Coping with NP-Completeness

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Examples of Hard Computational Problems

(*from Kevin Wayne's slides at Princeton University*)

- Aerospace engineering: optimal mesh partitioning for finite elements.
- Biology: protein folding.
- Chemical engineering: heat exchanger network synthesis.
- Civil engineering: equilibrium of urban traffic flow.
- Economics: computation of arbitrage in financial markets with friction.
- Electrical engineering: VLSI layout.
- Environmental engineering: optimal placement of contaminant sensors.
- Financial engineering: find minimum risk portfolio of given return.
- Game theory: find Nash equilibrium that maximizes social welfare.
- Genomics: phylogeny reconstruction.
- Mechanical engineering: structure of turbulence in sheared flows.
- Medicine: reconstructing 3-D shape from biplane angiocardiogram.
- Operations research: optimal resource allocation.
- Physics: partition function of 3-D Ising model in statistical mechanics.
- Politics: Shapley-Shubik voting power.
- Pop culture: Minesweeper consistency.
- Statistics: optimal experimental design.

How Do We Tackle an $\mathcal{NP}$-Complete Problem?

"I can't find an efficient algorithm, but neither can all these famous people."

(Garey and Johnson, *Computers and Intractability*)

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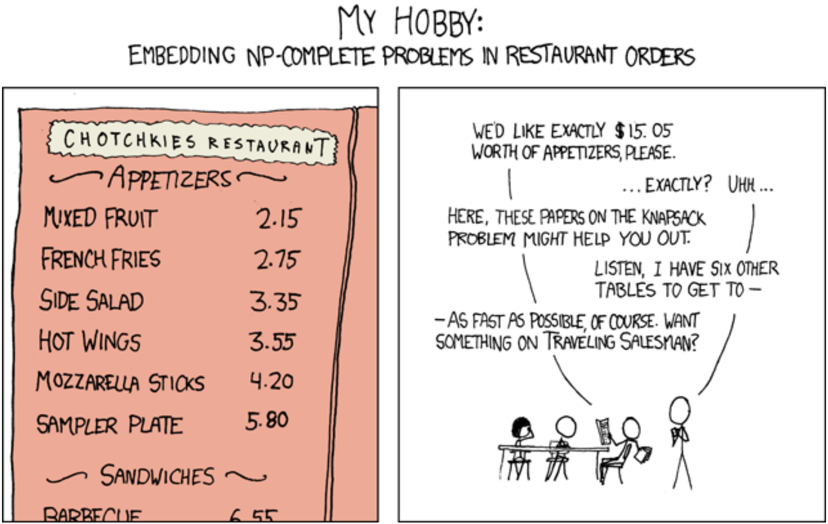

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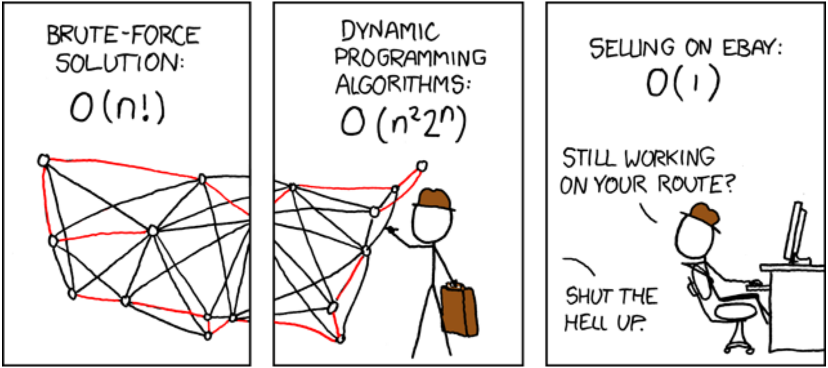

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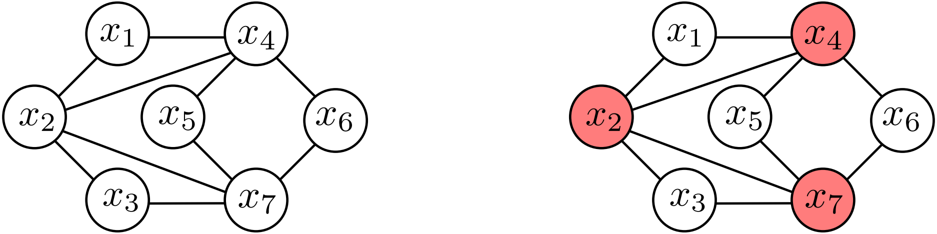
- These problems come up in real life.
- $\mathcal{NP}$-Complete means that a problem is hard to solve in the *worst case*. Can we come up with better solutions at least in *some* cases?
  - Develop algorithms that are exponential in one parameter in the problem.
  - Consider special cases of the input, e.g., graphs that “look like” trees.
  - Develop algorithms that can provably compute a solution close to the optimal.

Vertex Cover Problem



VERTEX COVER

INSTANCE: Undirected graph G and an integer k

QUESTION: Does G contain a vertex cover of size at most k?

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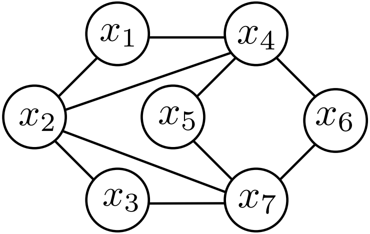

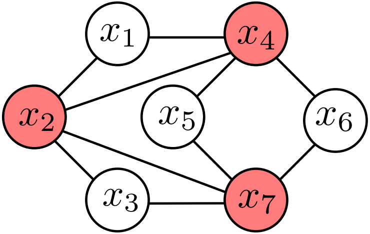

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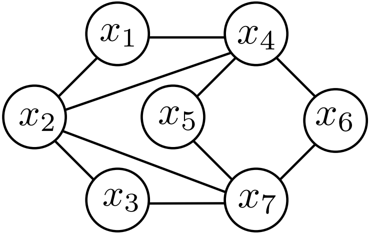

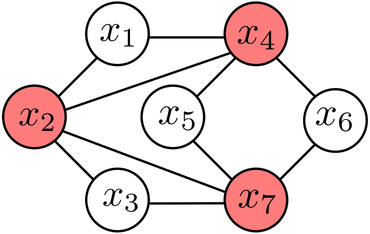

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- Can we devise an algorithm whose running time is exponential in k but polynomial in n, e.g., $O(2^k n)$?

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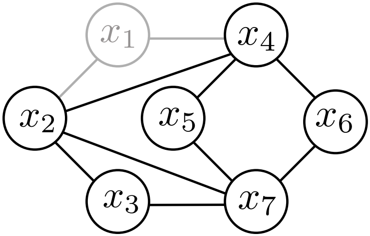

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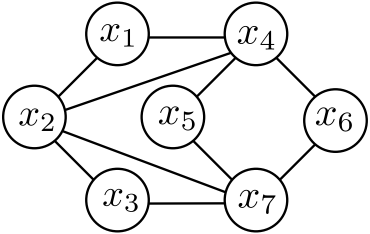

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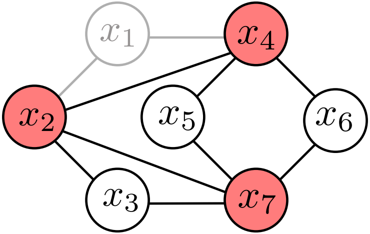
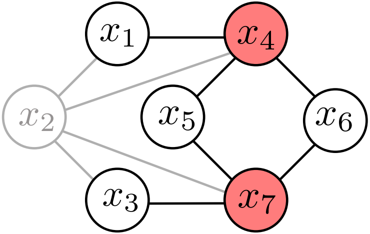
- $G - \{u\}$ is the graph G without node u and the edges incident on u.
- Consider an edge (u, v). Either u or v must be in the vertex cover.
- Claim: G has a vertex cover of size at most k iff for any edge (u, v) either $G - \{u\}$ or $G - \{v\}$ has a vertex cover of size at most $k - 1$.





Vertex Cover Algorithm

```
To search for a k-node vertex cover in G:
  If G contains no edges, then the empty set is a vertex cover
  If G contains > k |V| edges, then it has no k-node vertex cover
  Else let e = (u, v) be an edge of G
    Recursively check if either of G−{u} or G−{v}
                 has a vertex cover of size k − 1
    If neither of them does, then G has no k-node vertex cover
    Else, one of them (say, G−{u}) has a (k − 1)-node vertex cover T
       In this case, T ∪ {u} is a k-node vertex cover of G
    Endif
  Endif
```

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- $T(n, 1) \leq cn$.
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  - We need $O(kn)$ time to count the number of edges.

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  - We need $O(kn)$ time to count the number of edges.
- Claim: $T(n, k) = O(2^k kn)$.

Approximation Algorithms

- Methods for optimisation versions of $\mathcal{NP}$-Complete problems.
- Run in polynomial time.
- Solution returned is guaranteed to be within a small factor of the optimal solution

Approximation Algorithm for VertexCover

EasyVertexCover(G) (Gavril, 1974; Yannakakis)

```
C ← ∅                          {C will be the vertex cover}
while G has at least one edge do
    Let (u, v) be any edge in G
    ________________________   {Update C using u and/or v}
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end while
return C
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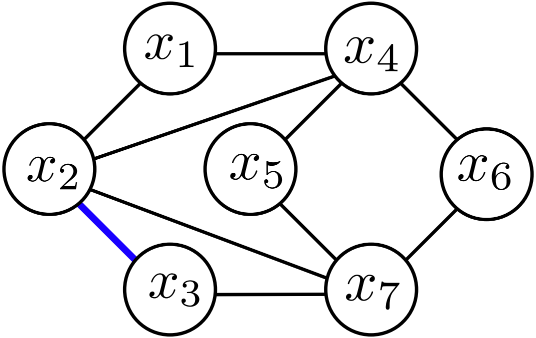

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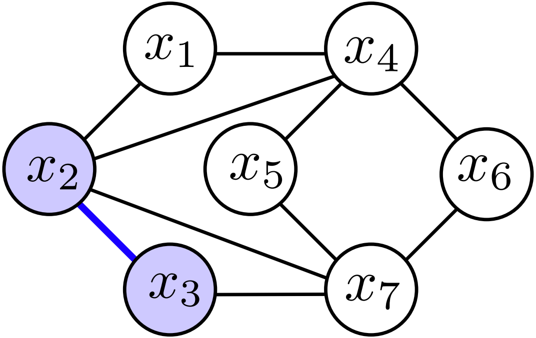

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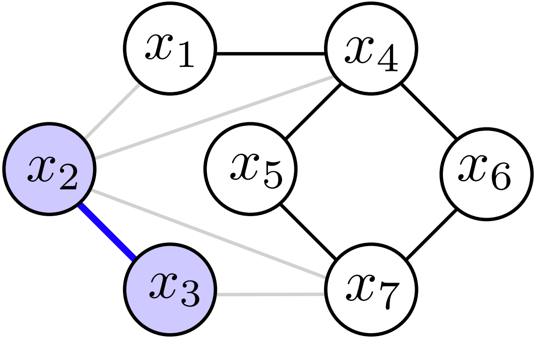

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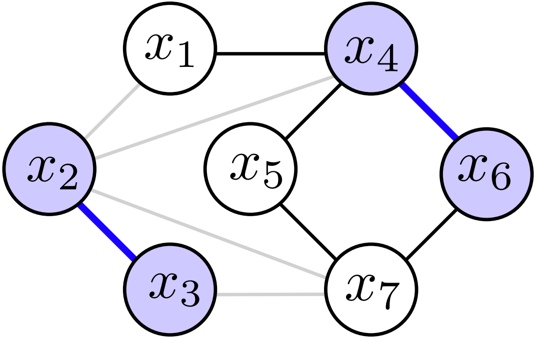

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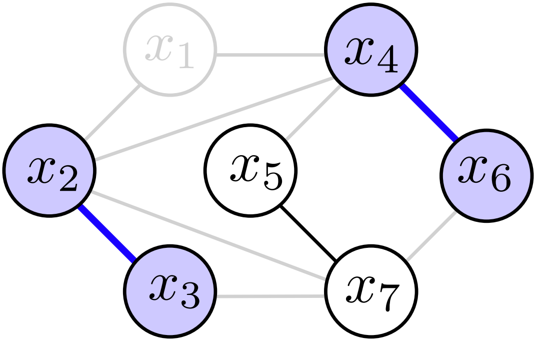

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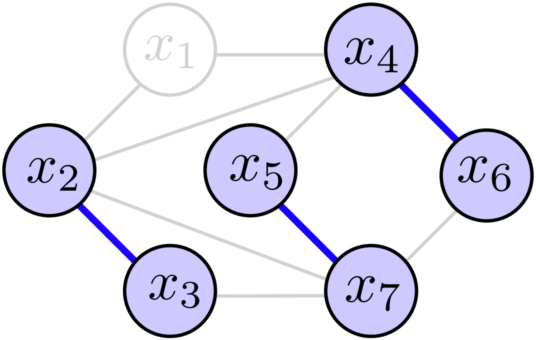

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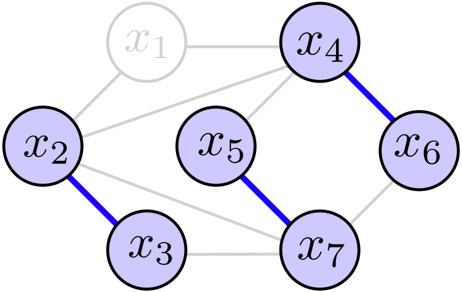

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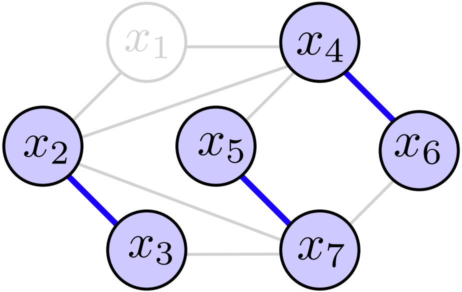


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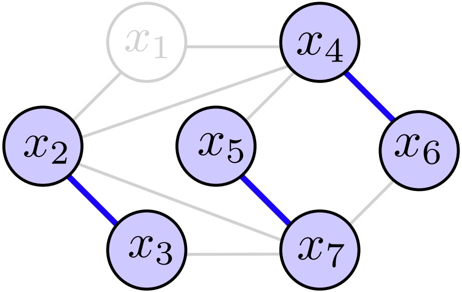

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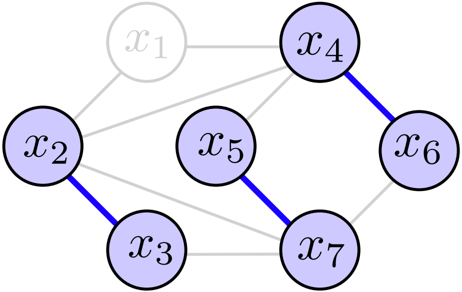


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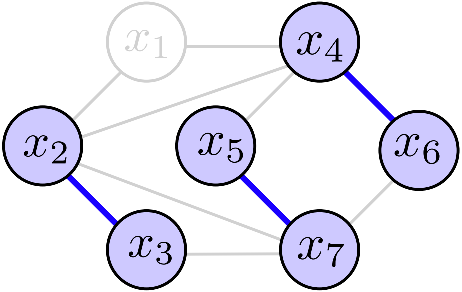

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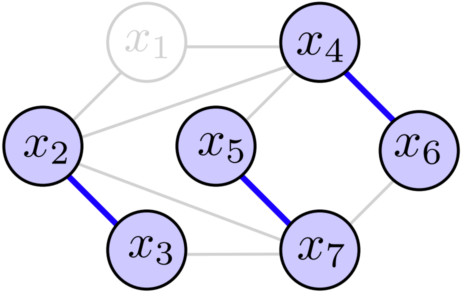

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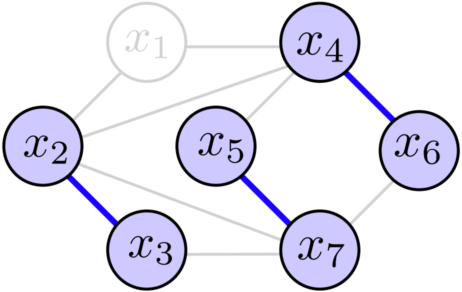


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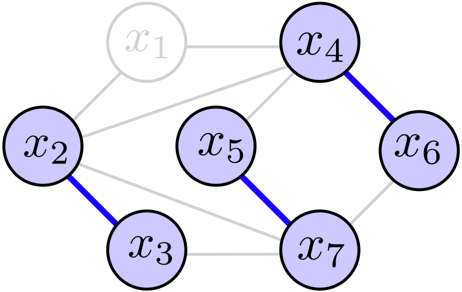

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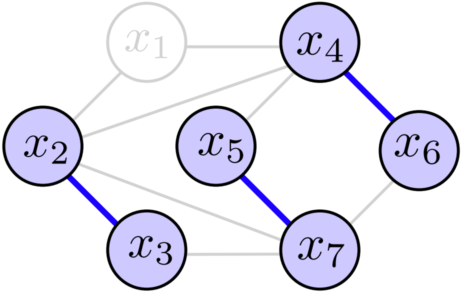

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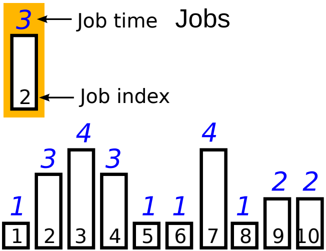
- No approximation algorithm with a factor better than $\sqrt{2} - \varepsilon$ is possible unless $\mathcal{P} = \mathcal{NP}$ (Dinur *et al.*, 2018).
- No approximation algorithm with a factor better than 2 is possible if the "unique games conjecture" is true (Khot and Regev, 2008).

Load Balancing Problem



- Given set of m machines $M_1, M_2, \ldots M_m$.
- Given a set of n jobs: job j has processing time t_j.
- Assign each job to one machine so that the total time spent is minimised.

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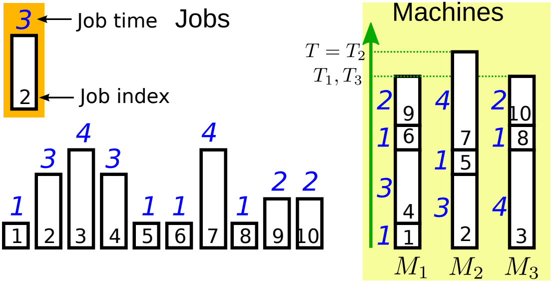

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- Let $A(i)$ be the set of jobs assigned to machine M_i.
- Total time spent on machine i is $T_i = \sum_{k \in A(i)} t_k$.
- Minimise *makespan* $T = \max_i T_i$, the largest load on any machine.

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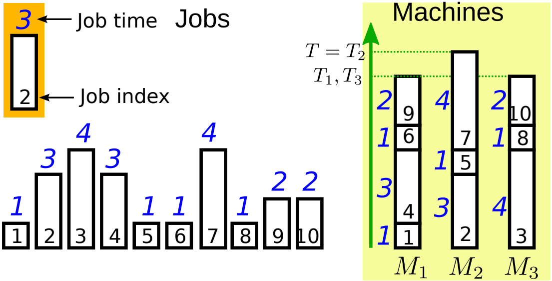

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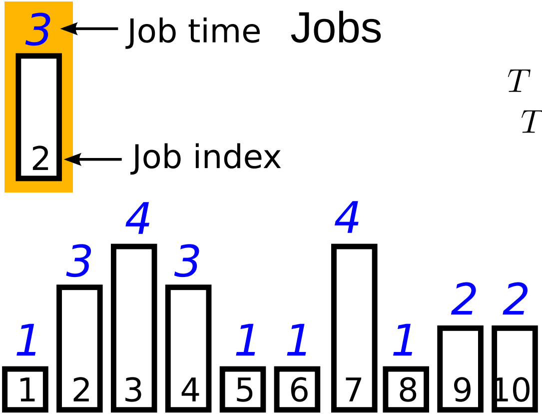
- Minimising makespan is $\mathcal{NP}$-Complete.

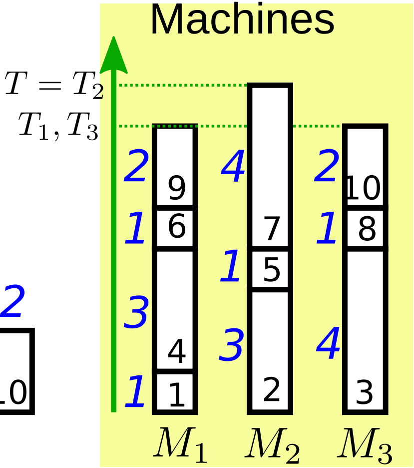
Greedy-Balance Algorithm

- Adopt a greedy approach (Graham, 1966).
- Process jobs in *any* order.
- Assign next job to the processor that has smallest total load so far.

Greedy-Balance:
Start with no jobs assigned
Set $T_i = 0$ and $A(i) = \emptyset$ for all machines M_i
For $j = 1, \ldots, n$
  Let M_i be a machine that achieves the minimum $\min_k T_k$
  Assign job j to machine M_i
  Set $A(i) \leftarrow A(i) \cup \{j\}$
  Set $T_i \leftarrow T_i + t_j$
EndFor

Example of Greedy-Balance Algorithm





Lower Bounds on the Optimal Makespan

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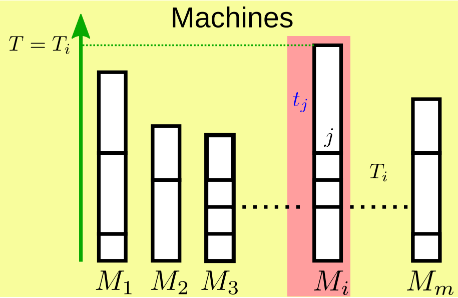
- The two bounds below will suffice:

$$T^* \geq \frac{1}{m} \sum_j t_j$$

$$T^* \geq \max_j t_j$$

Analysing Greedy-Balance



- Claim: Computed makespan $T \leq 2T^*$.

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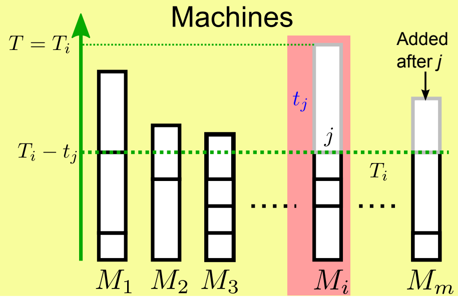

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- Let M_i be the machine whose load is T and j be the last job placed on M_i.
- What was the situation just before placing this job?

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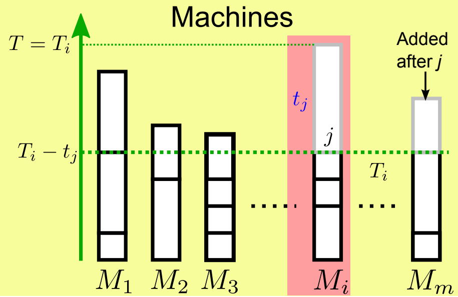

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- Let M_i be the machine whose load is T and j be the last job placed on M_i.
- What was the situation just before placing this job?
- M_i had the smallest load and its load was $T - t_j$.
- For every machine M_k, ▸ Poll

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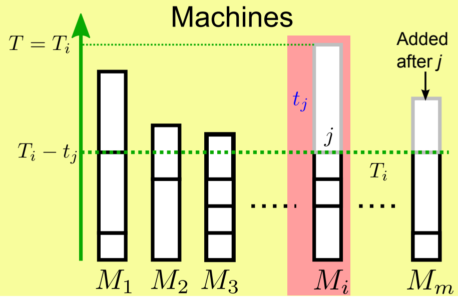

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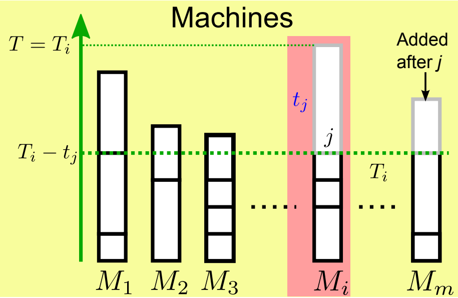

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$$\sum_k T_k \geq m(T - t_j), \text{ where } k \text{ ranges over all machines}$$

$$\sum_j t_j \geq m(T - t_j), \text{ where } j \text{ ranges over all jobs}$$

$$T - t_j \leq 1/m \sum_j t_j \leq T^*$$

$$T \leq 2T^*, \text{ since } t_j \leq T^*$$

Improving the Bound

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- It is easy to construct an example for which the greedy algorithm produces a solution close to a factor of 2 away from optimal.
- How can we improve the algorithm?
- What if we process the jobs in decreasing order of processing time? (Graham, 1969)

Sorted-Balance Algorithm

```
Sorted-Balance:
Start with no jobs assigned
Set T_i = 0 and A(i) = ∅ for all machines M_i
Sort jobs in decreasing order of processing times t_j
Assume that t_1 ≥ t_2 ≥ ... ≥ t_n
For j = 1, ..., n
  Let M_i be the machine that achieves the minimum min_k T_k
  Assign job j to machine M_i
  Set A(i) ← A(i) ∪ {j}
  Set T_i ← T_i + t_j
EndFor
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- This algorithm assigns the first m jobs to m distinct machines.

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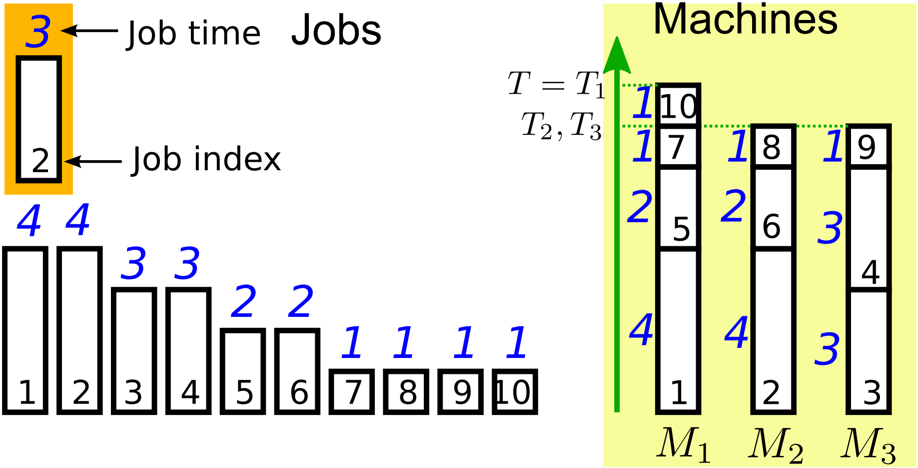

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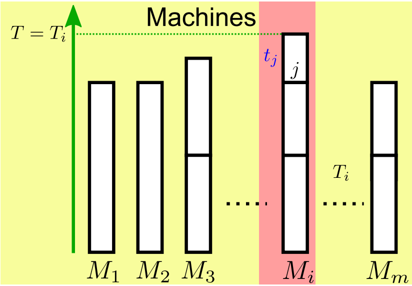

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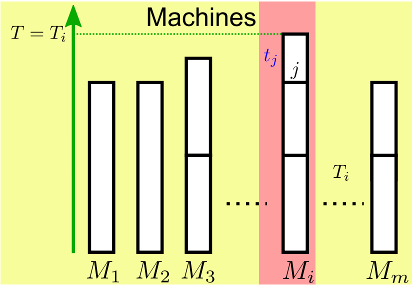

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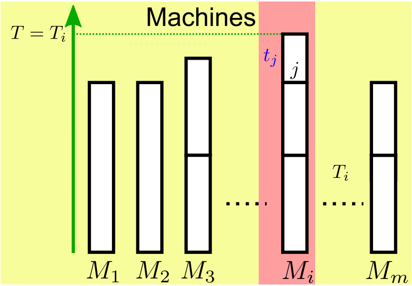

- Better bound: $T \leq 4T^*/3$ (Graham, 1969).

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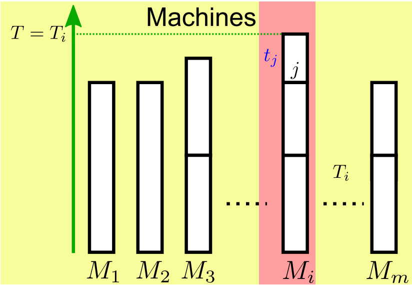

- Better bound: $T \leq 4T^*/3$ (Graham, 1969).
  - *Polynomial-time approximation scheme*: for every $\varepsilon > 0$, compute solution with makespan $T \leq (1+\varepsilon)T^*$ in $O((n/\varepsilon)^{(1/\varepsilon^2)})$ time (Hochbaum and Shmoys, 1987).

The Knapsack Problem

PARTITION

INSTANCE: A set of n natural numbers $w_1, w_2, \ldots, w_n$.

SOLUTION: A subset S of numbers such that $\sum_{i \in S} w_i = \sum_{i \notin S} w_i$.

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- 3D MATCHING $\leq_P$ PARTITION $\leq_P$ SUBSET SUM $\leq_P$ KNAPSACK
- All problems have dynamic programming algorithms with pseudo-polynomial running times.

Dynamic Programming for Subset Sum

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$$\text{OPT}(i, w) = \text{OPT}(i-1, w), \qquad i > 0, w_i > w$$

$$\text{OPT}(i, w) = \max\left(\text{OPT}(i-1, w), w_i + \text{OPT}(i-1, w - w_i)\right), \qquad i > 0, w_i \leq w$$

$$\text{OPT}(0, w) = 0$$

- Running time is $O(nW)$.

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- Can generalize the dynamic program for SUBSET SUM.
- But we will develop a different dynamic program that will be useful later.
- $OPT(i, v)$ is the smallest knapsack weight so that there is a solution with total value $\geq v$ that uses only the first i items.
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  - i ranges between 0 and n, the number of items.
  - Given i, v ranges between 0 and $\sum_{1 \leq j \leq i} v_j$.
  - Largest value of v is $\sum_{1 \leq j \leq n} v_j \leq nv^*$, where $v^* = \max_i v_i$.
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- Can find items in the solution by tracing back.
- Running time is $O(n^2 v^*)$, which is pseudo-polynomial in the input size.

Intuition Underlying Approximation Algorithm

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- What is the running time if all values are the same? Polynomial.
- What is the running time if all values are small integers? Also polynomial.
- Idea:
  - Round and scale all the values to lie in a smaller range.
  - Run the dynamic programming algorithm with the modified new values.
  - Return the items in this optimal solution.
  - Prove that the value of this solution is not much smaller than the true optimum.

Polynomial-Time Approximation Scheme for Knapsack

- $0 < \varepsilon < 1$ is a "precision" parameter; assume that $1/\varepsilon$ is an integer.
- Scaling factor $\theta = \frac{\varepsilon v^*}{2n}$.
- For every item i, set

$$\tilde{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil \theta, \qquad \hat{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil$$

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Solve the Knapsack problem using the dynamic program with the values $\hat{v}_i$.

Return the set S of items found.

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- What is the running time of `Knapsack-Approx`?

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`Knapsack-Approx(`ε`)`

Solve the Knapsack problem using the dynamic program with the values $\hat{v}_i$.
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- What is the running time of `Knapsack-Approx`?
  $O(n^2 \max_i \hat{v}_i) = O(n^2 v^*/\theta) = O(n^3/\varepsilon)$.
- We need to show that the value of the solution returned by `Knapsack-Approx` is good.

Approximation Guarantee for `Knapsack-Approx`

- Let S be the solution computed by `Knapsack-Approx`.
- Let S^* be any other solution satisfying $\sum_{j \in S^*} w_j \leq W$.

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- Claim: $\sum_{j \in S^*} v_j \leq (1 + \varepsilon) \sum_{i \in S} v_i$. Polynomial-time approximation scheme.

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- Since for each i, $v_i \leq \tilde{v}_i \leq v_i + \theta$,

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Approximation Guarantee for `Knapsack-Approx`

- Let S be the solution computed by `Knapsack-Approx`.
- Let S^* be any other solution satisfying $\sum_{j \in S^*} w_j \leq W$.
- Claim: $\sum_{j \in S^*} v_j \leq (1 + \varepsilon) \sum_{i \in S} v_i$. Polynomial-time approximation scheme.
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- Apply argument to S^* containing only the item with largest value: $v^* \leq \sum_{i \in S} v_i + \frac{\varepsilon v^*}{2} \leq \sum_{i \in S} v_i + \frac{v^*}{2}$, i.e., $v^* \leq 2 \sum_{i \in S} v_i$.

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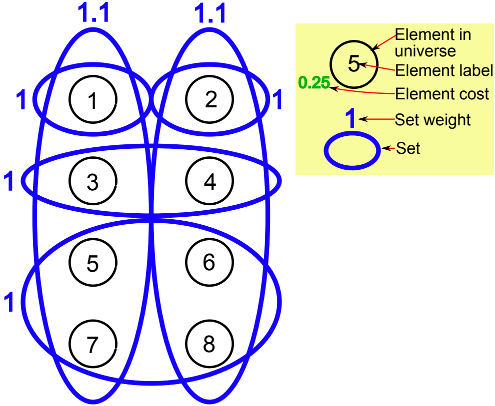
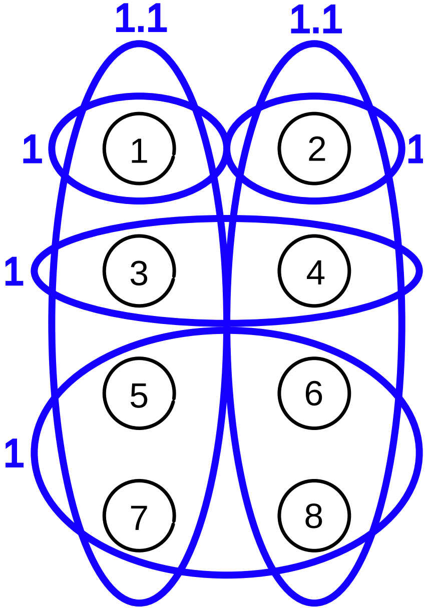
- Can Improve running time to $O(n \log_2 \frac{1}{\varepsilon} + \frac{1}{\varepsilon^4})$ (Lawler, 1979).

Set Cover

SET COVER
INSTANCE: A set U of n elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, each with an associated weight w.
SOLUTION: A collection $\mathcal{C}$ of sets in the collection such that $\bigcup_{S_i \in C} S_i = U$ and $\sum_{S_i \in C} w_i$ is minimised.



Greedy Approach



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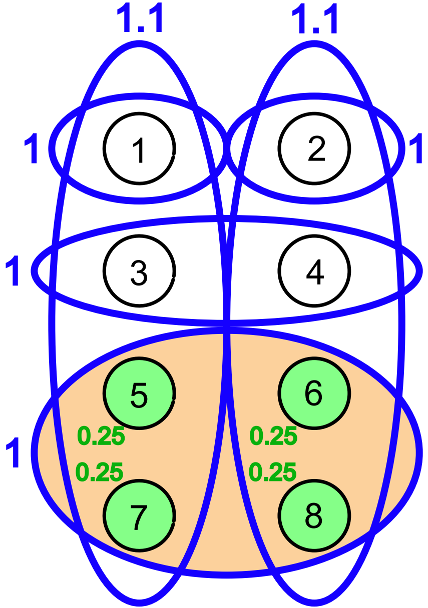

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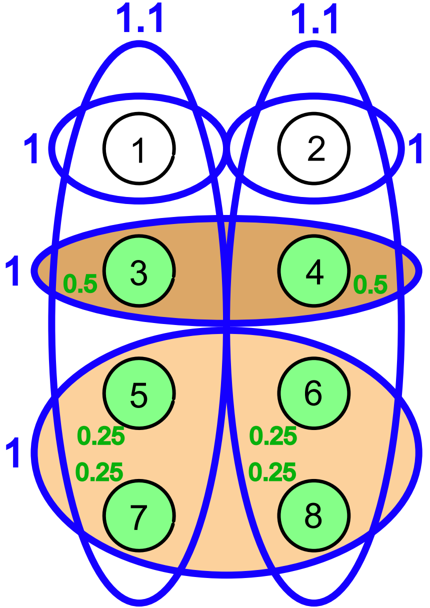

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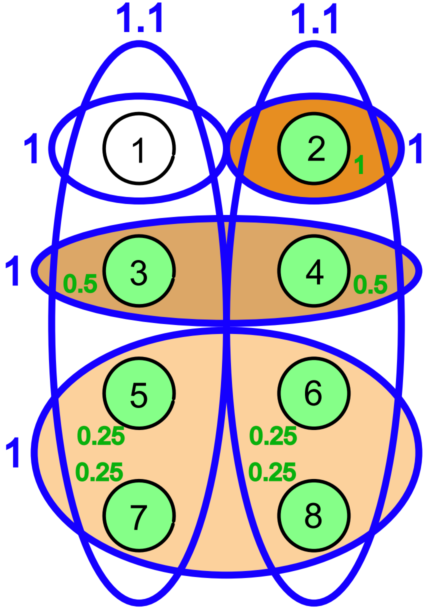

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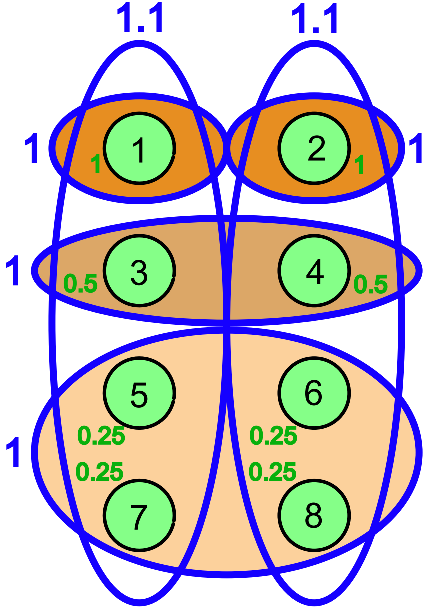

Greedy-Set-Cover

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Greedy-Set-Cover

- To get a greedy algorithm, in what order should we process the sets?
- Maintain set R of uncovered elements.
- Process set in decreasing order of $w_i/|S_i \cap R|$.

Greedy-Set-Cover:

Start with $R = U$ and no sets selected

While $R \neq \emptyset$

  Select set S_i that minimizes $w_i/|S_i \cap R|$

  Delete set S_i from R

EndWhile

Return the selected sets

Set Cover Problem

- Greedy algorithm achieves an approximation ratio of $H(d^*)$ (Johnson 1974, Lovász 1975, Chvatal 1979).
  - d^* is the size of the largest set in the collection
  - The *harmonic function*

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- No polynomial time algorithm can achieve an approximation bound better than $(1 - \Omega(1)) \ln n$ times optimal unless $\mathcal{P} = \mathcal{NP}$ (Dinur and Steurer, 2014)

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- 1-2 TSP: 8/7 approximation factor (Berman, Karpinski, 2006).
- Euclidean TSP (distances defined by points in d dimensions): PTAS in $O(n(\log n)^{1/\varepsilon})$ time (Arora, 1997; Mithcell, 1999) (second algorithm is slower).

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- Edit distance (sequence alighment) between two strings of length n: If it can be computed in $O(n^{2-\delta})$ time (for some constant $\delta >$), then SAT with n variables and m clauses can be solved in $m^{O(1)}2^{(1-\varepsilon)n}$ time, for some $\varepsilon > 0$ (Backurs, Indyk, 2015).