Review of Priority Queues and Graph Searches

T. M. Murali

February 1, 3, 2021

Motivation: Sort a List of Numbers

Sort

INSTANCE: Nonempty list $x_1, x_2, \ldots, x_n$ of integers.

SOLUTION: A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1},$ for all $1 \leq i < n$.

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- Possible algorithm:
  - Insert each numbers into a data structure D.
  - Repeatedly find the smallest number in D, output it, and remove it.
- To get $O(n \log n)$ running time, each "find minimum" step and each "remove" step must take $O(\log n)$ time.

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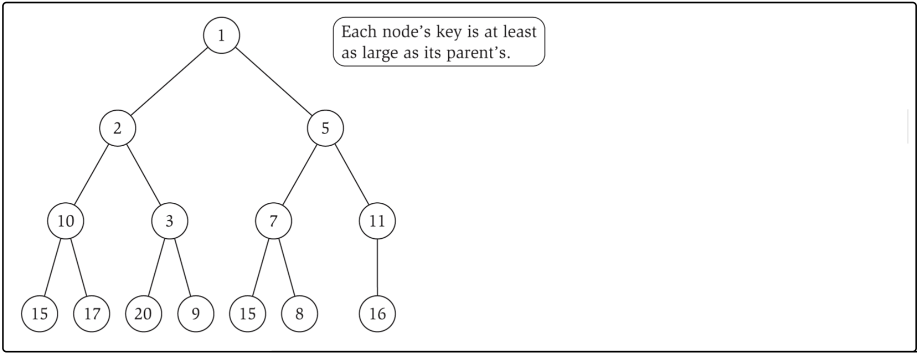
Candidate Data Structures for Sorting

- Possible algorithm:
  - Insert each number into a data structure D.
  - Repeatedly find the smallest number in D, output it, and remove it.
- Data structure must support three operations: insertion of a number, finding minimum, and deleting minimum, each in $O(\log n)$ time.

Priority Queue

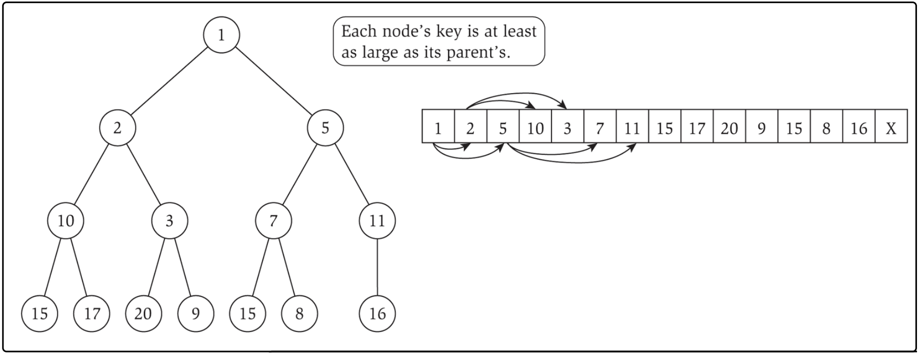
- Store a set S of elements, where each element v has a priority value $\texttt{key}(v)$.
- Smaller key values $\equiv$ higher priorities.
- Operations supported:
  - find the element with smallest key
  - remove the smallest element
  - insert an element
  - delete an element
  - update the key of an element
- Element deletion and key update require knowledge of the position of the element in the priority queue.

Heaps



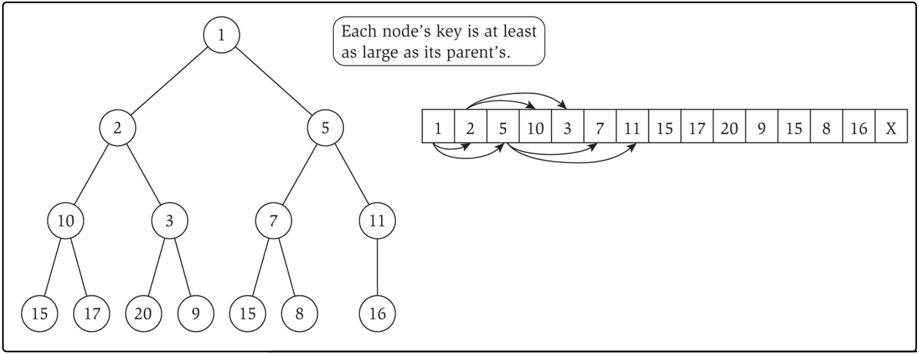
- Combine benefits of both lists and sorted arrays.
- Conceptually, a heap is a balanced binary tree.
- *Heap order*: For every element v at a node i, the element w at i's parent satisfies $\texttt{key}(w) \leq \texttt{key}(v)$.
- We can implement a heap in a pointer-based data structure.

Heaps



- Alternatively, assume maximum number N of elements is known in advance.
- Store nodes of the heap in an array.
  - Node at index i has children at indices $2i$ and $2i+1$ and parent at index $\lfloor i/2 \rfloor$.
  - Index 1 is the root.
  - How do you know that a node at index i is a leaf?

Heaps



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- Store nodes of the heap in an array.
  - Node at index i has children at indices $2i$ and $2i+1$ and parent at index $\lfloor i/2 \rfloor$.
  - Index 1 is the root.
  - How do you know that a node at index i is a leaf? If $2i > n$, where n is the current number of elements in the heap.

Inserting an Element: Heapify-up

1. Insert new element at index $n+1$.
2. Fix heap order using Heapify-up$(H, n+1)$.

```
Heapify-up(H,i):
  If i > 1 then
    let j = parent(i) = ⌊i/2⌋
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
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    Endif
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- Proof of correctness: read pages 61–62 of your textbook.

Example of `Heapify-up`

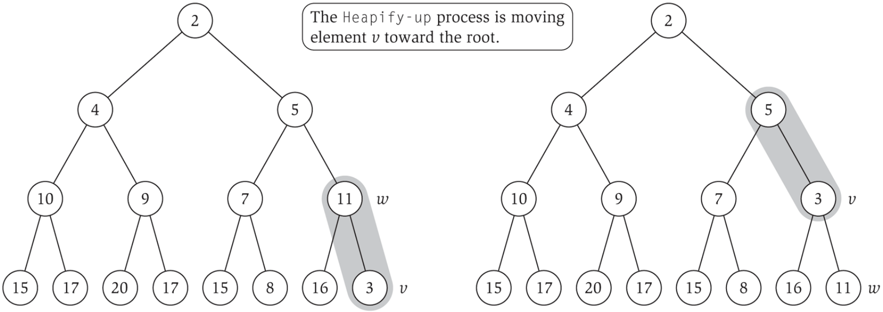

Figure 2.4 The `Heapify-up` process. Key 3 (at position 16) is too small (on the left). After swapping keys 3 and 11, the heap violation moves one step closer to the root of the tree (on the right).

Running time of `Heapify-up`

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- Running time of Heapify-up(i) is $O(\log i)$.
  - Each invocation decreases the second argument by a factor of at least 2. ▸ Poll

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```

- Running time of `Heapify-up`(i) is $O(\log i)$.
  - Each invocation decreases the second argument by a factor of at least 2. Poll
  - After k invocations, argument is at most $i/2^k$.
  - Therefore $i/2^k \geq 1$, which implies that $k \leq \log_2 i$.

Deleting an Element: `Heapify-down`

- Suppose H has $n+1$ elements.

1. Delete element at $H[i]$ by moving element at $H[n+1]$ to $H[i]$.
2. If element at $H[i]$ is too small, fix heap order using `Heapify-up`(H, i).
3. If element at $H[i]$ is too large, fix heap order using `Heapify-down`(H, i).

```
Heapify-down(H,i):
  Let n = length(H)
  If 2i > n then
    Terminate with H unchanged
  Else if 2i < n then
    Let left = 2i, and right = 2i + 1
    Let j be the index that minimizes key[H[left]] and key[H[right]]
  Else if 2i = n then
    Let j = 2i
  Endif
  If key[H[j]] < key[H[i]] then
     swap the array entries H[i] and H[j]
     Heapify-down(H,j)
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```

Deleting an Element: `Heapify-down`

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```

- Proof of correctness: read pages 63–64 of your textbook.

Example of Heapify-down

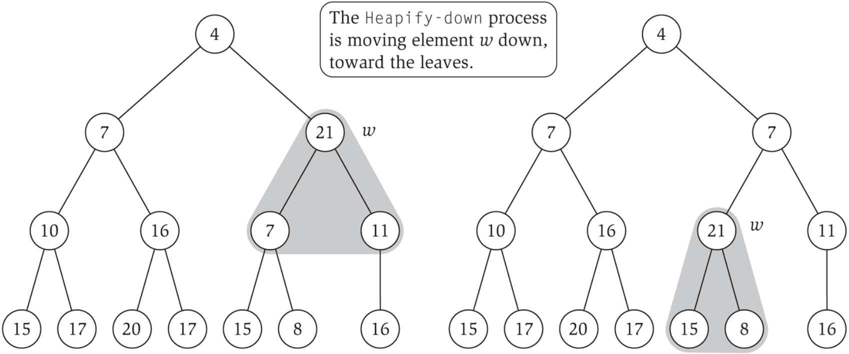

Figure 2.5 The Heapify-down process:. Key 21 (at position 3) is too big (on the left). After swapping keys 21 and 7, the heap violation moves one step closer to the bottom of the tree (on the right).

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Heapify-down(H,i):
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  Endif
  If key[H[j]] < key[H[i]] then
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```

- Each invocation of Heapify-down increases its second argument by a factor of at least two. ▸ Poll
- After k invocations argument must be at least $i2^k \leq n$, which implies that $k \leq \log_2 n/i$. Therefore running time is $O(\log_2 n/i)$.

Sorting Numbers with the Priority Queue

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- Final algorithm:
  - Insert each number in a priority queue H.
  - Repeatedly find the smallest number in H, output it, and delete it from H.

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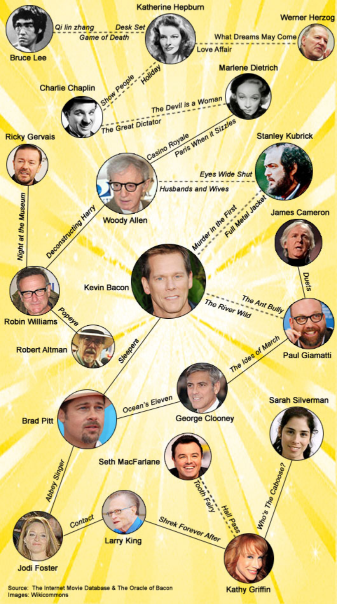
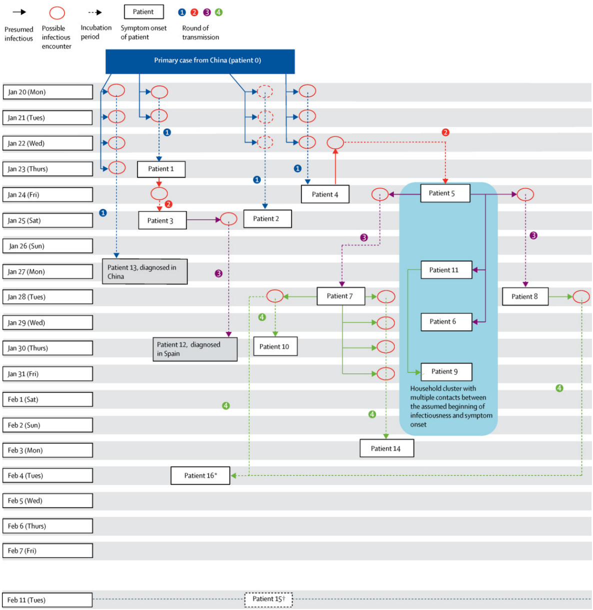
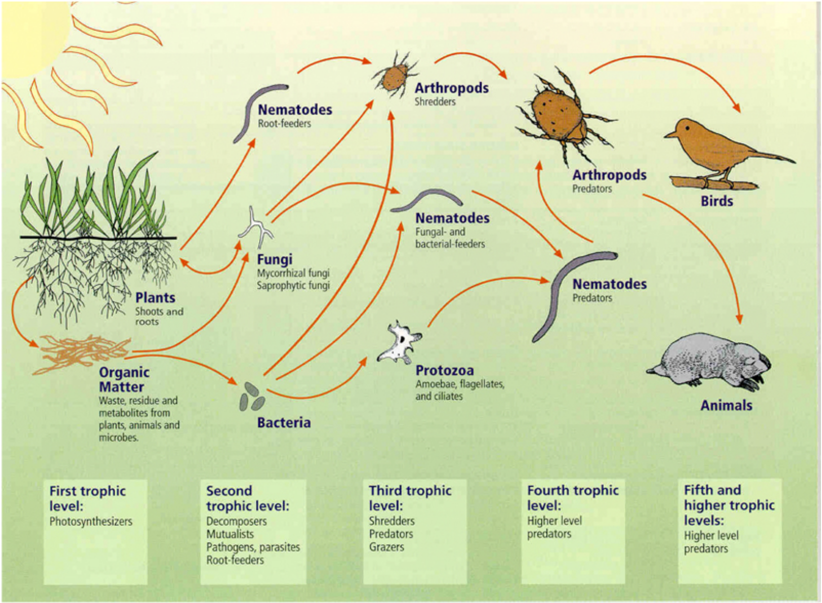
- Final algorithm:
  - Insert each number in a priority queue H.
  - Repeatedly find the smallest number in H, output it, and delete it from H.
- Each insertion and deletion takes $O(\log n)$ time for a total running time of $O(n \log n)$.

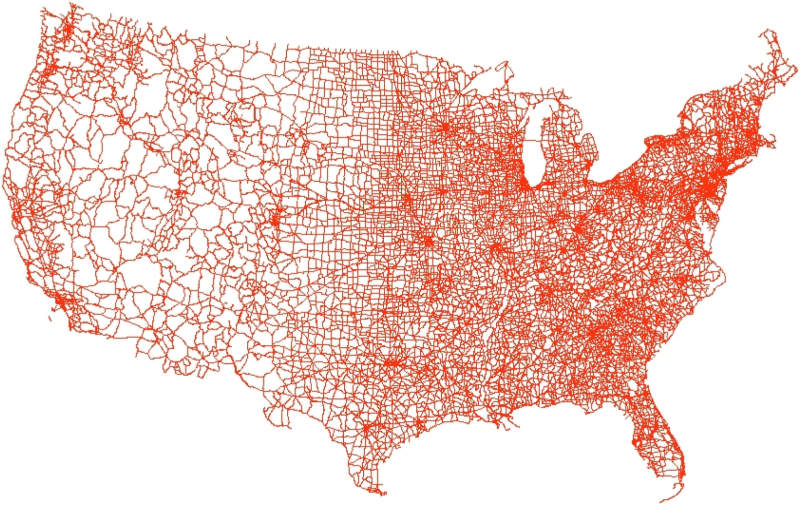


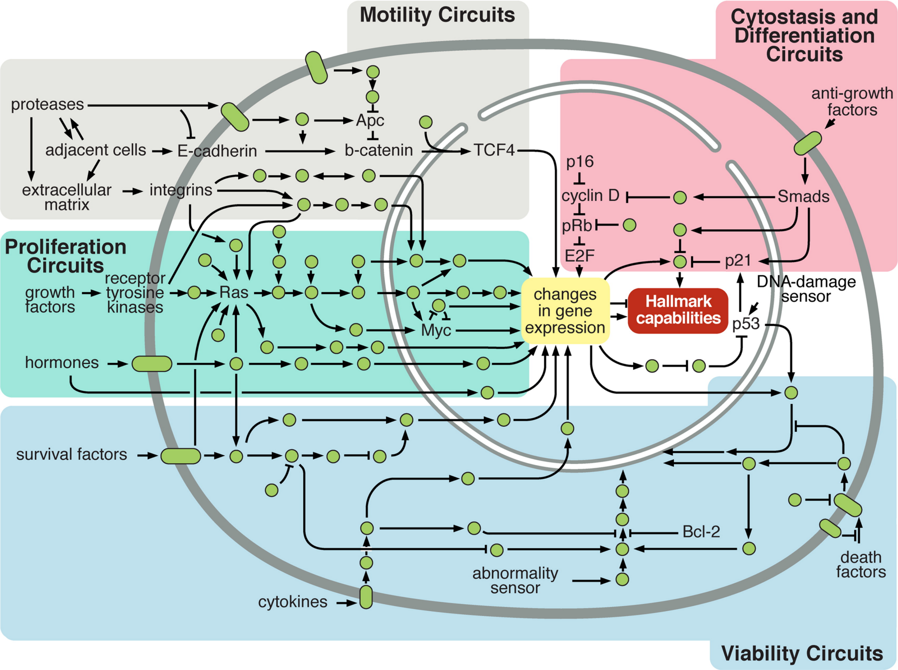
The Oracle of Bacon



(Böhmer *et al.*, *The Lancet*, May 15, 2020)


Nematodes
Root-feeders
Arthropods
Shredders
Arthropods
Predators
Birds
Fungi
Mycorrhizal fungi
Saprophytic fungi
Nematodes
Fungal- and bacterial-feeders
Plants
Shoots and roots
Nematodes
Predators
Organic Matter
Waste, residue and metabolites from plants, animals and microbes.
Protozoa
Amoebae, flagellates, and ciliates
Bacteria
Animals
First trophic level:
Photosynthesizers
Second trophic level:
Decomposers
Mutualists
Pathogens, parasites
Root-feeders
Third trophic level:
Shredders
Predators
Grazers
Fourth trophic level:
Higher level predators
Fifth and higher trophic levels:
Higher level predators




Motility Circuits
Cytostasis and Differentiation Circuits
proteases
adjacent cells
extracellular matrix
integrins
E-cadherin
Apc
b-catenin
TCF4
anti-growth factors
p16
cyclin D
Smads
pRb
E2F
p21
Proliferation Circuits
growth factors
receptor tyrosine kinases
Ras
Myc
changes in gene expression
Hallmark capabilities
DNA-damage sensor
p53
hormones
survival factors
Bcl-2
abnormality sensor
cytokines
death factors
Viability Circuits

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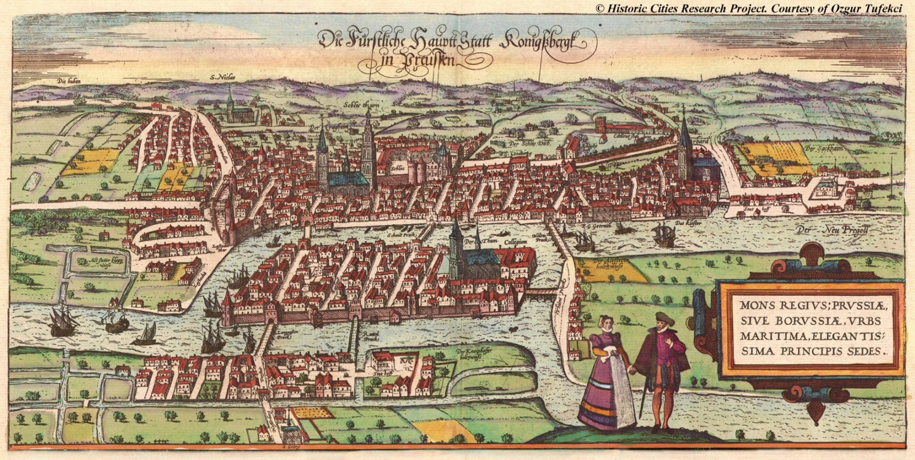
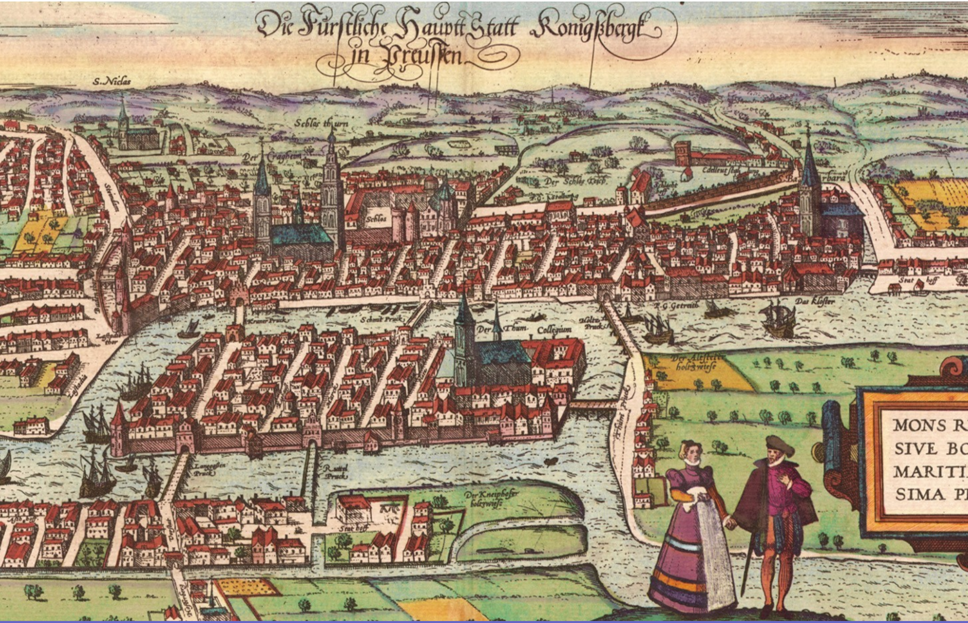
- Problems involving graphs have a rich history dating back to Euler.

Euler and Graphs

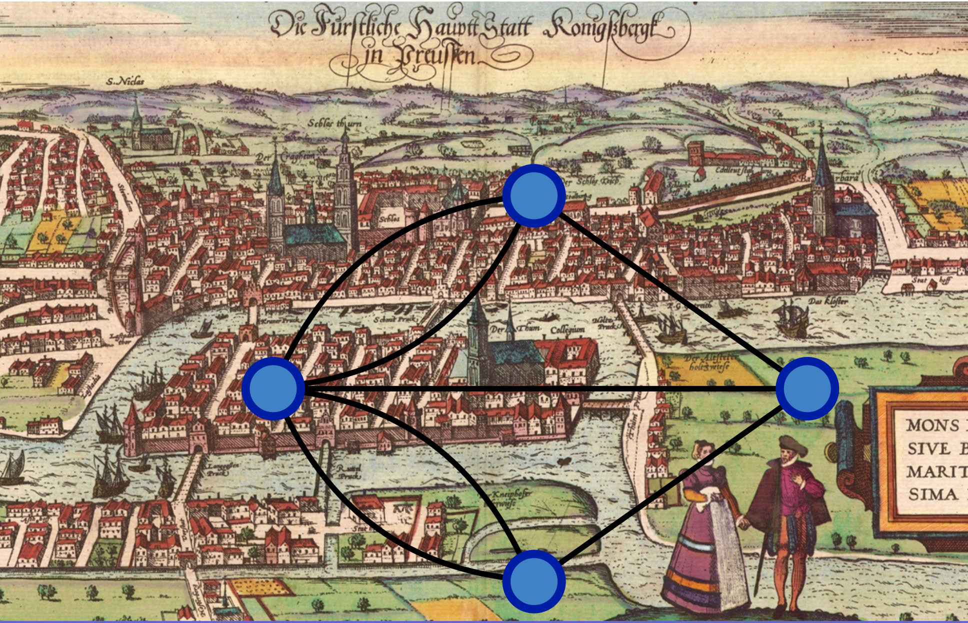


Devise a walk through the city that
crosses each of the seven bridges exactly once.

Euler and Graphs

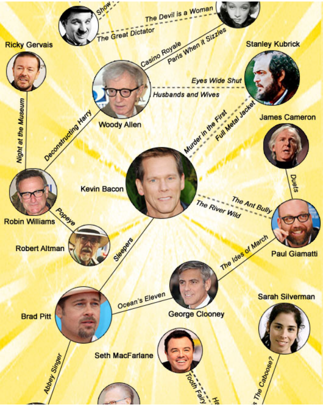


Euler and Graphs



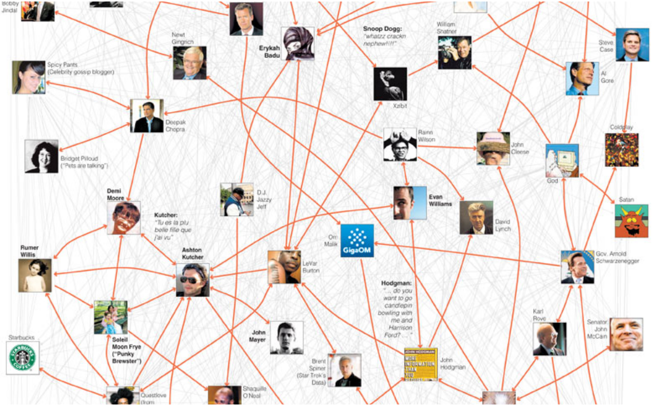
Definition of a Graph

- *Undirected graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$.
  - Elements of E are unordered pairs.
  - Edge (u, v) is *incident* on u, v; u and v are *neighbours* of each other.
  - Exactly one edge between any pair of nodes.
  - G contains no self loops, i.e., no edges of the form (u, u).

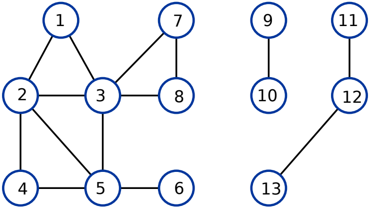


Definition of a Graph

- *Directed graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$.
  - Elements of E are ordered pairs.
  - $e = (u, v)$: u is the *tail* of the edge e, v is its *head*; e is *directed from u to v*.
  - A pair of nodes may be connected by two directed edges: (u, v) and (v, u).
  - G contains no self loops.



Paths and Connectivity



- A *v_1-v_k path* in an undirected graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k \in V$ such that every consecutive pair of nodes $v_i, v_{i+1}, 1 \leq i < k$ is connected by an edge in E.

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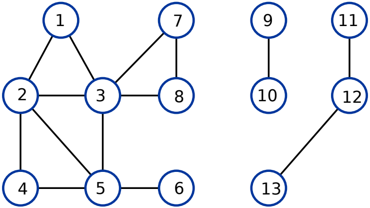

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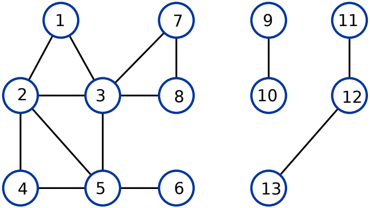

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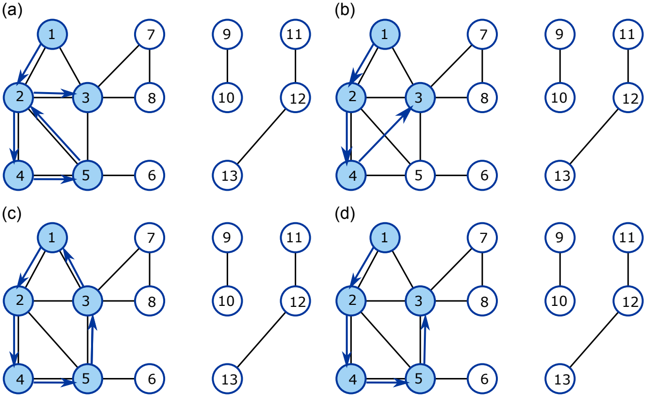

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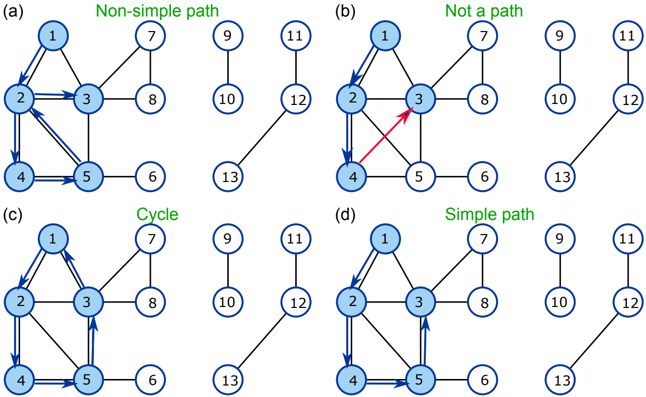
- Poll

Paths and Connectivity



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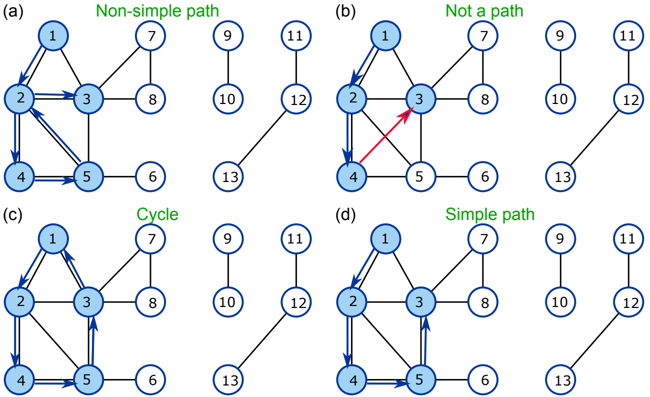

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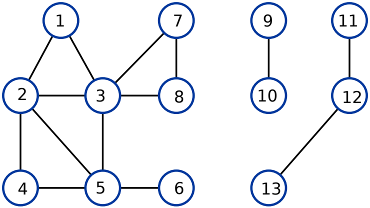
- A path is *simple* if all its nodes are distinct.
- A *cycle* is a path where $k > 2$, the first $k - 1$ nodes are distinct, and $v_1 = v_k$.
- Similar definitions carry over to directed graphs as well.

Connectivity



- An undirected graph G is *connected* if for every pair of nodes $u, v \in V$, there is a path from u to v in G.

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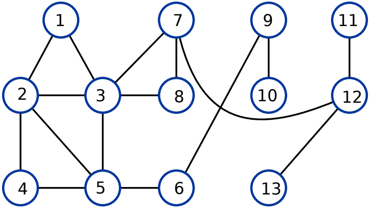

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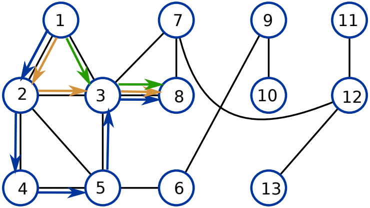
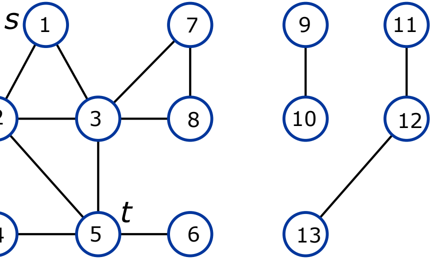


- An undirected graph G is *connected* if for every pair of nodes $u, v \in V$, there is a path from u to v in G.
- *Distance* $d(u, v)$ between two nodes u and v is the minimum number of edges in any u-v path.

s-t Connectivity



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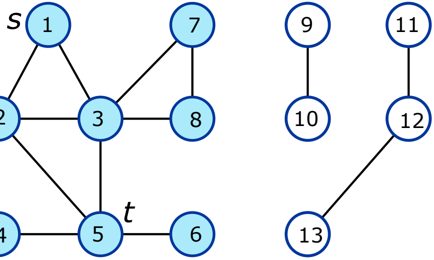

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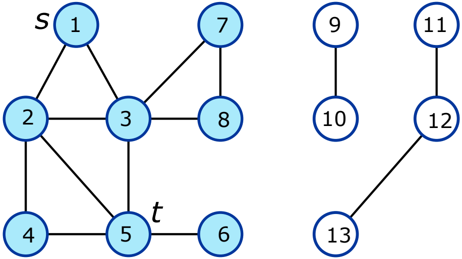

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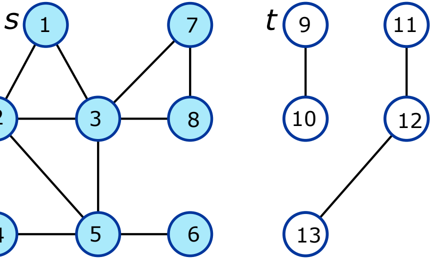

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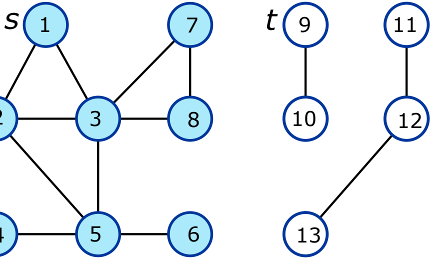

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- Appears to do more work than is strictly necessary.

Computing Connected Components

- Abstract idea for an algorithm, with details to be specified later.
- "Explore" G starting from s and maintain set R of visited nodes.

R will consist of nodes to which s has a path

Initially $R = \{s\}$

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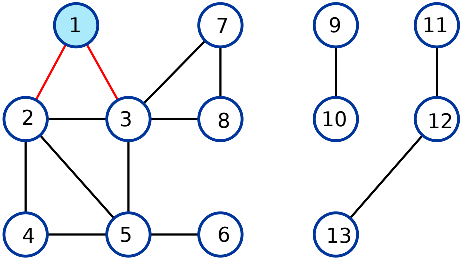

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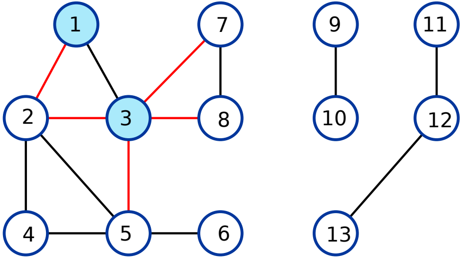

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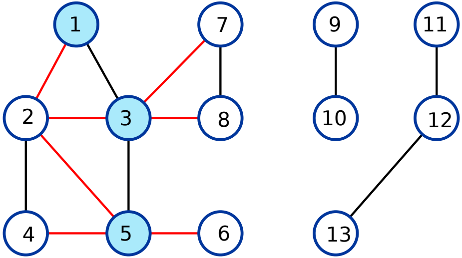

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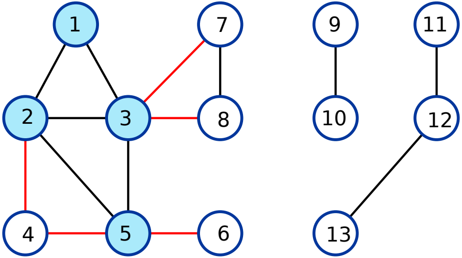

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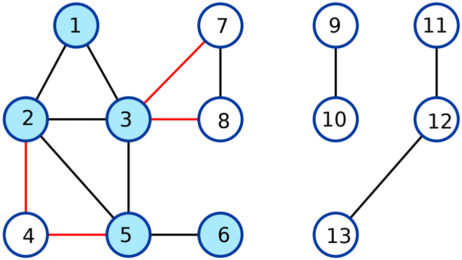

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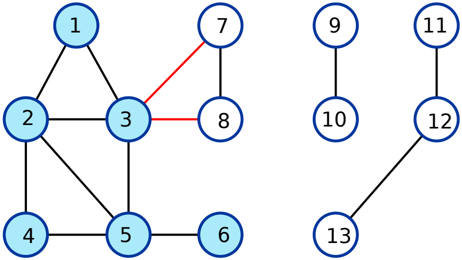

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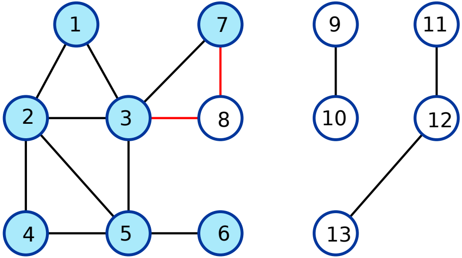

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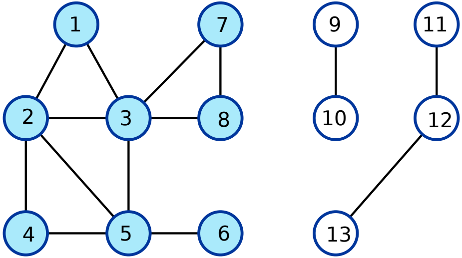

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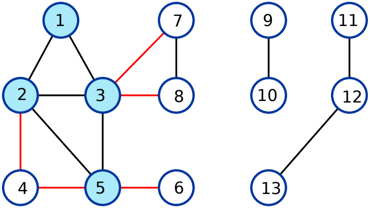

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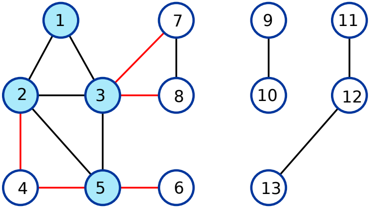

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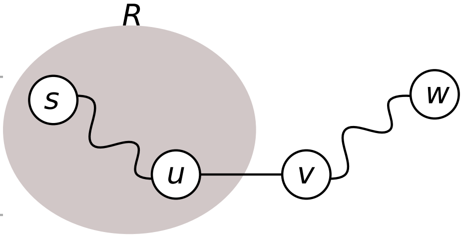
Correctness of the Algorithm

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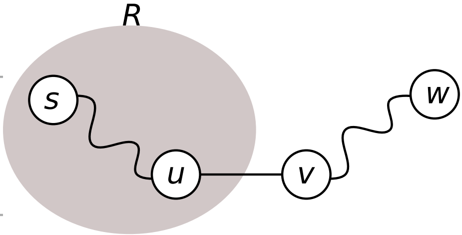

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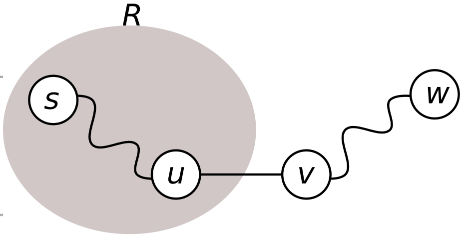

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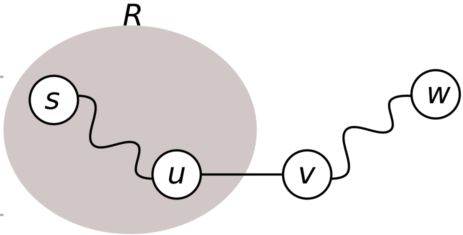

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  - Note: wrong to assume that predecessor of w in P is not in R.

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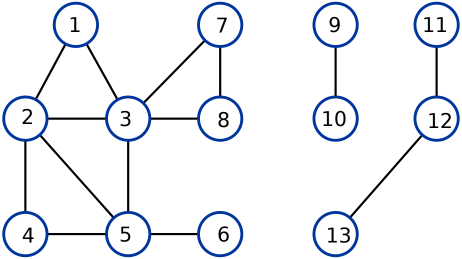
- How fast can we implement check in the `while` loop?
  - Naive approach: examine each edge in the graph.
  - Total running time is $O(mn)$.
- BFS and DFS improve the running time by processing edges more carefully.

Breadth-First Search (BFS)



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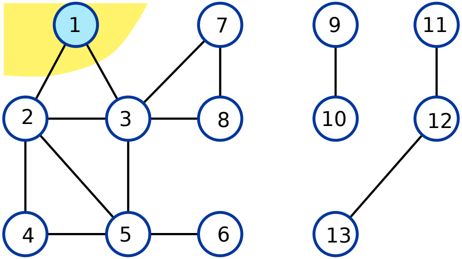

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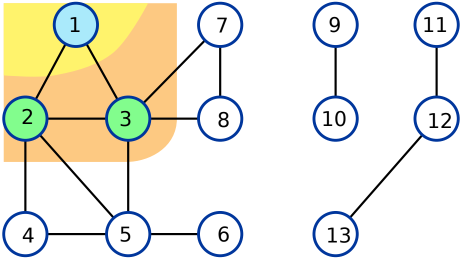

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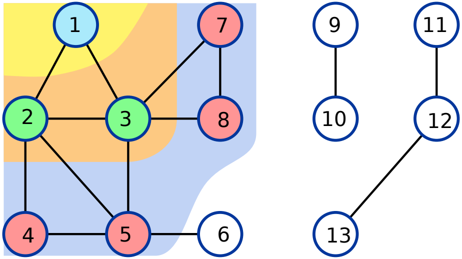

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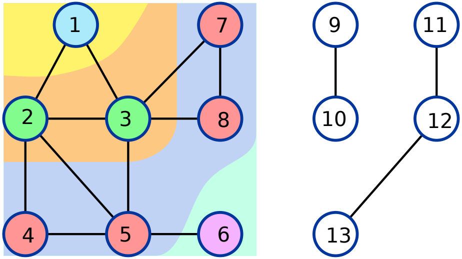

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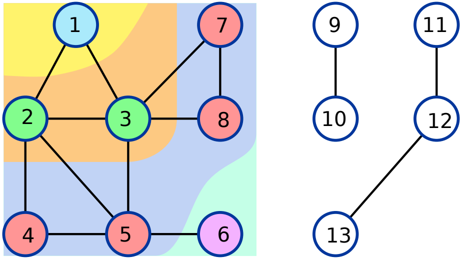
Properties of BFS



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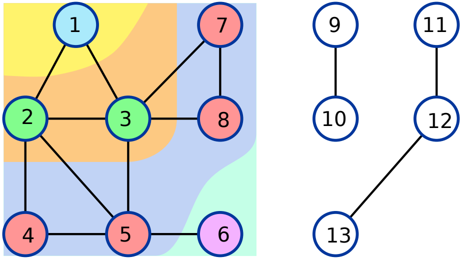

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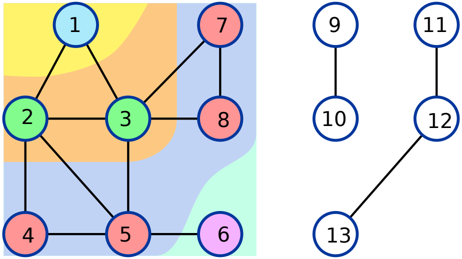

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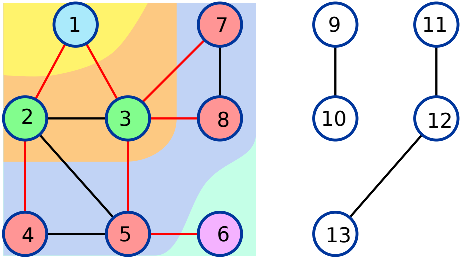

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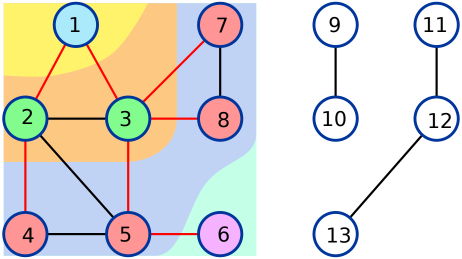

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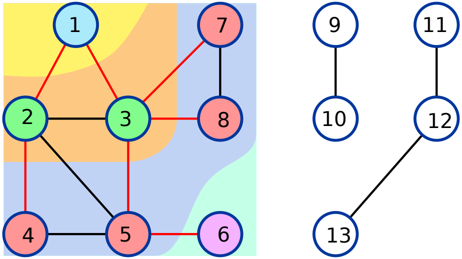

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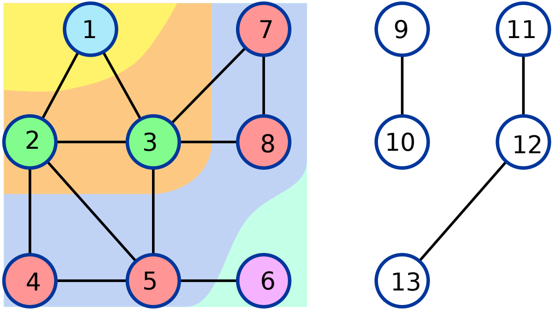
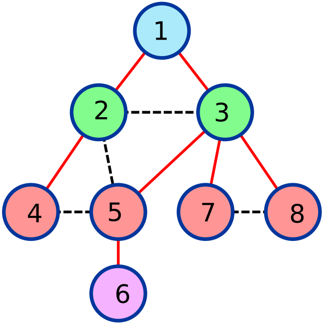
- Consider the graph T formed by all such edges, directed from u to v. Poll
  - Why is T a tree? It is connected. The number of edges in T is the number of nodes in all the layers minus 1.
  - T is called the *breadth-first search tree*.

BFS Trees





- *Non-tree edge*: an edge of G that does not belong to the BFS tree T.
- Claim: Let T be a BFS tree, let x and y be nodes in T belonging to layers L_i and L_j, respectively, and let (x, y) be an edge of G. Then $|i - j| \leq 1$.

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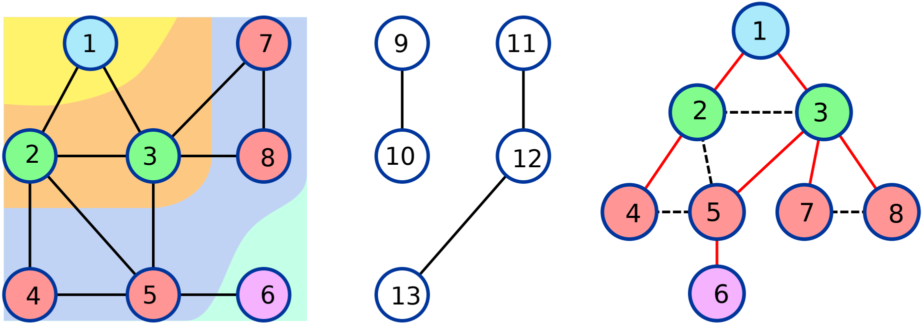

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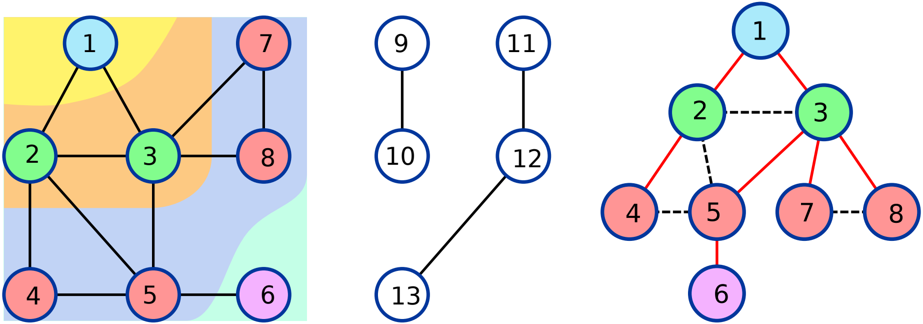

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- Still unresolved: an efficient implementation of BFS.

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2. Invoke DFS(s).

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DFS(u):
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  For each edge (u, v) incident to u
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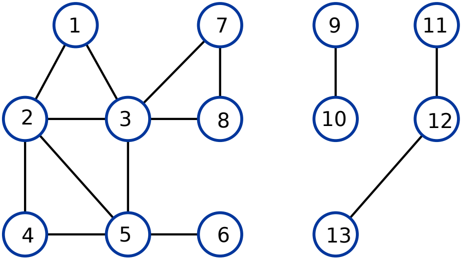
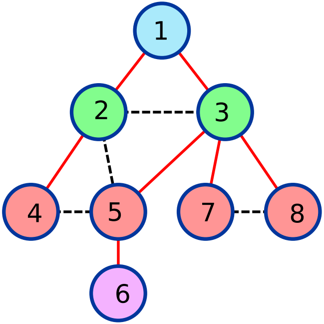
- *Depth-first search tree* is a tree T: when `DFS(`v`)` is invoked directly during the call to `DFS(`v`)`, add edge (u, v) to T.

Example of DFS





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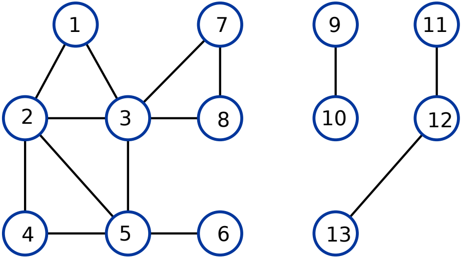

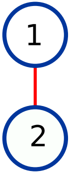

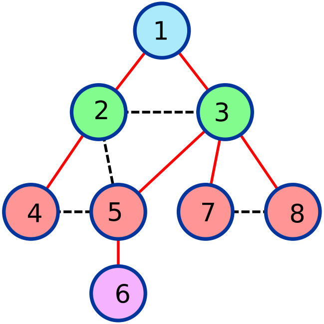

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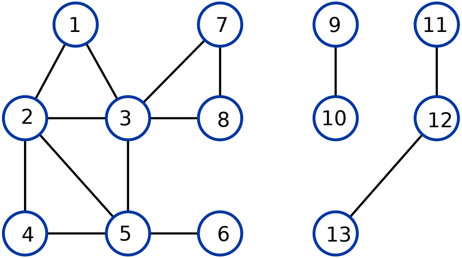

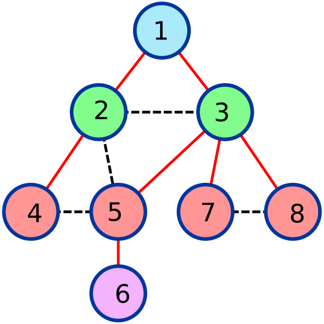

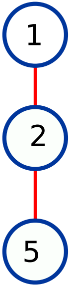

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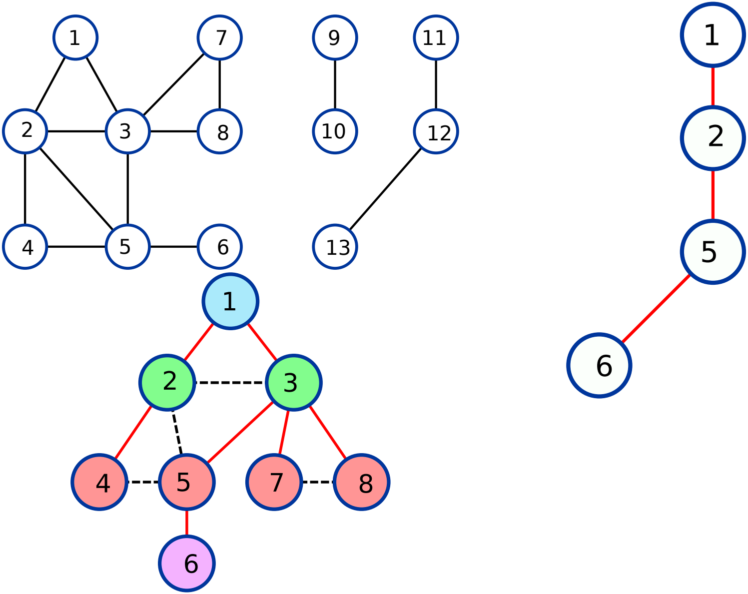

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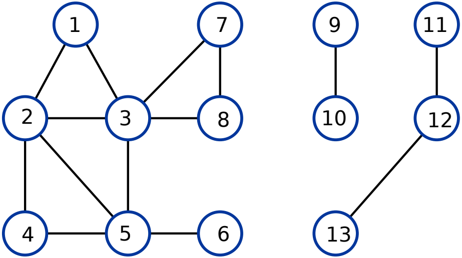

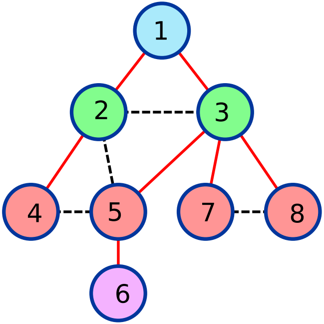

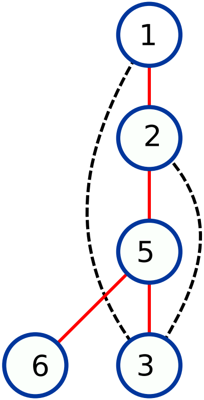

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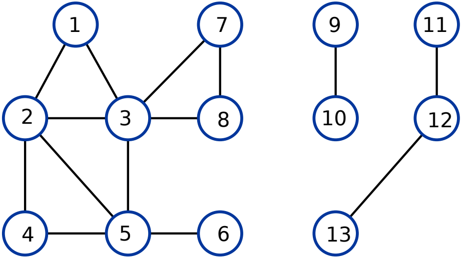

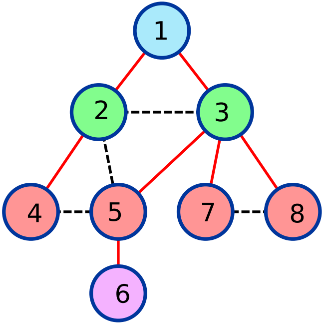

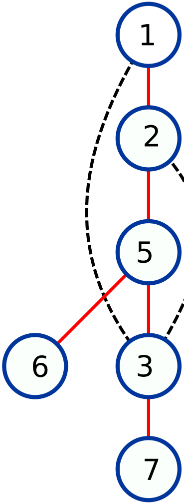

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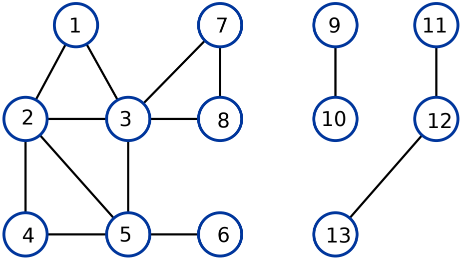

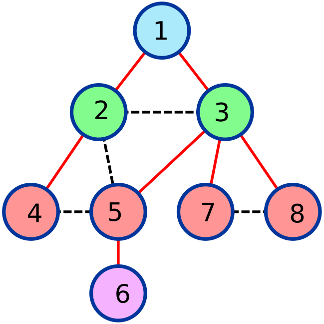

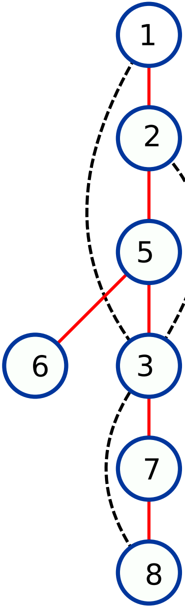

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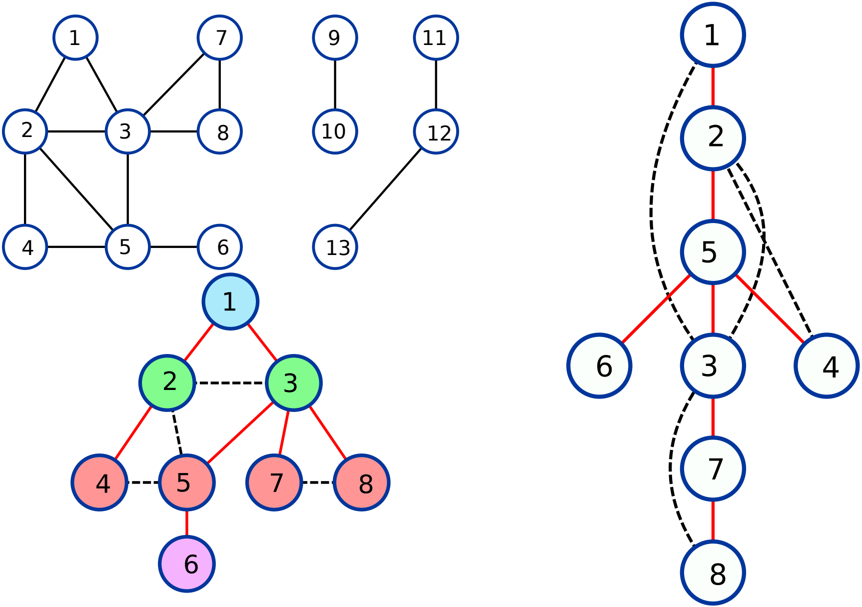
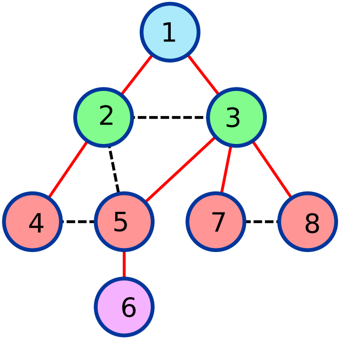
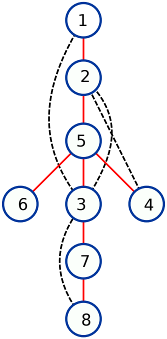


BFS vs. DFS





Poll

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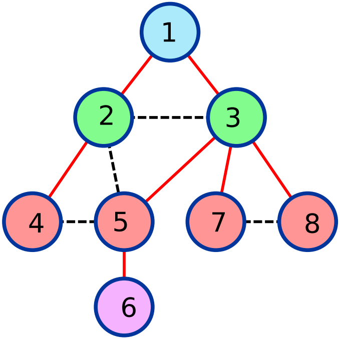

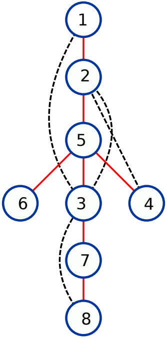

Poll

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- Both traverse all the edges in the connected component but in a different order.
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  BFS within the same level or between adjacent levels.

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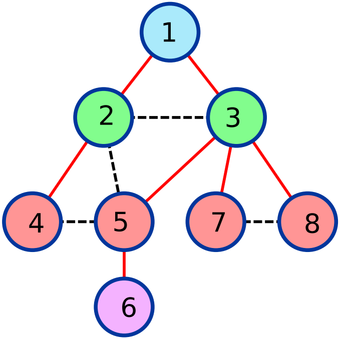

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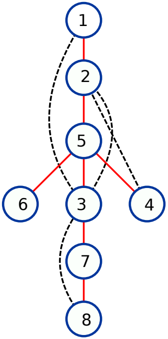

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  - DFS connect ancestors to descendants.

Properties of DFS Trees

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DFS(u):
  Mark u as "Explored" and add u to R
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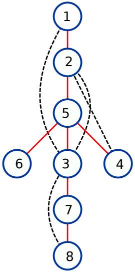

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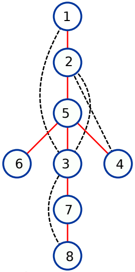

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- Claim: Let x and y be nodes in a DFS tree T such that (x, y) is an edge of G but not of T. Then one of x or y is an ancestor of the other in T. Read proof on page 86 of your textbook.

Representing Graphs

- Graph $G = (V, E)$ has two input parameters: $|V| = n, |E| = m$.
  - Size of the graph is defined to be $m + n$.
  - Strive for algorithms whose running time is linear in graph size, i.e., $O(m + n)$.

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Operation/Space	Adj. matrix	Adj. list
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  - Space used and time to iterate over neighbours are optimal for every graph. Poll

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Data Structures for Implementation

- "Implementation" of BFS and DFS: fully specify the algorithms and data structures so that we can obtain provably efficient times.
- Inner loop of both BFS and DFS: process the set of edges incident on a given node and the set of visited nodes.
- How do we store the set of visited nodes? Order in which we process the nodes is crucial.

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- How do we store the set of visited nodes? Order in which we process the nodes is crucial.
  - BFS: store visited nodes in a queue (first-in, first-out).
  - DFS: store visited nodes in a stack (last-in, first-out)

Using a Queue in BFS

- Maintain an array Discovered and set Discovered$[v]$ = *true* as soon as the algorithm sees v.
- Maintain all the layers in a single queue L.

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BFS(s):
   Set Discovered[s] = true
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   Initialize L to consist of the single element s
   While L is not empty
      Pop the node u at the head of L
      For each edge (u, v) incident on u
         If Discovered[v] = false then
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            Add edge (u, v) to the tree T
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         Endif
      Endfor
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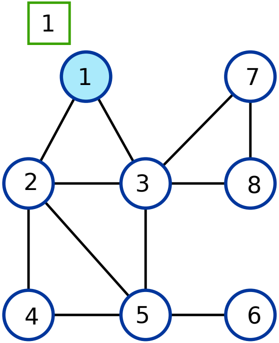

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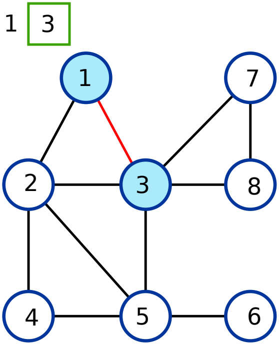

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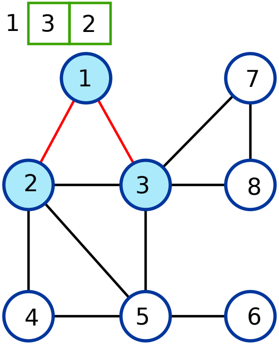

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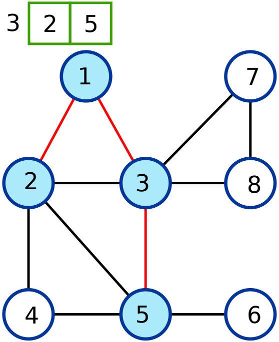

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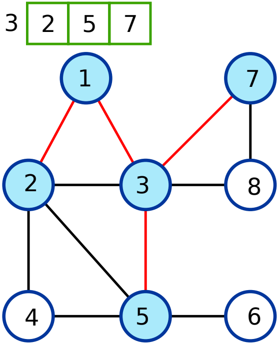

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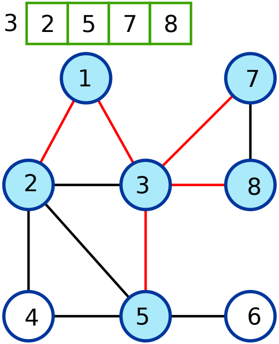

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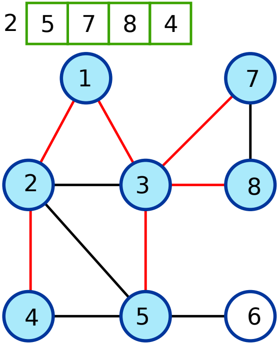

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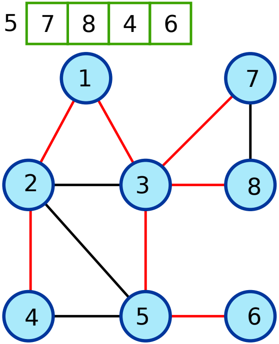

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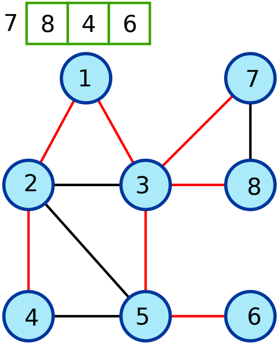

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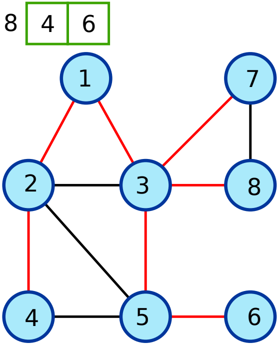

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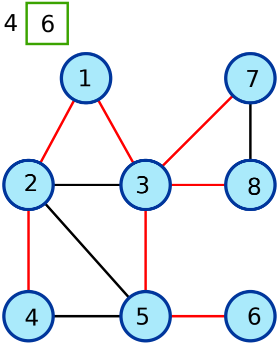

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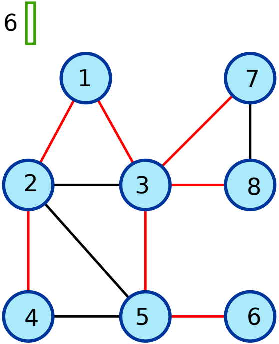

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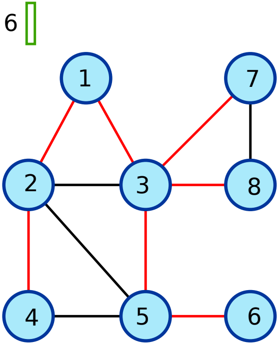

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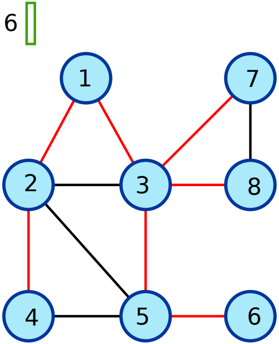

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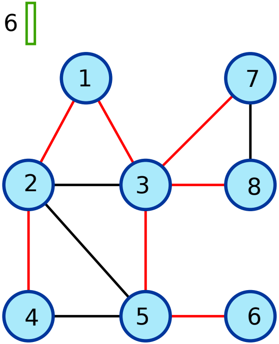

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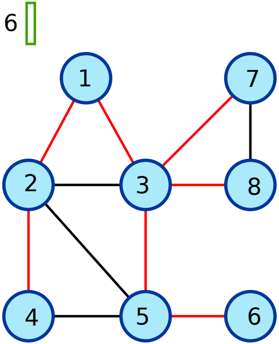

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- Total time is $O(n + m)$.

Recursive DFS to Stack-Based DFS

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DFS(u):
  Mark u as "Explored" and add u to R
  For each edge (u,v) incident to u
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- Procedure has "tail recursion": recursive call is the last step.
- Can replace the recursion by an iteration: use a stack to explicitly implement the recursion.

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- How many times is a node's adjacency list scanned?

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