Review of Priority Queues and Graph Searches

T. M. Murali

February 1, 3, 2021
Motivation: Sort a List of Numbers

Sort

**INSTANCE:** Nonempty list $x_1, x_2, \ldots, x_n$ of integers.

**SOLUTION:** A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1}$, for all $1 \leq i < n$. 
Motivation: Sort a List of Numbers

Sort

**INSTANCE:** Nonempty list $x_1, x_2, \ldots, x_n$ of integers.

**SOLUTION:** A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1}$, for all $1 \leq i < n$.

- Possible algorithm:
  - Insert each numbers into a data structure $D$.
  - Repeatedly find the smallest number in $D$, output it, and remove it.
Motivation: Sort a List of Numbers

Sort

**INSTANCE:** Nonempty list \(x_1, x_2, \ldots, x_n\) of integers.

**SOLUTION:** A permutation \(y_1, y_2, \ldots, y_n\) of \(x_1, x_2, \ldots, x_n\) such that \(y_i \leq y_{i+1}\), for all \(1 \leq i < n\).

- Possible algorithm:
  - Insert each numbers into a data structure \(D\).
  - Repeatedly find the smallest number in \(D\), output it, and remove it.

- To get \(O(n \log n)\) running time, each “find minimum” step and each “remove” step must take \(O(\log n)\) time.
Candidate Data Structures for Sorting

- Possible algorithm:
  - Insert each number into a data structure $D$.
  - Repeatedly find the smallest number in $D$, output it, and remove it.
- Data structure must support three operations:
Candidate Data Structures for Sorting

- Possible algorithm:
  - Insert each number into a data structure $D$.
  - Repeatedly find the smallest number in $D$, output it, and remove it.

- Data structure must support three operations: insertion of a number, finding minimum, and deleting minimum, each in $O(\log n)$ time.
Priority Queue

- Store a set $S$ of elements, where each element $v$ has a priority value $\text{key}(v)$.
- Smaller key values $\equiv$ higher priorities.
- Operations supported:
  - find the element with smallest key
  - remove the smallest element
  - insert an element
  - delete an element
  - update the key of an element
- Element deletion and key update require knowledge of the position of the element in the priority queue.
Heaps

- Combine benefits of both lists and sorted arrays.
- Conceptually, a heap is a balanced binary tree.
- **Heap order**: For every element \( v \) at a node \( i \), the element \( w \) at \( i \)'s parent satisfies \( \text{key}(w) \leq \text{key}(v) \).
- We can implement a heap in a pointer-based data structure.
Alternatively, assume maximum number $N$ of elements is known in advance.

Store nodes of the heap in an array.

- Node at index $i$ has children at indices $2i$ and $2i + 1$ and parent at index $\lfloor i/2 \rfloor$.
- Index 1 is the root.
- How do you know that a node at index $i$ is a leaf?
Heaps

- Alternatively, assume maximum number $N$ of elements is known in advance.
- Store nodes of the heap in an array.
  - Node at index $i$ has children at indices $2i$ and $2i + 1$ and parent at index $\lfloor i/2 \rfloor$.
  - Index 1 is the root.
  - How do you know that a node at index $i$ is a leaf? If $2i > n$, where $n$ is the current number of elements in the heap.
Inserting an Element: Heapify-up

1. Insert new element at index $n + 1$.
2. Fix heap order using $\text{Heapify-up}(H, n + 1)$.

$\text{Heapify-up}(H, i)$:

\begin{align*}
\text{If } i > 1 \text{ then} & \\
\text{let } j = \text{parent}(i) = \lfloor i/2 \rfloor & \\
\text{If key}[H[i]] < \text{key}[H[j]] \text{ then} & \\
\text{swap the array entries } H[i] \text{ and } H[j] & \\
\text{Heapify-up}(H, j) & \\
\text{Endif} & \\
\text{Endif} & \\
\end{align*}
Inserting an Element: Heapify-up

1. Insert new element at index $n+1$.
2. Fix heap order using $\text{Heapify-up}(H, n + 1)$.

Heapify-up$(H, i)$:

If $i > 1$ then

let $j = \text{parent}(i) = \lfloor i/2 \rfloor$

If $\text{key}[H[i]] < \text{key}[H[j]]$ then

swap the array entries $H[i]$ and $H[j]$

Heapify-up$(H, j)$

Endif

Endif

**Example of Heapify-up**

The *Heapify-up* process is moving element $v$ toward the root.

---

**Figure 2.4** The *Heapify-up* process. Key 3 (at position 16) is too small (on the left). After swapping keys 3 and 11, the heap violation moves one step closer to the root of the tree (on the right).
Running time of Heapify-up

Heapify-up(H,i):
    If \( i > 1 \) then
        let \( j = \text{parent}(i) = \lfloor i/2 \rfloor \)
        If \( \text{key}[H[i]] < \text{key}[H[j]] \) then
            swap the array entries \( H[i] \) and \( H[j] \)
            Heapify-up(H,j)
        Endif
    Endif

Running time of Heapify-up(i)
Running time of Heapify-up

Heapify-up(H,i):

If \( i > 1 \) then

let \( j = \text{parent}(i) = \lfloor i/2 \rfloor \)

If \( \text{key}[H[i]] < \text{key}[H[j]] \) then

swap the array entries \( H[i] \) and \( H[j] \)

Heapify-up(H,j)

Endif

Endif

- Running time of Heapify-up(i) is \( O(\log i) \).
  - Each invocation decreases the second argument by a factor of at least 2.
Running time of Heapify-up

Heapify-up(H,i):
   If i > 1 then
      let j = parent(i) = \lfloor i/2 \rfloor
      If key[H[i]] < key[H[j]] then
         swap the array entries H[i] and H[j]
         Heapify-up(H,j)
      Endif
   Endif

Running time of Heapify-up(i) is $O(\log i)$.
   - Each invocation decreases the second argument by a factor of at least 2.
   - After $k$ invocations, argument is at most $i/2^k$.
   - Therefore $i/2^k \geq 1$, which implies that $k \leq \log_2 i$. 
Deleting an Element: Heapify-down

Suppose $H$ has $n + 1$ elements.

1. Delete element at $H[i]$ by moving element at $H[n+1]$ to $H[i]$.
2. If element at $H[i]$ is too small, fix heap order using $\text{Heapify-up}(H, i)$.
3. If element at $H[i]$ is too large, fix heap order using $\text{Heapify-down}(H, i)$.

---

$\text{Heapify-down}(H, i)$:

Let $n = \text{length}(H)$

If $2i > n$ then
   Terminate with $H$ unchanged

Else if $2i < n$ then
   Let left = $2i$, and right = $2i + 1$
   Let $j$ be the index that minimizes $\text{key}[H[left]]$ and $\text{key}[H[right]]$

Else if $2i = n$ then
   Let $j = 2i$

Endif

If $\text{key}[H[j]] < \text{key}[H[i]]$ then
   swap the array entries $H[i]$ and $H[j]$
   $\text{Heapify-down}(H, j)$

Endif
Deleting an Element: Heapify-down

1. Suppose $H$ has $n + 1$ elements.
3. If element at $H[i]$ is too small, fix heap order using Heapify-up($H, i$).
4. If element at $H[i]$ is too large, fix heap order using Heapify-down($H, i$).

Proof of correctness: read pages 63–64 of your textbook.

Heapify-down($H, i$):

Let $n = \text{length}(H)$

If $2i > n$ then
   Terminate with $H$ unchanged

Else if $2i < n$ then
   Let $\text{left} = 2i$, and $\text{right} = 2i + 1$
   Let $j$ be the index that minimizes $\text{key}[H[\text{left}]]$ and $\text{key}[H[\text{right}]]$

Else if $2i = n$ then
   Let $j = 2i$

Endif

If $\text{key}[H[j]] < \text{key}[H[i]]$ then
   swap the array entries $H[i]$ and $H[j]$
   Heapify-down($H, j$)

Endif
Example of **Heapify-down**

The **Heapify-down** process is moving element $w$ down, toward the leaves.

**Figure 2.5** The Heapify-down process: Key 21 (at position 3) is too big (on the left). After swapping keys 21 and 7, the heap violation moves one step closer to the bottom of the tree (on the right).
Running time of Heapify-down

Heapify-down(H,i):
   Let \( n = \text{length}(H) \)
   If \( 2i > n \) then
      Terminate with \( H \) unchanged
   Else if \( 2i < n \) then
      Let \( \text{left} = 2i \), and \( \text{right} = 2i + 1 \)
      Let \( j \) be the index that minimizes \( \text{key}[H[\text{left}]] \) and \( \text{key}[H[\text{right}]] \)
   Else if \( 2i = n \) then
      Let \( j = 2i \)
   Endif
   If \( \text{key}[H[j]] < \text{key}[H[i]] \) then
      swap the array entries \( H[i] \) and \( H[j] \)
      Heapify-down(H,j)
   Endif

- Each invocation of Heapify-down increases its second argument by a factor of at least two.
Running time of `Heapify-down`

---

**Heapify-down**\((H,i)\):

Let \(n = \text{length}(H)\)

If \(2i > n\) then

Terminate with \(H\) unchanged

Else if \(2i < n\) then

Let \(\text{left} = 2i\), and \(\text{right} = 2i + 1\)

Let \(j\) be the index that minimizes \(\text{key}[H[\text{left}]]\) and \(\text{key}[H[\text{right}]]\)

Else if \(2i = n\) then

Let \(j = 2i\)

Endif

If \(\text{key}[H[j]] < \text{key}[H[i]]\) then

swap the array entries \(H[i]\) and \(H[j]\)

Heapify-down\((H,j)\)

Endif

---

- Each invocation of `Heapify-down` increases its second argument by a factor of at least two.
- After \(k\) invocations argument must be at least \(\frac{n}{i}\)
Running time of Heapify-down

Heapify-down(H,i):
   Let n = length(H)
   If 2i > n then
      Terminate with H unchanged
   Else if 2i < n then
      Let left = 2i, and right = 2i + 1
      Let j be the index that minimizes key[H[left]] and key[H[right]]
   Else if 2i = n then
      Let j = 2i
   Endif
   If key[H[j]] < key[H[i]] then
      swap the array entries H[i] and H[j]
      Heapify-down(H,j)
   Endif

- Each invocation of Heapify-down increases its second argument by a factor of at least two.
- After \( k \) invocations argument must be at least \( i2^k \leq n \), which implies that \( k \leq \log_2 n/i \). Therefore running time is \( O(\log_2 n/i) \).
Sort

**INSTANCE:** Nonempty list \( x_1, x_2, \ldots, x_n \) of integers.

**SOLUTION:** A permutation \( y_1, y_2, \ldots, y_n \) of \( x_1, x_2, \ldots, x_n \) such that \( y_i \leq y_{i+1} \), for all \( 1 \leq i < n \).
Sort

**INSTANCE:** Nonempty list $x_1, x_2, \ldots, x_n$ of integers.

**SOLUTION:** A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1}$, for all $1 \leq i < n$.

- **Final algorithm:**
  - Insert each number in a priority queue $H$.
  - Repeatedly find the smallest number in $H$, output it, and delete it from $H$. 

---

T. M. Murali February 1, 3, 2021 Review of Priority Queues and Graph Searches
Sorting Numbers with the Priority Queue

Sort

**INSTANCE:** Nonempty list \( x_1, x_2, \ldots, x_n \) of integers.

**SOLUTION:** A permutation \( y_1, y_2, \ldots, y_n \) of \( x_1, x_2, \ldots, x_n \) such that
\[
y_i \leq y_{i+1}, \text{ for all } 1 \leq i < n.
\]

- **Final algorithm:**
  - Insert each number in a priority queue \( H \).
  - Repeatedly find the smallest number in \( H \), output it, and delete it from \( H \).

- Each insertion and deletion takes \( O(\log n) \) time for a total running time of \( O(n \log n) \).
The Oracle of Bacon
(Böhmer et al., *The Lancet*, May 15, 2020)
Graphs

- Model pairwise relationships (edges) between objects (nodes).
Graphs

- Model pairwise relationships (edges) between objects (nodes).
- Useful in a large number of applications:
Graphs

- Model pairwise relationships (edges) between objects (nodes).
- Useful in a large number of applications: computer networks, the World Wide Web, ecology (food webs), social networks, software systems, job scheduling, VLSI circuits, cellular networks, gene and protein networks, our bodies (nervous and circulatory systems, brains), buildings, transportation networks, ...
Graphs

- Model pairwise relationships (edges) between objects (nodes).
- Useful in a large number of applications: computer networks, the World Wide Web, ecology (food webs), social networks, software systems, job scheduling, VLSI circuits, cellular networks, gene and protein networks, our bodies (nervous and circulatory systems, brains), buildings, transportation networks, ...
- Problems involving graphs have a rich history dating back to Euler.
Euler and Graphs

Devise a walk through the city that crosses each of the seven bridges exactly once.
Euler and Graphs
Euler and Graphs
Definition of a Graph

- **Undirected graph** \( G = (V, E) \): set \( V \) of nodes and set \( E \) of edges, where \( E \subseteq V \times V \).
  - Elements of \( E \) are unordered pairs.
  - Edge \((u, v)\) is *incident* on \( u, v \); \( u \) and \( v \) are *neighbours* of each other.
  - Exactly one edge between any pair of nodes.
  - \( G \) contains no self loops, i.e., no edges of the form \((u, u)\).
Definition of a Graph

- **Directed graph** $G = (V, E)$: set $V$ of nodes and set $E$ of edges, where $E \subseteq V \times V$.
  - Elements of $E$ are ordered pairs.
  - $e = (u, v)$: $u$ is the *tail* of the edge $e$, $v$ is its *head*; $e$ is directed from $u$ to $v$.
  - A pair of nodes may be connected by two directed edges: $(u, v)$ and $(v, u)$.
  - $G$ contains no self loops.
A \( v_1 \)-\( v_k \) path in an undirected graph \( G = (V, E) \) is a sequence \( P \) of nodes \( v_1, v_2, \ldots, v_{k-1}, v_k \in V \) such that every consecutive pair of nodes \( v_i, v_{i+1}, 1 \leq i < k \) is connected by an edge in \( E \).
A $v_1$-$v_k$ path in an undirected graph $G = (V, E)$ is a sequence $P$ of nodes $v_1, v_2, \ldots, v_{k-1}, v_k \in V$ such that every consecutive pair of nodes $v_i, v_{i+1}, 1 \leq i < k$ is connected by an edge in $E$.

A path is *simple* if all its nodes are distinct.
A $v_1$-$v_k$ path in an undirected graph $G = (V, E)$ is a sequence $P$ of nodes $v_1, v_2, \ldots, v_{k-1}, v_k \in V$ such that every consecutive pair of nodes $v_i, v_{i+1}, 1 \leq i < k$ is connected by an edge in $E$.

- A path is *simple* if all its nodes are distinct.
- A *cycle* is a path where $k > 2$, the first $k - 1$ nodes are distinct, and $v_1 = v_k$. 
A \( v_1 \)-\( v_k \) path in an undirected graph \( G = (V, E) \) is a sequence \( P \) of nodes \( v_1, v_2, \ldots, v_{k-1}, v_k \in V \) such that every consecutive pair of nodes \( v_i, v_{i+1}, 1 \leq i < k \) is connected by an edge in \( E \).

A path is \textit{simple} if all its nodes are distinct.

A \textit{cycle} is a path where \( k > 2 \), the first \( k - 1 \) nodes are distinct, and \( v_1 = v_k \).
A \( v_1-v_k \) path in an undirected graph \( G = (V, E) \) is a sequence \( P \) of nodes \( v_1, v_2, \ldots, v_{k-1}, v_k \in V \) such that every consecutive pair of nodes \( v_i, v_{i+1}, 1 \leq i < k \) is connected by an edge in \( E \).

A path is \textit{simple} if all its nodes are distinct.

A \textit{cycle} is a path where \( k > 2 \), the first \( k - 1 \) nodes are distinct, and \( v_1 = v_k \).
A \( v_1 \)-\( v_k \) path in an undirected graph \( G = (V, E) \) is a sequence \( P \) of nodes \( v_1, v_2, \ldots, v_{k-1}, v_k \in V \) such that every consecutive pair of nodes \( v_i, v_{i+1}, 1 \leq i < k \) is connected by an edge in \( E \).

A path is simple if all its nodes are distinct.

A cycle is a path where \( k > 2 \), the first \( k - 1 \) nodes are distinct, and \( v_1 = v_k \).

Similar definitions carry over to directed graphs as well.
Connectivity

An undirected graph $G$ is *connected* if for every pair of nodes $u, v \in V$, there is a path from $u$ to $v$ in $G$. 
An undirected graph $G$ is *connected* if for every pair of nodes $u, v \in V$, there is a path from $u$ to $v$ in $G$. 
An undirected graph $G$ is \textit{connected} if for every pair of nodes $u, v \in V$, there is a path from $u$ to $v$ in $G$.

\textit{Distance} $d(u, v)$ between two nodes $u$ and $v$ is the minimum number of edges in any $u$-$v$ path.
**s-t Connectivity**

**INSTANCE:** An undirected graph $G = (V, E)$ and two nodes $s, t \in V$.

**QUESTION:** Is there an $s-t$ path in $G$?
\textbf{s-t Connectivity}

\textbf{INSTANCE:} An undirected graph $G = (V, E)$ and two nodes $s, t \in V$.

\textbf{QUESTION:} Is there an $s$-$t$ path in $G$?

- The \textit{connected component of} $G$ \textit{containing} $s$ is the set of all nodes $u$ such that there is an $s$-$u$ path in $G$. 
s-t Connectivity

**INSTANCE:** An undirected graph $G = (V, E)$ and two nodes $s, t \in V$.

**QUESTION:** Is there an $s$-$t$ path in $G$?

- The *connected component of $G$ containing $s$* is the set of all nodes $u$ such that there is an $s$-$u$ path in $G$.
- Algorithm for the $s$-$t$ Connectivity problem: compute the connected component of $G$ that contains $s$ and check if $t$ is in that component.
s-t Connectivity

**INSTANCE:** An undirected graph $G = (V, E)$ and two nodes $s, t \in V$.

**QUESTION:** Is there an $s$-$t$ path in $G$?

- The *connected component of $G$ containing $s$* is the set of all nodes $u$ such that there is an $s$-$u$ path in $G$.
- Algorithm for the $s$-$t$ Connectivity problem: compute the connected component of $G$ that contains $s$ and check if $t$ is in that component.
**s-t Connectivity**

**INSTANCE:** An undirected graph $G = (V, E)$ and two nodes $s, t \in V$.

**QUESTION:** Is there an $s$-$t$ path in $G$?

- The *connected component of $G$ containing $s$* is the set of all nodes $u$ such that there is an $s$-$u$ path in $G$.

- Algorithm for the $s$-$t$ Connectivity problem: compute the connected component of $G$ that contains $s$ and check if $t$ is in that component.

- Appears to do more work than is strictly necessary.
Computing Connected Components

- Abstract idea for an algorithm, with details to be specified later.
- “Explore” $G$ starting from $s$ and maintain set $R$ of visited nodes.

---

$R$ will consist of nodes to which $s$ has a path

Initially $R = \{s\}$

While there is an edge $(u,v)$ where $u \in R$ and $v \notin R$
  
  Add $v$ to $R$

Endwhile
Computing Connected Components

- Abstract idea for an algorithm, with details to be specified later.
- “Explore” $G$ starting from $s$ and maintain set $R$ of visited nodes.

$R$ will consist of nodes to which $s$ has a path
Initially $R = \{s\}$
While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$
  Add $v$ to $R$
Endwhile
Computing Connected Components

- Abstract idea for an algorithm, with details to be specified later.
- “Explore” $G$ starting from $s$ and maintain set $R$ of visited nodes.

---

$R$ will consist of nodes to which $s$ has a path
Initially $R = \{s\}$
While there is an edge $(u,v)$ where $u \in R$ and $v \notin R$
  Add $v$ to $R$
Endwhile
Computing Connected Components

- Abstract idea for an algorithm, with details to be specified later.
- “Explore” $G$ starting from $s$ and maintain set $R$ of visited nodes.

$R$ will consist of nodes to which $s$ has a path.
Initially $R = \{s\}$
While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$
    Add $v$ to $R$
Endwhile
Computing Connected Components

- Abstract idea for an algorithm, with details to be specified later.
- “Explore” $G$ starting from $s$ and maintain set $R$ of visited nodes.

$R$ will consist of nodes to which $s$ has a path

Initially $R = \{s\}$

While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$
  \begin{itemize}
    \item Add $v$ to $R$
  \end{itemize}
Endwhile
Computing Connected Components

- Abstract idea for an algorithm, with details to be specified later.
- “Explore” $G$ starting from $s$ and maintain set $R$ of visited nodes.

---

$R$ will consist of nodes to which $s$ has a path
Initially $R = \{s\}$
While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$
    Add $v$ to $R$
Endwhile
Computing Connected Components

- Abstract idea for an algorithm, with details to be specified later.
- “Explore” $G$ starting from $s$ and maintain set $R$ of visited nodes.

$R$ will consist of nodes to which $s$ has a path
Initially $R = \{s\}$
While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$
    Add $v$ to $R$
Endwhile
Computing Connected Components

- Abstract idea for an algorithm, with details to be specified later.
- “Explore” $G$ starting from $s$ and maintain set $R$ of visited nodes.

$R$ will consist of nodes to which $s$ has a path
Initially $R = \{s\}$
While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$
  Add $v$ to $R$
Endwhile
Computing Connected Components

- Abstract idea for an algorithm, with details to be specified later.
- “Explore” $G$ starting from $s$ and maintain set $R$ of visited nodes.

$R$ will consist of nodes to which $s$ has a path
Initially $R = \{s\}$
While there is an edge $(u,v)$ where $u \in R$ and $v \notin R$
  Add $v$ to $R$
Endwhile
Issues in Computing Connected Components

$R$ will consist of nodes to which $s$ has a path
Initially $R = \{s\}$
While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$
    Add $v$ to $R$
Endwhile

- Why does the algorithm terminate?
- Does the algorithm truly compute connected component of $G$ containing $s$?
- What is the running time of the algorithm?
Issues in Computing Connected Components

$R$ will consist of nodes to which $s$ has a path
Initially $R = \{s\}$
While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$
    Add $v$ to $R$
Endwhile

- Why does the algorithm terminate? Each iteration adds a new node to $R$.
- Does the algorithm truly compute connected component of $G$ containing $s$?
- What is the running time of the algorithm?
Claim: at the end of the algorithm, the set $R$ is exactly the connected component of $G$ containing $s$. 

\[ R \text{ will consist of nodes to which } s \text{ has a path} \]
\[ \text{Initially } R = \{s\} \]
\[ \text{While there is an edge } (u, v) \text{ where } u \in R \text{ and } v \notin R \]
\[ \quad \text{Add } v \text{ to } R \]
\[ \text{Endwhile} \]
Correctness of the Algorithm

\[ R \text{ will consist of nodes to which } s \text{ has a path} \]
Initially \( R = \{s\} \)
While there is an edge \((u,v)\) where \( u \in R \) and \( v \notin R \)
   Add \( v \) to \( R \)
Endwhile

Claim: at the end of the algorithm, the set \( R \) is exactly the connected component of \( G \) containing \( s \).

Proof: At termination, suppose \( w \notin R \) but there is an \( s-w \) path \( P \) in \( G \).
   ▶ Consider first node \( v \) in \( P \) not in \( R \) \((v \neq s)\).
   ▶ Let \( u \) be the predecessor of \( v \) in \( P \):
Correctness of the Algorithm

Claim: at the end of the algorithm, the set \( R \) is exactly the connected component of \( G \) containing \( s \).

Proof: At termination, suppose \( w \not\in R \) but there is an \( s-w \) path \( P \) in \( G \).

\begin{itemize}
  \item Consider first node \( v \) in \( P \) not in \( R \) (\( v \neq s \)).
  \item Let \( u \) be the predecessor of \( v \) in \( P \): \( u \) is in \( R \).
  \item \((u, v)\) is an edge with \( u \in R \) but \( v \not\in R \), contradicting the stopping rule.
\end{itemize}
### Correctness of the Algorithm

Claim: at the end of the algorithm, the set $R$ is exactly the connected component of $G$ containing $s$.

Proof: At termination, suppose $w \not\in R$ but there is an $s$-$w$ path $P$ in $G$.

- Consider first node $v$ in $P$ not in $R$ ($v \neq s$).
- Let $u$ be the predecessor of $v$ in $P$: $u$ is in $R$.
- $(u, v)$ is an edge with $u \in R$ but $v \not\in R$, contradicting the stopping rule.
- Note: wrong to assume that predecessor of $w$ in $P$ is not in $R$.

---

$R$ will consist of nodes to which $s$ has a path
Initially $R = \{s\}$
While there is an edge $(u, v)$ where $u \in R$ and $v \not\in R$
  Add $v$ to $R$
Endwhile
Running Time of the Algorithm

\[ R \text{ will consist of nodes to which } s \text{ has a path} \]

Initially \( R = \{s\} \)

While there is an edge \((u, v)\) where \( u \in R \) and \( v \not\in R \)

\hspace{1cm} Add \( v \) to \( R \)

Endwhile
Running Time of the Algorithm

\[ R \text{ will consist of nodes to which } s \text{ has a path} \]

Initially \( R = \{s\} \)

While there is an edge \((u, v)\) where \( u \in R \) and \( v \not\in R \)

\begin{itemize}
  \item Add \( v \) to \( R \)
\end{itemize}

Endwhile

- Analyse algorithm in terms of two parameters: the number of nodes \( n \) and the number of edges \( m \).
- How fast can we implement check in the while loop?
Running Time of the Algorithm

R will consist of nodes to which s has a path
Initially R={s}
While there is an edge (u,v) where \( u \in R \) and \( v \notin R \)
  Add v to R
Endwhile

- Analyse algorithm in terms of two parameters: the number of nodes \( n \) and the number of edges \( m \).
- How fast can we implement check in the while loop?
  - Naive approach: examine each edge in the graph.
  - Total running time is
Running Time of the Algorithm

---

**R** will consist of nodes to which **s** has a path

Initially $R = \{ s \}$

While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$
  
  Add $v$ to $R$

Endwhile

- Analyse algorithm in terms of two parameters: the number of nodes $n$ and the number of edges $m$.

- How fast can we implement check in the while loop?
  - **Naive approach**: examine each edge in the graph.
  - Total running time is $O(mn)$.

- BFS and DFS improve the running time by processing edges more carefully.
Breadth-First Search (BFS)

Idea: explore $G$ starting at $s$ and going “outward” in all directions, adding nodes one layer at a time.
Idea: explore $G$ starting at $s$ and going “outward” in all directions, adding nodes one layer at a time.

Layer $L_0$ contains only $s$. 
Breadth-First Search (BFS)

- Idea: explore $G$ starting at $s$ and going “outward” in all directions, adding nodes one layer at a time.
- Layer $L_0$ contains only $s$.
- Layer $L_1$ contains all neighbours of $s$. 

Given layers $L_0, L_1, \ldots, L_j$, layer $L_{j+1}$ contains all nodes that do not belong to an earlier layer and are connected by an edge to a node in layer $L_j$. 

T. M. Murali February 1, 3, 2021 Review of Priority Queues and Graph Searches
Breadth-First Search (BFS)

Idea: explore $G$ starting at $s$ and going “outward” in all directions, adding nodes one layer at a time.

Layer $L_0$ contains only $s$.

Layer $L_1$ contains all neighbours of $s$.

Given layers $L_0, L_1, \ldots, L_j$, layer $L_{j+1}$ contains all nodes that

1. do not belong to an earlier layer and
2. are connected by an edge to a node in layer $L_j$. 
Breadth-First Search (BFS)

Idea: explore $G$ starting at $s$ and going “outward” in all directions, adding nodes one layer at a time.

- Layer $L_0$ contains only $s$.
- Layer $L_1$ contains all neighbours of $s$.
- Given layers $L_0, L_1, \ldots, L_j$, layer $L_{j+1}$ contains all nodes that
  1. do not belong to an earlier layer and
  2. are connected by an edge to a node in layer $L_j$. 

T. M. Murali February 1, 3, 2021 Review of Priority Queues and Graph Searches
We have not yet described how to compute these layers.

Claim: For each $j \geq 1$, layer $L_j$ consists of all nodes
We have not yet described how to compute these layers.

Claim: For each \( j \geq 1 \), layer \( L_j \) consists of all nodes exactly at distance \( j \) from \( S \). Proof
We have not yet described how to compute these layers.

Claim: For each \( j \geq 1 \), layer \( L_j \) consists of all nodes exactly at distance \( j \) from \( S \). Proof by induction on \( j \).

Claim: There is a path from \( s \) to \( t \) if and only if \( t \) is a member of some layer.
We have not yet described how to compute these layers.

- **Claim:** For each \( j \geq 1 \), layer \( L_j \) consists of all nodes exactly at distance \( j \) from \( S \). Proof by induction on \( j \).

- **Claim:** There is a path from \( s \) to \( t \) if and only if \( t \) is a member of some layer.

- **Poll**

- For each node \( v \) in layer \( L_{j+1} \), select one node \( u \) in \( L_j \) such that \((u, v)\) is an edge in \( G \).

- **Poll**

- Consider the graph \( T \) formed by all such edges, directed from \( u \) to \( v \).
We have not yet described how to compute these layers.

Claim: For each $j \geq 1$, layer $L_j$ consists of all nodes exactly at distance $j$ from $S$. Proof by induction on $j$.

Claim: There is a path from $s$ to $t$ if and only if $t$ is a member of some layer.

For each node $v$ in layer $L_{j+1}$, select one node $u$ in $L_j$ such that $(u, v)$ is an edge in $G$.

Consider the graph $T$ formed by all such edges, directed from $u$ to $v$.

Why is $T$ a tree?
Properties of BFS

- We have not yet described how to compute these layers.
- Claim: For each $j \geq 1$, layer $L_j$ consists of all nodes exactly at distance $j$ from $S$. Proof by induction on $j$.
- Claim: There is a path from $s$ to $t$ if and only if $t$ is a member of some layer.
- For each node $v$ in layer $L_{j+1}$, select one node $u$ in $L_j$ such that $(u, v)$ is an edge in $G$.
- Consider the graph $T$ formed by all such edges, directed from $u$ to $v$.
  - Why is $T$ a tree? It is connected. The number of edges in $T$ is the number of nodes in all the layers minus 1.
  - $T$ is called the **breadth-first search tree**.
**Non-tree edge**: an edge of \( G \) that does not belong to the BFS tree \( T \).

Claim: Let \( T \) be a BFS tree, let \( x \) and \( y \) be nodes in \( T \) belonging to layers \( L_i \) and \( L_j \), respectively, and let \( (x, y) \) be an edge of \( G \). Then \( |i - j| \leq 1 \).
Non-tree edge: an edge of $G$ that does not belong to the BFS tree $T$.

Claim: Let $T$ be a BFS tree, let $x$ and $y$ be nodes in $T$ belonging to layers $L_i$ and $L_j$, respectively, and let $(x, y)$ be an edge of $G$. Then $|i - j| \leq 1$.

Proof by contradiction: Suppose $i < j - 1$. Node $x \in L_i \Rightarrow$ all nodes adjacent to $x$ are in layers $L_1, L_2, \ldots L_{i+1}$. Hence $y$ must be in layer $L_{i+1}$ or earlier.
BFS Trees

- **Non-tree edge**: an edge of $G$ that does not belong to the BFS tree $T$.
- **Claim**: Let $T$ be a BFS tree, let $x$ and $y$ be nodes in $T$ belonging to layers $L_i$ and $L_j$, respectively, and let $(x, y)$ be an edge of $G$. Then $|i - j| \leq 1$.
- **Proof by contradiction**: Suppose $i < j - 1$. Node $x \in L_i \Rightarrow$ all nodes adjacent to $x$ are in layers $L_1, L_2, \ldots L_{i+1}$. Hence $y$ must be in layer $L_{i+1}$ or earlier.
- **Still unresolved**: an efficient implementation of BFS.
Depth-First Search (DFS)

- Explore $G$ as if it were a maze: start from $s$, traverse first edge out (to node $v$), traverse first edge out of $v$, . . . , reach a dead-end, backtrack, . . . .
Depth-First Search (DFS)

- Explore $G$ as if it were a maze: start from $s$, traverse first edge out (to node $v$), traverse first edge out of $v$, ..., reach a dead-end, backtrack, .......

1. Mark all nodes as “Unexplored”.
2. Invoke DFS$(s)$.

---

DFS$(u)$:

- Mark $u$ as "Explored" and add $u$ to $R$
- For each edge $(u, v)$ incident to $u$
  - If $v$ is not marked "Explored" then
    - Recursively invoke DFS$(v)$
  Endif
Endfor
Depth-First Search (DFS)

- Explore $G$ as if it were a maze: start from $s$, traverse first edge out (to node $v$), traverse first edge out of $v$, ..., reach a dead-end, backtrack, .......

1. Mark all nodes as “Unexplored”.
2. Invoke DFS($s$).

DFS($u$):

- Mark $u$ as "Explored" and add $u$ to $R$
- For each edge $(u, v)$ incident to $u$
  - If $v$ is not marked "Explored" then
    - Recursively invoke DFS($v$)
  - Endif
- Endfor

- Depth-first search tree is a tree $T$: when DFS($v$) is invoked directly during the call to DFS($v$), add edge $(u, v)$ to $T$. 
Example of DFS
Example of DFS
Example of DFS

T. M. Murali February 1, 3, 2021 Review of Priority Queues and Graph Searches
Example of DFS
Example of DFS
Example of DFS
Example of DFS
Example of DFS

The example shows a directed graph with nodes numbered 1 to 13. The DFS traversal starts from node 1 and follows the edges as follows:

1. 2
2. 3
3. 4
4. 5
5. 6
6. 7
7. 8
8. 9
9. 10
10. 11
11. 12
12. 13

The edges are colored to indicate the order of traversal:
- Red edges indicate the order of visiting nodes.
- Blue edges indicate the reversing of the order of visiting nodes.
- Dashed edges indicate the connected components.

This example demonstrates the depth-first search (DFS) algorithm on a graph.
BFS vs. DFS

Both visit the same set of nodes but in a different order.

Both traverse all the edges in the connected component but in a different order.

BFS trees have root-to-leaf paths that look as short as possible while paths in DFS trees tend to be long and deep.

Non-tree edges
- BFS within the same level or between adjacent levels.
- DFS connect ancestors to descendants.
Both visit the same set of nodes but in a different order.
Both traverse all the edges in the connected component but in a different order.
BFS trees have root-to-leaf paths that look as short as possible while paths in DFS trees tend to be long and deep.
Non-tree edges

BFS within the same level or between adjacent levels.
Both visit the same set of nodes but in a different order.
Both traverse all the edges in the connected component but in a different order.
BFS trees have root-to-leaf paths that look as short as possible while paths in DFS trees tend to be long and deep.
Non-tree edges

**BFS** within the same level or between adjacent levels.
**DFS** connect ancestors to descendants.
Properties of DFS Trees

DFS(u):
Mark u as "Explored" and add u to R
For each edge (u, v) incident to u
  If v is not marked "Explored" then
    Recursively invoke DFS(v)
  Endif
Endfor

- Observation: All nodes marked as "Explored" between the start of DFS(u) and its end are descendants of u in the DFS tree T.
Properties of DFS Trees

DFS(u):
Mark u as "Explored" and add u to R
For each edge (u, v) incident to u
    If v is not marked "Explored" then
        Recursively invoke DFS(v)
    Endif
Endfor

- Observation: All nodes marked as “Explored” between the start of DFS(u) and its end are descendants of u in the DFS tree T.

- Claim: Let x and y be nodes in a DFS tree T such that (x, y) is an edge of G but not of T. Then one of x or y is an ancestor of the other in T. Read proof on page 86 of your textbook.
Representing Graphs

- Graph $G = (V, E)$ has two input parameters: $|V| = n, |E| = m$.
  - Size of the graph is defined to be $m + n$.
  - Strive for algorithms whose running time is linear in graph size, i.e., $O(m + n)$. 

<table>
<thead>
<tr>
<th>Operation/Space</th>
<th>Adj. matrix</th>
<th>Adj. list</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is $(i, j)$ an edge?</td>
<td>$O(1)$ time</td>
<td>$O(n_i)$ time</td>
</tr>
<tr>
<td>Iterate over all edges incident on node $i$</td>
<td>$O(n)$ time</td>
<td>$O(n_i)$ time</td>
</tr>
<tr>
<td>Space used</td>
<td>$O(n^2)$</td>
<td>$O(n + \sum_{v \in G} n_v) = O(n + m)$</td>
</tr>
</tbody>
</table>
Representing Graphs

- Graph $G = (V, E)$ has two input parameters: $|V| = n, |E| = m$.
  - Size of the graph is defined to be $m + n$.
  - Strive for algorithms whose running time is linear in graph size, i.e., $O(m + n)$.
- Assume $V = \{1, 2, \ldots, n - 1, n\}$.
- **Adjacency matrix**: $n \times n$ Boolean matrix, where the entry in row $i$ and column $j$ is 1 iff the graph contains the edge $(i, j)$.
Representing Graphs

- **Graph** $G = (V,E)$ has two input parameters: $|V| = n, |E| = m$.
  - Size of the graph is defined to be $m + n$.
  - Strive for algorithms whose running time is linear in graph size, i.e., $O(m + n)$.
- Assume $V = \{1, 2, \ldots, n-1, n\}$.
- **Adjacency matrix**: $n \times n$ Boolean matrix, where the entry in row $i$ and column $j$ is 1 iff the graph contains the edge $(i,j)$.
- **Adjacency list**: array $\text{Adj}$, where $\text{Adj}[v]$ stores the list of all nodes adjacent to $v$.
  - An edge $e = (u,v)$ appears twice: in $\text{Adj}[u]$ and $\text{Adj}[v]$. 
Representing Graphs

- Graph $G = (V, E)$ has two input parameters: $|V| = n, |E| = m$.
  - Size of the graph is defined to be $m + n$.
  - Strive for algorithms whose running time is linear in graph size, i.e., $O(m + n)$.
- Assume $V = \{1, 2, \ldots, n - 1, n\}$.
- **Adjacency matrix**: $n \times n$ Boolean matrix, where the entry in row $i$ and column $j$ is 1 iff the graph contains the edge $(i, j)$.
- **Adjacency list**: array Adj, where Adj[$v$] stores the list of all nodes adjacent to $v$.
  - An edge $e = (u, v)$ appears twice: in Adj[$u$] and Adj[$v$].
  - $n_v$ = the number of neighbours of node $v$.

<table>
<thead>
<tr>
<th>Operation/Space</th>
<th>Adj. matrix</th>
<th>Adj. list</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is $(i, j)$ an edge?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Iterate over all edges incident on node $i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Space used</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Representing Graphs

- Graph $G = (V, E)$ has two input parameters: $|V| = n, |E| = m$.
  - Size of the graph is defined to be $m + n$.
  - Strive for algorithms whose running time is linear in graph size, i.e., $O(m + n)$.
- Assume $V = \{1, 2, \ldots, n - 1, n\}$.
- **Adjacency matrix**: $n \times n$ Boolean matrix, where the entry in row $i$ and column $j$ is 1 iff the graph contains the edge $(i, j)$.
- **Adjacency list**: array $\text{Adj}$, where $\text{Adj}[v]$ stores the list of all nodes adjacent to $v$.
  - An edge $e = (u, v)$ appears twice: in $\text{Adj}[u]$ and $\text{Adj}[v]$.
  - $n_v = \text{the number of neighbours of node } v$.

<table>
<thead>
<tr>
<th>Operation/Space</th>
<th>Adj. matrix</th>
<th>Adj. list</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is $(i, j)$ an edge?</td>
<td>$O(1)$ time</td>
<td>$O(n_i)$ time</td>
</tr>
<tr>
<td>Iterate over all edges incident on node $i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Space used</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Representing Graphs

- Graph $G = (V, E)$ has two input parameters: $|V| = n, |E| = m$.
  - Size of the graph is defined to be $m + n$.
  - Strive for algorithms whose running time is linear in graph size, i.e., $O(m + n)$.

- Assume $V = \{1, 2, \ldots, n - 1, n\}$.

- **Adjacency matrix**: $n \times n$ Boolean matrix, where the entry in row $i$ and column $j$ is 1 iff the graph contains the edge $(i, j)$.

- **Adjacency list**: array $\text{Adj}$, where $\text{Adj}[v]$ stores the list of all nodes adjacent to $v$.
  - An edge $e = (u, v)$ appears twice: in $\text{Adj}[u]$ and $\text{Adj}[v]$.
  - $n_v =$ the number of neighbours of node $v$.

<table>
<thead>
<tr>
<th>Operation/Space</th>
<th>Adj. matrix</th>
<th>Adj. list</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is $(i, j)$ an edge?</td>
<td>$O(1)$ time</td>
<td>$O(n_i)$ time</td>
</tr>
<tr>
<td>Iterate over all edges incident on node $i$</td>
<td>$O(n)$ time</td>
<td>$O(n_i)$ time</td>
</tr>
<tr>
<td>Space used</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Representing Graphs

- **Graph** $G = (V, E)$ has two input parameters: $|V| = n$, $|E| = m$.
  - Size of the graph is defined to be $m + n$.
  - Strive for algorithms whose running time is linear in graph size, i.e., $O(m + n)$.

- Assume $V = \{1, 2, \ldots, n - 1, n\}$.

- **Adjacency matrix**: $n \times n$ Boolean matrix, where the entry in row $i$ and column $j$ is 1 iff the graph contains the edge $(i, j)$.

- **Adjacency list**: array $\text{Adj}$, where $\text{Adj}[v]$ stores the list of all nodes adjacent to $v$.
  - An edge $e = (u, v)$ appears twice: in $\text{Adj}[u]$ and $\text{Adj}[v]$.
  - $n_v$ = the number of neighbours of node $v$.

<table>
<thead>
<tr>
<th>Operation/Space</th>
<th>Adj. matrix</th>
<th>Adj. list</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is $(i, j)$ an edge?</td>
<td>$O(1)$ time</td>
<td>$O(n_i)$ time</td>
</tr>
<tr>
<td>Iterate over all edges incident on node $i$</td>
<td>$O(n)$ time</td>
<td>$O(n_i)$ time</td>
</tr>
<tr>
<td>Space used</td>
<td>$O(n^2)$</td>
<td>$O(n + \sum_{v \in G} n_v)$</td>
</tr>
</tbody>
</table>
Representing Graphs

- Graph \( G = (V, E) \) has two input parameters: \(|V| = n, |E| = m\).
  - Size of the graph is defined to be \( m + n \).
  - Strive for algorithms whose running time is linear in graph size, i.e., \( O(m + n) \).

- Assume \( V = \{1, 2, \ldots, n - 1, n\} \).

- \textit{Adjacency matrix}: \( n \times n \) Boolean matrix, where the entry in row \( i \) and column \( j \) is 1 iff the graph contains the edge \((i, j)\).

- \textit{Adjacency list}: array \( \text{Adj} \), where \( \text{Adj}[v] \) stores the list of all nodes adjacent to \( v \).
  - An edge \( e = (u, v) \) appears twice: in \( \text{Adj}[u] \) and \( \text{Adj}[v] \).
  - \( n_v \) = the number of neighbours of node \( v \).

<table>
<thead>
<tr>
<th>Operation/Space</th>
<th>Adj. matrix</th>
<th>Adj. list</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is ((i, j)) an edge?</td>
<td>( O(1) ) time</td>
<td>( O(n_i) ) time</td>
</tr>
<tr>
<td>Iterate over all edges incident on node ( i )</td>
<td>( O(n) ) time</td>
<td>( O(n_i) ) time</td>
</tr>
<tr>
<td>Space used</td>
<td>( O(n^2) )</td>
<td>( O(n + \sum_{v \in G} n_v) = O(n + m) )</td>
</tr>
</tbody>
</table>
Representing Graphs

- Graph $G = (V, E)$ has two input parameters: $|V| = n, |E| = m$.
  - Size of the graph is defined to be $m + n$.
  - Strive for algorithms whose running time is linear in graph size, i.e., $O(m + n)$.
- Assume $V = \{1, 2, \ldots, n - 1, n\}$.
- **Adjacency matrix**: $n \times n$ Boolean matrix, where the entry in row $i$ and column $j$ is 1 iff the graph contains the edge $(i, j)$.
- **Adjacency list**: array $\text{Adj}$, where $\text{Adj}[v]$ stores the list of all nodes adjacent to $v$.
  - An edge $e = (u, v)$ appears twice: in $\text{Adj}[u]$ and $\text{Adj}[v]$.
  - $n_v = \text{the number of neighbours of node } v$.
  - Space used and time to iterate over neighbours are optimal for every graph.

<table>
<thead>
<tr>
<th>Operation/Space</th>
<th>Adj. matrix</th>
<th>Adj. list</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is $(i, j)$ an edge?</td>
<td>$O(1)$ time</td>
<td>$O(n_i)$ time</td>
</tr>
<tr>
<td>Iterate over all edges incident on node $i$</td>
<td>$O(n)$ time</td>
<td>$O(n_i)$ time</td>
</tr>
<tr>
<td></td>
<td>$O(n^2)$</td>
<td>$O(n + \sum_{v \in G} n_v)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= O(n + m)$</td>
</tr>
<tr>
<td>Space used</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
“Implementation” of BFS and DFS: fully specify the algorithms and data structures so that we can obtain provably efficient times.

Inner loop of both BFS and DFS: process the set of edges incident on a given node and the set of visited nodes.

How do we store the set of visited nodes? Order in which we process the nodes is crucial.
Data Structures for Implementation

“Implementation” of BFS and DFS: fully specify the algorithms and data structures so that we can obtain provably efficient times.

Inner loop of both BFS and DFS: process the set of edges incident on a given node and the set of visited nodes.

How do we store the set of visited nodes? Order in which we process the nodes is crucial.
  - BFS: store visited nodes in a queue (first-in, first-out).
  - DFS: store visited nodes in a stack (last-in, first-out)
Using a Queue in BFS

- Maintain an array `Discovered` and set `Discovered[v] = true` as soon as the algorithm sees `v`.
- Maintain all the layers in a single queue `L`.

\[\text{BFS}(s):\]

1. Set `Discovered[s] = true`
2. Set `Discovered[v] = false`, for all other nodes `v`
3. Initialize `L` to consist of the single element `s`
4. While `L` is not empty
   - Pop the node `u` at the head of `L`
   - Consider each edge `(u, v)` incident on `u`
   - If `Discovered[v] = false`
     - Set `Discovered[v] = true`
     - Add edge `(u, v)` to the tree `T`
     - Push `v` to the back of `L`
5. Endif
6. Endwhile

Can modify this procedure to keep track of distance to `s` (layer numbers) as well.

Store the pair `(u, l_u)`, where `l_u` is the index of the layer containing `u`.

Claim: More formally: If `BFS(s)` pops `(v, l_v)` from `L` immediately after it pops `(u, l_u)`, then either `l_v = l_u` or `l_v = l_u + 1`. 
Using a Queue in BFS

- Maintain an array `Discovered` and set `Discovered[v] = true` as soon as the algorithm sees `v`.
- Maintain all the layers in a single queue `L`.

**BFS(s):**
- Set `Discovered[s] = true`
- Set `Discovered[v] = false`, for all other nodes `v`
- Initialize `L` to consist of the single element `s`
- While `L` is not empty
  - Pop the node `u` at the head of `L`
  - Consider each edge `(u, v)` incident on `u`
  - If `Discovered[v] = false` then
    - Set `Discovered[v] = true`
    - Add edge `(u, v)` to the tree `T`
    - Push `v` to the back of `L`
- Endif
- Endwhile
Using a Queue in BFS

- Maintain an array Discovered and set $\text{Discovered}[v] = true$ as soon as the algorithm sees $v$.
- Maintain all the layers in a single queue $L$.

**BFS(s):**
- Set $\text{Discovered}[s] = true$
- Set $\text{Discovered}[v] = false$, for all other nodes $v$
- Initialize $L$ to consist of the single element $s$
- While $L$ is not empty
  - Pop the node $u$ at the head of $L$
  - Consider each edge $(u, v)$ incident on $u$
  - If $\text{Discovered}[v] = false$ then
    - Set $\text{Discovered}[v] = true$
    - Add edge $(u, v)$ to the tree $T$
    - Push $v$ to the back of $L$
- Endif
- Endwhile

Can modify this procedure to keep track of distance to $s$ (layer numbers) as well.

Store the pair $(u, l_u)$, where $l_u$ is the index of the layer containing $u$.

Claim: More formally: If $\text{BFS}(s)$ pops $(v, l_v)$ from $L$ immediately after it pops $(u, l_u)$, then either $l_v = l_u$ or $l_v = l_u + 1$. 
Using a Queue in BFS

- Maintain an array `Discovered` and set `Discovered[v] = true` as soon as the algorithm sees `v`.
- Maintain all the layers in a single queue `L`.

**BFS(s):**
- Set `Discovered[s] = true`
- Set `Discovered[v] = false`, for all other nodes `v`
- Initialize `L` to consist of the single element `s`
- While `L` is not empty
  - Pop the node `u` at the head of `L`
  - Consider each edge `(u, v)` incident on `u`
  - If `Discovered[v] = false` then
    - Set `Discovered[v] = true`
    - Add edge `(u, v)` to the tree `T`
    - Push `v` to the back of `L`
- Endif
- Endwhile

Can modify this procedure to keep track of distance to `s` (layer numbers) as well.

Store the pair `(u, l_u)`, where `l_u` is the index of the layer containing `u`.

**Claim:** More formally: If `BFS(s)` pops `(v, l_v)` from `L` immediately after it pops `(u, l_u)`, then either `l_v = l_u` or `l_v = l_u + 1`.
Using a Queue in BFS

- Maintain an array `Discovered` and set `Discovered[v] = true` as soon as the algorithm sees `v`.
- Maintain all the layers in a single queue `L`.

**BFS(s):**

1. Set `Discovered[s] = true`
2. Set `Discovered[v] = false`, for all other nodes `v`
3. Initialize `L` to consist of the single element `s`
4. While `L` is not empty
   - Pop the node `u` at the head of `L`
   - Consider each edge `(u, v)` incident on `u`
   - If `Discovered[v] = false`
     - Set `Discovered[v] = true`
     - Add edge `(u, v)` to the tree `T`
     - Push `v` to the back of `L`
   - Endif
5. Endwhile

---

Can modify this procedure to keep track of distance to `s` (layer numbers) as well.

Store the pair `(u, l_u)`, where `l_u` is the index of the layer containing `u`.

Claim: More formally: If `BFS(s)` pops `(v, l_v)` from `L` immediately after it pops `(u, l_u)`, then either `l_v = l_u` or `l_v = l_u + 1`. 
Using a Queue in BFS

- Maintain an array Discovered and set $\text{Discovered}[v] = true$ as soon as the algorithm sees $v$.
- Maintain all the layers in a single queue $L$.

BFS($s$):

1. Set $\text{Discovered}[s] = true$
2. Set $\text{Discovered}[v] = false$, for all other nodes $v$
3. Initialize $L$ to consist of the single element $s$
4. While $L$ is not empty
   - Pop the node $u$ at the head of $L$
   - Consider each edge $(u, v)$ incident on $u$
   - If $\text{Discovered}[v] = false$ then
     - Set $\text{Discovered}[v] = true$
     - Add edge $(u, v)$ to the tree $T$
     - Push $v$ to the back of $L$
   - Endif
5. Endwhile

Can modify this procedure to keep track of distance to $s$ (layer numbers) as well.

Store the pair $(u, l_u)$, where $l_u$ is the index of the layer containing $u$.

Claim: More formally: If BFS($s$) pops $(v, l_v)$ from $L$ immediately after it pops $(u, l_u)$, then either $l_v = l_u$ or $l_v = l_u + 1$. 

T. M. Murali February 1, 3, 2021 Review of Priority Queues and Graph Searches
Using a Queue in BFS

- Maintain an array Discovered and set Discovered[v] = true as soon as the algorithm sees v.
- Maintain all the layers in a single queue L.

BFS(s):
- Set Discovered[s] = true
- Set Discovered[v] = false, for all other nodes v
- Initialize L to consist of the single element s
- While L is not empty
  - Pop the node u at the head of L
  - Consider each edge (u, v) incident on u
  - If Discovered[v] = false then
    - Set Discovered[v] = true
    - Add edge (u, v) to the tree T
    - Push v to the back of L
  - Endif
- Endwhile

Can modify this procedure to keep track of distance to s (layer numbers) as well.
Store the pair (u, lv), where lv is the index of the layer containing u.
Claim: More formally: If BFS(s) pops (v, lv) from L immediately after it pops (u, lu), then either lv = lu or lv = lu + 1.
Using a Queue in BFS

- Maintain an array Discovered and set $\text{Discovered}[v] = \text{true}$ as soon as the algorithm sees $v$.
- Maintain all the layers in a single queue $L$.

BFS($s$):
- Set $\text{Discovered}[s] = \text{true}$
- Set $\text{Discovered}[v] = \text{false}$, for all other nodes $v$
- Initialize $L$ to consist of the single element $s$
- While $L$ is not empty
  - Pop the node $u$ at the head of $L$
  - Consider each edge $(u, v)$ incident on $u$
  - If $\text{Discovered}[v] = \text{false}$ then
    - Set $\text{Discovered}[v] = \text{true}$
    - Add edge $(u, v)$ to the tree $T$
    - Push $v$ to the back of $L$
  - Endif
- Endwhile

Can modify this procedure to keep track of distance to $s$ (layer numbers) as well. Store the pair $(u, l_u)$, where $l_u$ is the index of the layer containing $u$.

Claim: More formally: If BFS($s$) pops $(v, l_v)$ from $L$ immediately after it pops $(u, l_u)$, then either $l_v = l_u$ or $l_v = l_u + 1$. 

T. M. Murali February 1, 3, 2021 Review of Priority Queues and Graph Searches
Using a Queue in BFS

- Maintain an array Discovered and set $\text{Discovered}[v] = \text{true}$ as soon as the algorithm sees $v$.
- Maintain all the layers in a single queue $L$.

BFS($s$):

1. Set $\text{Discovered}[s] = \text{true}$
2. Set $\text{Discovered}[v] = \text{false}$, for all other nodes $v$
3. Initialize $L$ to consist of the single element $s$
4. While $L$ is not empty
   - Pop the node $u$ at the head of $L$
   - Consider each edge $(u, v)$ incident on $u$
   - If $\text{Discovered}[v] = \text{false}$ then
     - Set $\text{Discovered}[v] = \text{true}$
     - Add edge $(u, v)$ to the tree $T$
     - Push $v$ to the back of $L$
   - Endif
5. Endwhile
Using a Queue in BFS

- Maintain an array Discovered and set Discovered[v] = true as soon as the algorithm sees v.
- Maintain all the layers in a single queue L.

BFS(s):

1. Set Discovered[s] = true
2. Set Discovered[v] = false, for all other nodes v
3. Initialize L to consist of the single element s
4. While L is not empty
   a. Pop the node u at the head of L
   b. Consider each edge (u, v) incident on u
   c. If Discovered[v] = false then
      i. Set Discovered[v] = true
      ii. Add edge (u, v) to the tree T
      iii. Push v to the back of L
5. Endwhile

Can modify this procedure to keep track of distance to s (layer numbers) as well.

Store the pair (u, l_u), where l_u is the index of the layer containing u.

Claim: More formally: If BFS(s) pops (v, l_v) from L immediately after it pops (u, l_u), then either l_v = l_u or l_v = l_u + 1.
Using a Queue in BFS

- Maintain an array `Discovered` and set `Discovered[v] = true` as soon as the algorithm sees `v`.
- Maintain all the layers in a single queue `L`.

**BFS(s):**

Set `Discovered[s] = true`  
Set `Discovered[v] = false`, for all other nodes `v`  
Initialize `L` to consist of the single element `s`  
While `L` is not empty  
  Pop the node `u` at the head of `L`  
  Consider each edge `(u, v)` incident on `u`  
  If `Discovered[v] = false` then  
    Set `Discovered[v] = true`  
    Add edge `(u, v)` to the tree `T`  
    Push `v` to the back of `L`  
Endif

Endwhile
Using a Queue in BFS

- Maintain an array `Discovered` and set `Discovered[v] = true` as soon as the algorithm sees `v`.
- Maintain all the layers in a single queue `L`.

**BFS(s):**

1. Set `Discovered[s] = true`
2. Set `Discovered[v] = false`, for all other nodes `v`
3. Initialize `L` to consist of the single element `s`
4. While `L` is not empty
   - Pop the node `u` at the head of `L`
   - Consider each edge `(u, v)` incident on `u`
   - If `Discovered[v] = false` then
     - Set `Discovered[v] = true`
     - Add edge `(u, v)` to the tree `T`
     - Push `v` to the back of `L`
   - Endif
5. Endwhile

Can modify this procedure to keep track of distance to `s` (layer numbers) as well.

Store the pair `(u, lv)`, where `lv` is the index of the layer containing `u`.

**Claim:** More formally: If `BFS(s)` pops `(v, lv)` from `L` immediately after it pops `(u, lu)`, then either `lv = lu` or `lv = lu + 1`.

T. M. Murali February 1, 3, 2021 Review of Priority Queues and Graph Searches
Using a Queue in BFS

- Maintain an array Discovered and set Discovered[v] = true as soon as the algorithm sees v.
- Maintain all the layers in a single queue L.

BFS(s):
- Set Discovered[s] = true
- Set Discovered[v] = false, for all other nodes v
- Initialize L to consist of the single element s
- While L is not empty
  - Pop the node u at the head of L
  - Consider each edge (u, v) incident on u
  - If Discovered[v] = false then
    - Set Discovered[v] = true
    - Add edge (u, v) to the tree T
    - Push v to the back of L
- Endif
- Endwhile

- Can modify this procedure to keep track of distance to s (layer numbers) as well.
Using a Queue in BFS

- Maintain an array Discovered and set Discovered[v] = true as soon as the algorithm sees v.
- Maintain all the layers in a single queue L.

**BFS(s):**

1. Set Discovered[s] = true
2. Set Discovered[v] = false, for all other nodes v
3. Initialize L to consist of the single element s
4. While L is not empty
   - Pop the node u at the head of L
   - Consider each edge (u, v) incident on u
   - If Discovered[v] = false then
     - Set Discovered[v] = true
     - Add edge (u, v) to the tree T
     - Push v to the back of L
   - Endif
5. Endwhile

- Can modify this procedure to keep track of distance to s (layer numbers) as well. Store the pair (u, l_u), where l_u is the index of the layer containing u.
Using a Queue in BFS

- Maintain an array Discovered and set Discovered[v] = true as soon as the algorithm sees v.
- Maintain all the layers in a single queue L.

BFS(s):
- Set Discovered[s] = true
- Set Discovered[v] = false, for all other nodes v
- Initialize L to consist of the single element s
- While L is not empty
  - Pop the node u at the head of L
  - Consider each edge (u, v) incident on u
  - If Discovered[v] = false then
    - Set Discovered[v] = true
    - Add edge (u, v) to the tree T
    - Push v to the back of L
  - Endif
- Endwhile

- Can modify this procedure to keep track of distance to s (layer numbers) as well. Store the pair (u, l_u), where l_u is the index of the layer containing u.
- Claim: More formally: If BFS(s) pops (v, l_v) from L immediately after it pops (u, l_u), then either l_v = l_u or l_v = l_u + 1.
Analysis of BFS Implementation

BFS(s):

Set Discovered[s] = true
Set Discovered[v] = false, for all other nodes v
Initialize L to consist of the single element s
While L is not empty
    Pop the node u at the head of L
    Consider each edge (u, v) incident on u
    If Discovered[v] = false then
        Set Discovered[v] = true
        Add edge (u, v) to the tree T
        Push v to the back of L
    Endif
Endwhile

How many times is a node popped from L?
Analysis of BFS Implementation

BFS(s):
Set Discovered[s] = true
Set Discovered[v] = false, for all other nodes v
Initialize L to consist of the single element s
While L is not empty
    Pop the node u at the head of L
    Consider each edge (u, v) incident on u
    If Discovered[v] = false then
        Set Discovered[v] = true
        Add edge (u, v) to the tree T
        Push v to the back of L
    Endif
Endwhile

● How many times is a node popped from L? Exactly once.
Analysis of BFS Implementation

BFS(s):

Set Discovered[s] = true
Set Discovered[v] = false, for all other nodes v
Initialize L to consist of the single element s
While L is not empty
    Pop the node u at the head of L
    Consider each edge (u, v) incident on u
    If Discovered[v] = false then
        Set Discovered[v] = true
        Add edge (u, v) to the tree T
        Push v to the back of L
    Endif
Endwhile

- How many times is a node popped from L? Exactly once.
- Time used by for loop for a node u:
Analysis of BFS Implementation

BFS(s):
Set Discovered[s] = true
Set Discovered[v] = false, for all other nodes v
Initialize L to consist of the single element s
While L is not empty
    Pop the node u at the head of L
    Consider each edge (u, v) incident on u
    If Discovered[v] = false then
        Set Discovered[v] = true
        Add edge (u, v) to the tree T
        Push v to the back of L
    Endif
Endwhile

- How many times is a node popped from L? Exactly once.
- Time used by for loop for a node u: $O(n_u)$ time.
Analysis of BFS Implementation

BFS(s):
Set Discovered[s] = true
Set Discovered[v] = false, for all other nodes v
Initialize $L$ to consist of the single element $s$
While $L$ is not empty
    Pop the node $u$ at the head of $L$
    Consider each edge $(u, v)$ incident on $u$
    If Discovered[v] = false then
        Set Discovered[v] = true
        Add edge $(u, v)$ to the tree $T$
        Push $v$ to the back of $L$
    Endif
Endwhile

- How many times is a node popped from $L$? Exactly once.
- Time used by for loop for a node $u$: $O(n_u)$ time.
- Total time for all for loops: $\sum_{u \in G} O(n_u) = O(m)$ time.
- Maintaining layer information:
Analysis of BFS Implementation

BFS(s):
Set Discovered[s] = true
Set Discovered[v] = false, for all other nodes v
Initialize L to consist of the single element s
While L is not empty
    Pop the node u at the head of L
    Consider each edge \((u, v)\) incident on u
    If Discovered[v] = false then
        Set Discovered[v] = true
        Add edge \((u, v)\) to the tree \(T\)
        Push v to the back of L
    Endif
Endwhile

- How many times is a node popped from \(L\)? Exactly once.
- Time used by for loop for a node \(u\): \(O(n_u)\) time.
- Total time for all for loops: \(\sum_{u \in G} O(n_u) = O(m)\) time.
- Maintaining layer information: \(O(1)\) time per node.
- Total time is \(O(n + m)\).
Recursive DFS to Stack-Based DFS

DFS(u):
    Mark \( u \) as "Explored" and add \( u \) to \( R \)
    For each edge \((u, v)\) incident to \( u \)
        If \( v \) is not marked "Explored" then
            Recursively invoke DFS(v)
        Endif
    Endfor

Procedure has “tail recursion”: recursive call is the last step.
Recursive DFS to Stack-Based DFS

DFS($u$):

Mark $u$ as "Explored" and add $u$ to $R$

For each edge $(u, v)$ incident to $u$

  If $v$ is not marked "Explored" then
    Recursively invoke DFS($v$)
  Endif

Endfor

- Procedure has “tail recursion”: recursive call is the last step.
- Can replace the recursion by an iteration: use a stack to explicitly implement the recursion.
Analysing DFS

DFS(s):

Initialize S to be a stack with one element s
While S is not empty
    Take a node u from S
    If Explored[u] = false then
        Set Explored[u] = true
        For each edge (u, v) incident to u
            Add v to the stack S
    Endfor
Endif
Endwhile

● How many times is a node’s adjacency list scanned?
Analysing DFS

DFS(s):
  Initialize S to be a stack with one element s
  While S is not empty
    Take a node u from S
    If Explored[u] = false then
      Set Explored[u] = true
      For each edge (u, v) incident to u
        Add v to the stack S
    Endfor
  Endif
Endwhile

- How many times is a node’s adjacency list scanned? Exactly once.
Analyzing DFS

DFS(s):

  Initialize $S$ to be a stack with one element $s$
  While $S$ is not empty
    Take a node $u$ from $S$
    If $\text{Explored}[u] = \text{false}$ then
      Set $\text{Explored}[u] = \text{true}$
      For each edge $(u, v)$ incident to $u$
        Add $v$ to the stack $S$
    Endfor
  Endif
Endwhile

- How many times is a node’s adjacency list scanned? Exactly once.
- The total amount of time to process edges incident on node $u$’s is
Analysing DFS

DFS(s):

1. Initialize S to be a stack with one element s
2. While S is not empty
   a. Take a node u from S
   b. If Explored[u] = false then
      i. Set Explored[u] = true
      ii. For each edge (u, v) incident to u
          a. Add v to the stack S
   c. Endfor
3. Endif
4. Endwhile

- How many times is a node’s adjacency list scanned? Exactly once.
- The total amount of time to process edges incident on node u’s is $O(n_u)$.
- The total running time of the algorithm is $O(n + m)$. 
Analysing DFS

DFS(s):
  Initialize S to be a stack with one element s
  While S is not empty
    Take a node u from S
    If Explored[u] = false then
      Set Explored[u] = true
      For each edge (u, v) incident to u
        Add v to the stack S
    Endfor
  Endif
Endwhile

- How many times is a node’s adjacency list scanned? Exactly once.
- The total amount of time to process edges incident on node u’s is $O(n_u)$.
- The total running time of the algorithm is $O(n + m)$. 