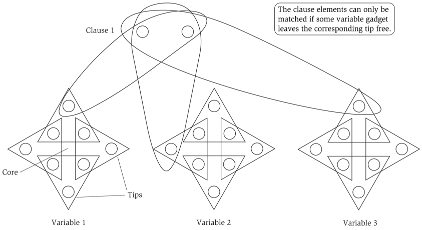
- A triple (tip) is *even* if j is even. Otherwise, the triple (tip) is *odd*.
- Only these triples contain elements in A_i.
- In any perfect matching, we can cover the elements in A_i

3-SAT $\leq_P$ 3-Dimensional Matching: Variables



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- For each $1 \leq j \leq 2k$, variable gadget i includes a triple $t_{ij} = (a_{i,j}, a_{i,j+1}, b_{i,j})$.
- A triple (tip) is *even* if j is even. Otherwise, the triple (tip) is *odd*.
- Only these triples contain elements in A_i.
- In any perfect matching, we can cover the elements in A_i either using all the even triples in gadget i or all the odd triples in the gadget.
- Even triples used, odd tips free $\equiv x_i = 0$; odd triples used, even tips free $\equiv x_i = 1$.

3-SAT $\leq_P$ 3-Dimensional Matching: Clauses

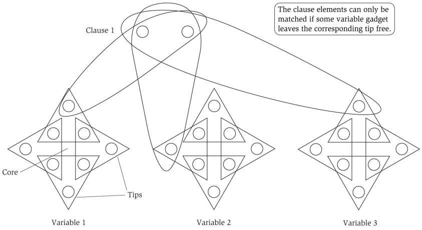

Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.

3-SAT $\leq_P$ 3-Dimensional Matching: Clauses

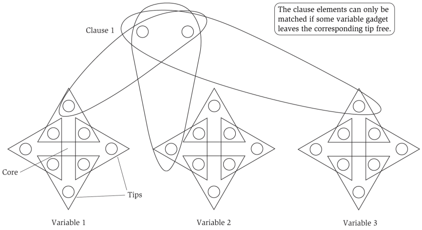

Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.
- C_1 says "The matching on the cores of the gadgets should leave the even tips of gadget 1 free; or it should leave the odd tips of gadget 2 free; or it should leave the even tips of gadget 3 free."

3-SAT $\leq_P$ 3-Dimensional Matching: Clauses

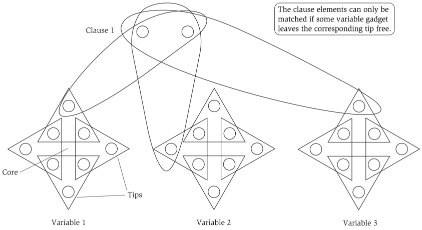

Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

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- C_1 says "The matching on the cores of the gadgets should leave the even tips of gadget 1 free; or it should leave the odd tips of gadget 2 free; or it should leave the even tips of gadget 3 free."
- *Clause gadget* j for clause C_j contains two core elements $P_j = \{p_j, p'_j\}$ and three triples:
  - C_j contains x_i: add triple $(p_j, p'_j, b_{i,2j})$.
  - C_j contains $\overline{x_i}$: add triple $(p_j, p'_j, b_{i,2j-1})$.

3-SAT $\leq_P$ 3-Dimensional Matching: Example

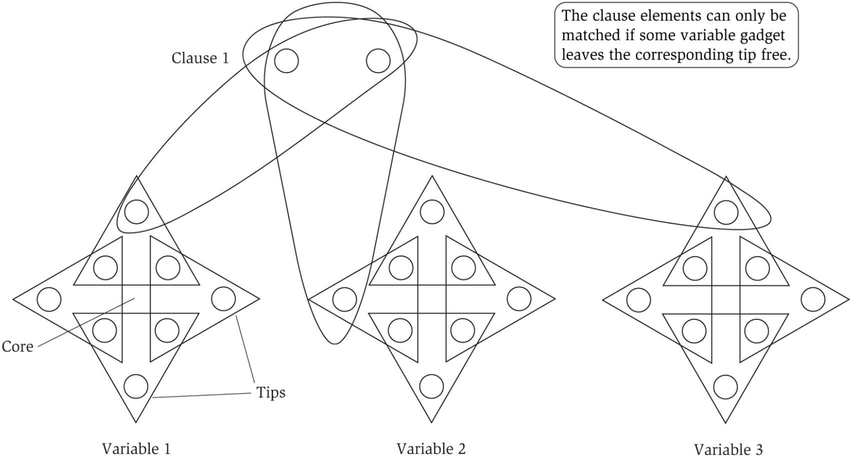

Figure 8.9 The reduction from 3-SAT to 3-Dimensional Matching.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.
  - Make appropriate choices for the core of each variable gadget.
  - At least one free tip available for each clause gadget, allowing core elements of clause gadgets to be covered.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.
  - Make appropriate choices for the core of each variable gadget.
  - At least one free tip available for each clause gadget, allowing core elements of clause gadgets to be covered.
  - We have not covered all the tips!

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

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  - Make appropriate choices for the core of each variable gadget.
  - At least one free tip available for each clause gadget, allowing core elements of clause gadgets to be covered.
  - We have not covered all the tips!
  - Add $(n-1)k$ *cleanup gadgets* to allow the remaining $(n-1)k$ tips to be covered: cleanup gadget i contains two core elements $Q = \{q_i, q'_i\}$ and triple (q_i, q'_i, b) for every tip b in variable gadget i.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.
  - Make appropriate choices for the core of each variable gadget.
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- Matching $\rightarrow$ satisfying assignment.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.
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- Matching $\rightarrow$ satisfying assignment.
  - Matching chooses all even a_{ij} $(x_i = 0)$ or all odd a_{ij} $(x_i = 1)$.

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.
  - Make appropriate choices for the core of each variable gadget.
  - At least one free tip available for each clause gadget, allowing core elements of clause gadgets to be covered.
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- Matching $\rightarrow$ satisfying assignment.
  - Matching chooses all even a_{ij} $(x_i = 0)$ or all odd a_{ij} $(x_i = 1)$.
  - Is clause C_j satisfied?

3-SAT $\leq_P$ 3-Dimensional Matching: Proof

- Satisfying assignment $\rightarrow$ matching.
  - Make appropriate choices for the core of each variable gadget.
  - At least one free tip available for each clause gadget, allowing core elements of clause gadgets to be covered.
  - We have not covered all the tips!
  - Add $(n-1)k$ *cleanup gadgets* to allow the remaining $(n-1)k$ tips to be covered: cleanup gadget i contains two core elements $Q = \{q_i, q_i'\}$ and triple (q_i, q_i', b) for every tip b in variable gadget i.
- Matching $\rightarrow$ satisfying assignment.
  - Matching chooses all even a_{ij} $(x_i = 0)$ or all odd a_{ij} $(x_i = 1)$.
  - Is clause C_j satisfied? Core in clause gadget j is covered by some triple $\Rightarrow$ other element in the triple must be a tip element from the correct odd/even set in the three variable gadgets corresponding to a term in C_j.

3-SAT $\leq_P$ 3-Dimensional Matching: Finale

- Did we create an input to 3-DIMENSIONAL MATCHING?

3-SAT $\leq_P$ 3-Dimensional Matching: Finale

- Did we create an input to 3-DIMENSIONAL MATCHING?
- We need three sets X, Y, and Z of equal size.

3-SAT $\leq_P$ 3-Dimensional Matching: Finale

- Did we create an input to 3-DIMENSIONAL MATCHING?
- We need three sets X, Y, and Z of equal size.
- How many elements do we have?
  - $2nk$ a_{ij} elements.
  - $2nk$ b_{ij} elements.
  - k p_j elements.
  - k p'_j elements.
  - $(n-1)k$ q_i elements.
  - $(n-1)k$ q'_i elements.

3-SAT $\leq_P$ 3-Dimensional Matching: Finale

- Did we create an input to 3-DIMENSIONAL MATCHING?
- We need three sets X, Y, and Z of equal size.
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  - $2nk$ a_{ij} elements.
  - $2nk$ b_{ij} elements.
  - k p_j elements.
  - k p'_j elements.
  - $(n-1)k$ q_i elements.
  - $(n-1)k$ q'_i elements.
- X is the union of a_{ij} with even j, the set of all p_j and the set of all q_i.
- Y is the union of a_{ij} with odd j, the set if all p'_j and the set of all q'_i.
- Z is the set of all b_{ij}.

3-SAT $\leq_P$ 3-Dimensional Matching: Finale

- Did we create an input to 3-Dimensional Matching?
- We need three sets X, Y, and Z of equal size.
- How many elements do we have?
  - $2nk$ a_{ij} elements.
  - $2nk$ b_{ij} elements.
  - k p_j elements.
  - k p'_j elements.
  - $(n-1)k$ q_i elements.
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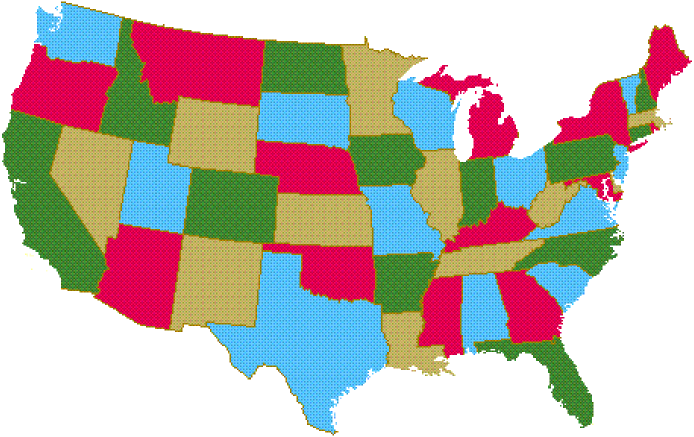
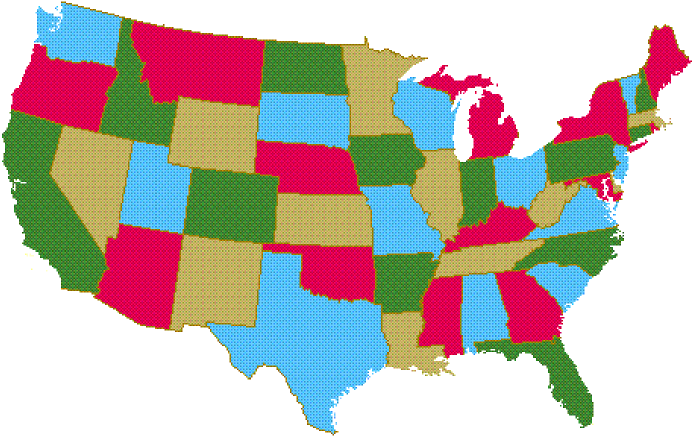
- X is the union of a_{ij} with even j, the set of all p_j and the set of all q_i.
- Y is the union of a_{ij} with odd j, the set if all p'_j and the set of all q'_i.
- Z is the set of all b_{ij}.
- Each triple contains exactly one element from X, Y, and Z.

Colouring maps

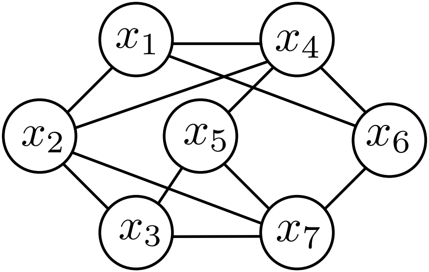


Colouring maps



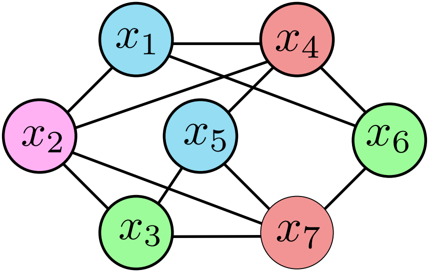
- Any map can be coloured with four colours (Appel and Hakken, 1976).

Graph Colouring



- Given an undirected graph $G(V, E)$, a *k-colouring* of G is a function $f : V \to \{1, 2, \ldots k\}$ such that for every edge $(u, v) \in E$, $f(u) \neq f(v)$.

Graph Colouring



- Given an undirected graph $G(V, E)$, a *k-colouring* of G is a function $f : V \rightarrow \{1, 2, \ldots k\}$ such that for every edge $(u, v) \in E$, $f(u) \neq f(v)$.

GRAPH COLOURING (k-COLOURING)

INSTANCE: An undirected graph $G(V, E)$ and an integer $k > 0$.

QUESTION: Does G have a k-colouring?

Applications of Graph Colouring

1. Job scheduling: assign jobs to n processors under constraints that certain pairs of jobs cannot be scheduled at the same time.
2. Compiler design: assign variables to k registers but two variables being used at the same time cannot be assigned to the same register.
3. Wavelength assignment: assign one of k transmitting wavelengths to each of n wireless devices. If two devices are close to each other, they must get different wavelengths.

2-Colouring

- How hard is 2-COLOURING?

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- Claim: A graph is 2-colourable if and only if it is bipartite.

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- Testing 2-colourability is possible in $O(|V| + |E|)$ time.

2-Colouring

- How hard is 2-COLOURING?
- Claim: A graph is 2-colourable if and only if it is bipartite.
- Testing 2-colourability is possible in $O(|V| + |E|)$ time.
- What about 3-COLOURING? Is it easy to exhibit a certificate that a graph *cannot* be coloured with three colours?

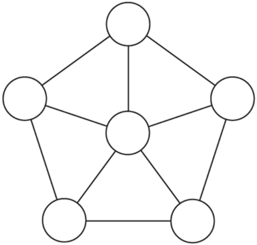

Figure 8.10 A graph that is not 3-colorable.

3-Colouring is $\mathcal{NP}$-Complete

- Why is 3-Colouring in $\mathcal{NP}$?

3-Colouring is $\mathcal{NP}$-Complete

- Why is 3-Colouring in $\mathcal{NP}$?
- 3-SAT $\leq_P$ 3-COLOURING. Jump to other problems

3-SAT $\leq_P$ 3-Colouring: Encoding Variables

3-SAT	3-COLOURING
Boolean variables	
True or *False*	
Clauses	
Is there a satisfying assignment?	Does a 3-colouring exist?

3-SAT $\leq_P$ 3-Colouring: Encoding Variables

3-SAT	3-COLOURING
Boolean variables	Nodes
True or *False*	Colours called *True*, *False*, and *Base*
Clauses	
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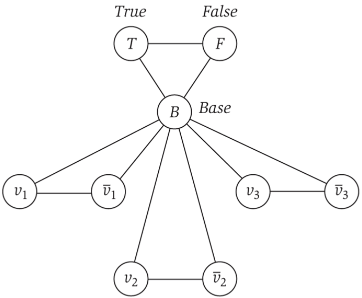

Figure 8.11 The beginning of the reduction for 3-Coloring.

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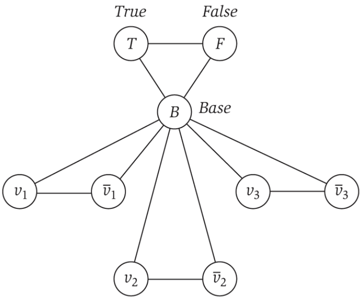

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- x_i corresponds to node v_i and $\overline{x_i}$ corresponds to node $\overline{v_i}$.

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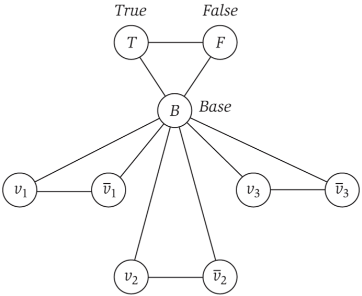


Figure 8.11 The beginning of the reduction for 3-Coloring.

- x_i corresponds to node v_i and $\overline{x_i}$ corresponds to node $\overline{v_i}$.
- In any 3-Colouring, nodes v_i and $\overline{v_i}$ get a colour different from *Base*.
- *True colour*: colour assigned to the *True* node; *False colour*: colour assigned to the *False* node.
- Set x_i to 1 iff v_i gets the *True* colour.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses

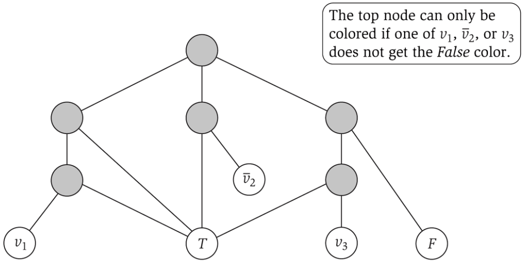
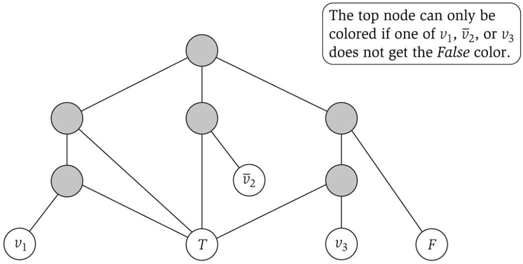


Figure 8.12 Attaching a subgraph to represent the clause $x_1 \vee \overline{x}_2 \vee x_3$.

- Consider the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3$.
- Attach a six-node subgraph for this clause to the rest of the graph.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses



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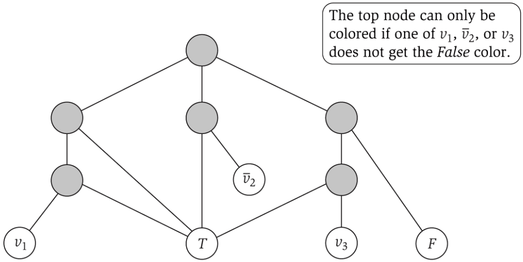

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- Claim: If all of of v_1, $\overline{v_2}$, or v_3 get the *False* colour, then the top node in the subgraph cannot be coloured in a 3-colouring.

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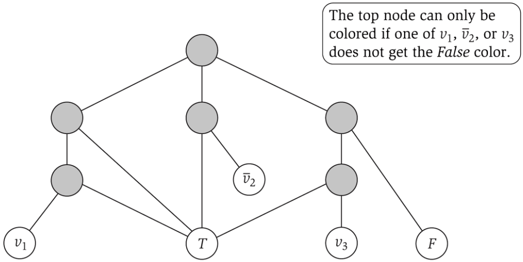

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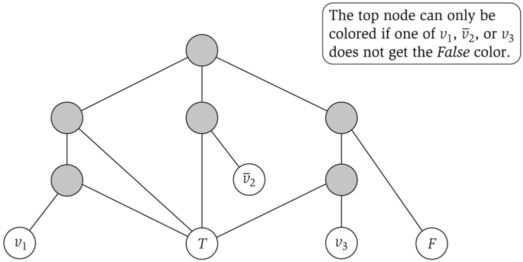
- Claim: If at least one of v_1, $\overline{v_2}$, or v_3 does not get the *False* colour, then the top node in the subgraph can be coloured in a 3-colouring.

3-SAT $\leq_P$ 3-Colouring: Encoding Clauses



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- Claim: If at least one of v_1, $\overline{v_2}$, or v_3 does not get the *False* colour, then the top node in the subgraph can be coloured in a 3-colouring.
- Claim: Graph is 3-colourable iff input to 3-SAT is satisfiable.

Subset Sum

SUBSET SUM

INSTANCE: A set of n natural numbers $w_1, w_2, \ldots, w_n$ and a target W.

QUESTION: Is there a subset of $\{w_1, w_2, \ldots, w_n\}$ whose sum is W?

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- Claim: SUBSET SUM is $\mathcal{NP}$-Complete,
  3-DIMENSIONAL MATCHING $\leq_P$ SUBSET SUM.
- Caveat: Special case of SUBSET SUM in which W is bounded by a polynomial function of n is not $\mathcal{NP}$-Complete (read pages 494–495 of your textbook).

Examples of Hard Computational Problems

Examples of Hard Computational Problems

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Computer Science > Computational Complexity

Bejeweled, Candy Crush and other Match-Three Games are (NP-)Hard

Luciano Gualà, Stefano Leucci, Emanuele Natale

(Submitted on 24 Mar 2014)

The twentieth century has seen the rise of a new type of video games targeted at a mass audience of "casual" gamers. Many of these games require the player to swap items in order to form matches of three and are collectively known as \emph{tile-matching match-three games}. Among these, the most influential one is arguably \emph{Bejeweled} in which the matched items (gems) pop and the above gems fall in their place. Bejeweled has been ported to many different platforms and influenced an incredible number of similar games. Very recently one of them, named \emph{Candy Crush Saga} enjoyed a huge popularity and quickly went viral on social networks. We generalize this kind of games by only parameterizing the size of the board, while all the other elements (such as the rules or the number of gems) remain unchanged. Then, we prove that answering many natural questions regarding such games is actually \NP-Hard. These questions include determining if the player can reach a certain score, play for a certain number of turns, and others.

Examples of Hard Computational Problems

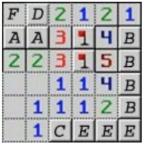
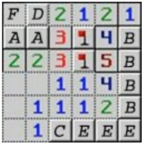


Fig.1 A Typical Minesweeper Position

Fig.2 Impossible Minesweeper position.

Examples of Hard Computational Problems



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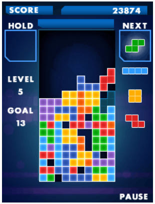
RICHARD KAYE

Minesweeper is NP-complete

Examples of Hard Computational Problems



Computer Science > Computational Complexity

Tetris is Hard, Even to Approximate

Erik D. Demaine, Susan Hohenberger, David Liben-Nowell

(Submitted on 21 Oct 2002)

In the popular computer game of Tetris, the player is given a sequence of tetromino pieces and must pack them into a rectangular gameboard initially occupied by a given configuration of filled squares; any completely filled row of the gameboard is cleared and all pieces above it drop by one row. We prove that in the offline version of Tetris, it is NP-complete to maximize the number of cleared rows, maximize the number of tetrises (quadruples of rows simultaneously filled and cleared), minimize the maximum height of an occupied square, or maximize the number of pieces placed before the game ends. We furthermore show the extreme inapproximability of the first and last of these objectives to within a factor of p^(1-epsilon), when given a sequence of p pieces, and the inapproximability of the third objective to within a factor of (2 - epsilon), for any epsilon>0. Our results hold under several variations on the rules of Tetris, including different models of rotation, limitations on player agility, and restricted piece sets.

More Examples of Hard Computational Problems

(*from Kevin Wayne's slides at Princeton University*)

- Aerospace engineering: optimal mesh partitioning for finite elements.
- Biology: protein folding.
- Chemical engineering: heat exchanger network synthesis.
- Civil engineering: equilibrium of urban traffic flow.
- Economics: computation of arbitrage in financial markets with friction.
- Electrical engineering: VLSI layout.
- Environmental engineering: optimal placement of contaminant sensors.
- Financial engineering: find minimum risk portfolio of given return.
- Game theory: find Nash equilibrium that maximizes social welfare.
- Genomics: phylogeny reconstruction.
- Mechanical engineering: structure of turbulence in sheared flows.
- Medicine: reconstructing 3-D shape from biplane angiocardiogram.
- Operations research: optimal resource allocation.
- Physics: partition function of 3-D Ising model in statistical mechanics.
- Politics: Shapley-Shubik voting power.
- Pop culture: Minesweeper consistency.
- Statistics: optimal experimental design.