

Greedy Algorithms

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February 10, 15, 2021

Algorithm Design

- Start discussion of different ways of designing algorithms.
- Greedy algorithms, divide and conquer, dynamic programming.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.

Algorithm Design

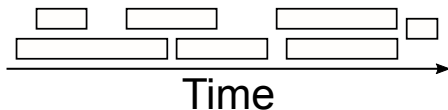
- Start discussion of different ways of designing algorithms.
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- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.
- Greedy algorithms: make the current best choice.





- Input: Start and end time of each ride.
- Constraint: Cannot be in two places at one time.
- Goal: Compute the largest number of rides you can be on in one day.

Interval Scheduling



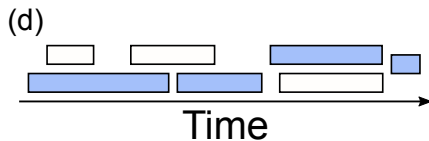
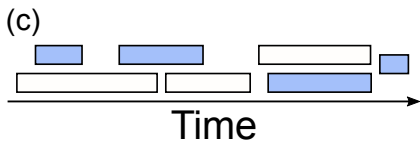
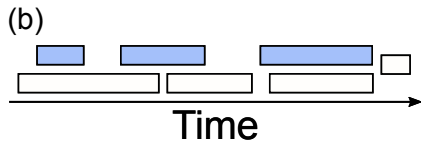
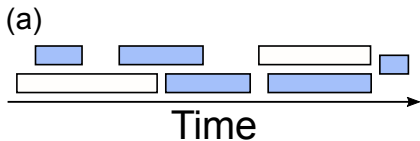
INTERVAL SCHEDULING

INSTANCE: Set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION: The largest subset of mutually compatible jobs.

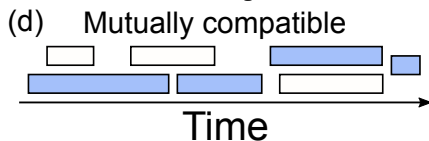
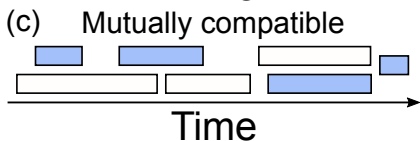
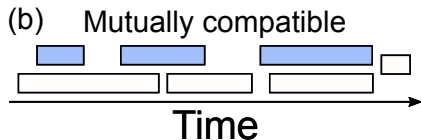
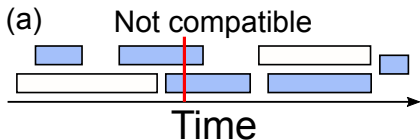
- Two jobs are *compatible* if they do not overlap.
- This problem models the situation where you have a resource, a set of fixed jobs, and you want to schedule as many jobs as possible.
- For any input set of jobs, algorithm must provably compute the **largest** set of compatible jobs.

Interval Scheduling Example



► Poll

Interval Scheduling Example



- Solutions (c) and (d) are optimal.
 - ▶ Each contains four jobs.
 - ▶ No set of mutually compatible jobs can contain more than four jobs.

Template for Greedy Algorithm

- Process jobs in some order. Add next job to the result if it is compatible with the jobs already in the result.
- Key question: in what order should we process the jobs?

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 - **Earliest finish time** Increasing order of finish time $f(i)$.
 - **Shortest interval** Increasing order of length $f(i) - s(i)$.
 - **Fewest conflicts** Increasing order of the number of conflicting jobs.

Template for Greedy Algorithm

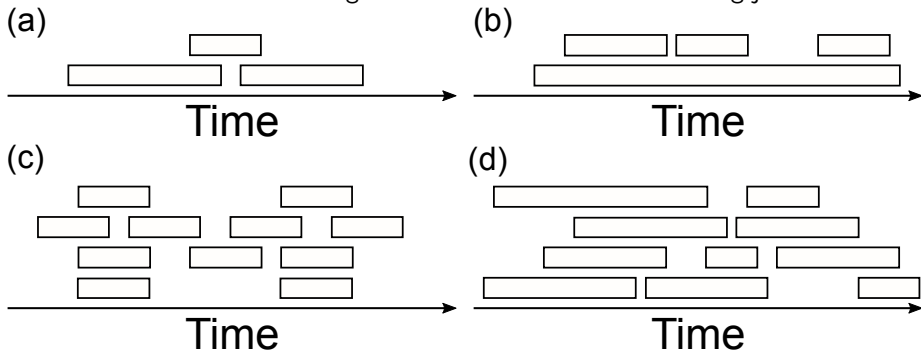
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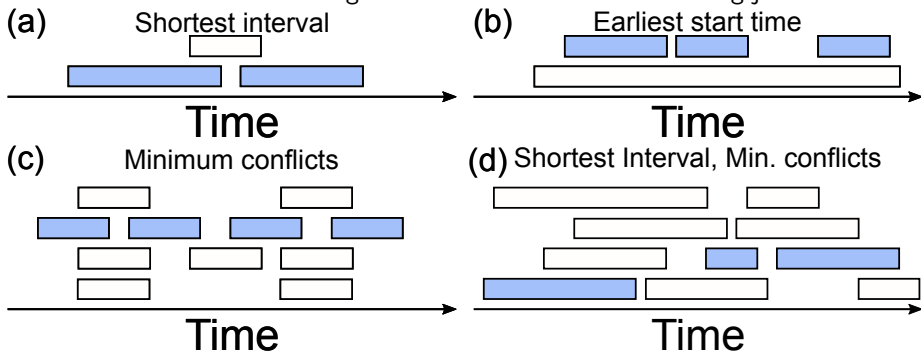
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Interval Scheduling Algorithm: Earliest Finish Time

- Schedule jobs in order of earliest finish time (EFT).

Initially let R be the set of all requests, and let A be empty

While R is not yet empty

 Choose a request $i \in R$ that has the smallest finishing time

 Add request i to A

 Delete all requests from R that are not compatible with request i

EndWhile

Return the set A as the set of accepted requests

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- Claim: A is a compatible set of jobs. Proof follows by construction, i.e., the algorithm computes a compatible set of jobs.

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- Proof idea 2: at each step, can we show algorithm has the “better” solution than any other answer?
 - ▶ What does “better” mean?
 - ▶ How do we measure progress of the algorithm?
- Basic idea of proof:
 - ▶ We can sort jobs in any solution in increasing order of their finishing time.
 - ▶ Finishing time of job number r selected by $A \leq$ finishing time of job number r selected by any other algorithm.

Analysing the EFT Algorithm

- Let O be an optimal set of jobs. We will show that $|A| = |O|$.
- Let i_1, i_2, \dots, i_k be the set of jobs in A in order.
- Let j_1, j_2, \dots, j_m be the set of jobs in O in order, $m \geq k$.
- Claim: For all indices $r \leq k$, $f(i_r) \leq f(j_r)$.

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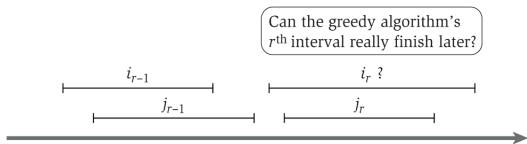


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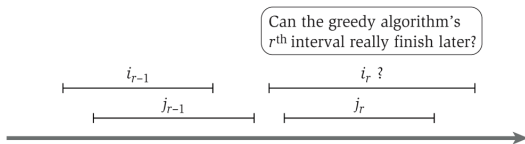


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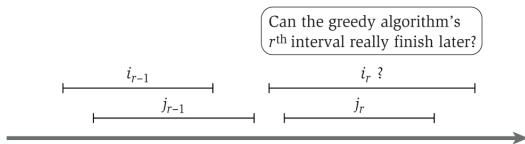
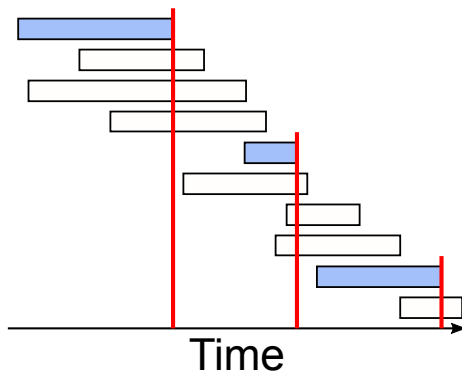


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- Claim: $m = k$.
- Claim: The greedy algorithm returns an optimal set A .

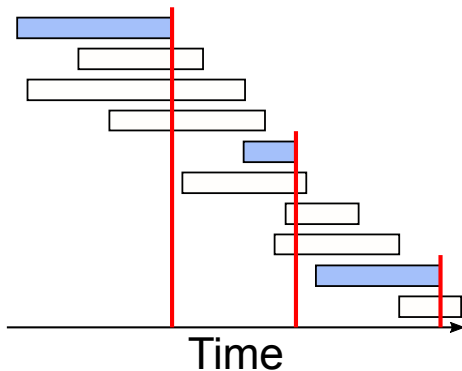
Implementing the EFT Algorithm

- 1 Reorder jobs so that they are in increasing order of finish time.
- 2 Store starting time of jobs in an array S .
- 3 $k = 1$.
- 4 While $k \leq |S|$,
 - 1 Output job k .
 - 2 Let finish time of job k be f .
 - 3 Iterate over S from index k onwards to find the first index i such that $S[i] \geq f$.
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- Must be careful to iterate over S such that we never scan same index more than once.
- Running time is $O(n \log n)$, dominated by sorting.



Interval Scheduling		Interval Partitioning					Minimising Lateness				
12616	CS-4104	Data and Algorithm Analysis	L	3	75	CA Shaffer	T R	2:00PM	3:15PM	SURGE 107	14T
18154	CS-4104	Data and Algorithm Analysis	L	3	70	TM Murali	M W	2:30PM	3:45PM	SURGE 104C	14M
12617	CS-4114	Formal Languages	L	3	75	L Zhang	T R	9:30AM	10:45AM	MCB 129	09T
18155	CS-4204	Computer Graphics	L	3	36	D Gracanin	T R	11:00AM	12:15PM	MCB 224	11T
19593	CS-4264	Principles Computer Security	L	3	50	KE Giles	M W	2:30PM	3:45PM	GOODW 135	14M
12618	CS-4284	Systems & Networking Capstone	L	3	40	GV Back	M W	2:30PM	3:45PM	MCB 238	14M
18156	CS-4304	Compiler Design	L	3	50	C Jung	T R	8:00AM	9:15AM	GOODW 125	08T
12620	CS-4604	Int Data Base Mgt Sys	L	3	55	RJ Quintin	M W	4:00PM	5:15PM	SURGE 109	16M
12621	CS-4624	Multimedia/Hypertext	L	3	70	EA Fox	T R	3:30PM	4:45PM	SURGE 109	15T
12622	CS-4644	Creative Computing Studio	L	3	25	SR Harrison	W	2:30PM	5:15PM	MAC 253A	14W
Comments for CRN 12622:		Prerequisite: C or better in CS 3724 OR CS 3744									
12623	CS-4654	Intermed Data Analytics & ML	L	3	50	RB Gramacy	M W	4:00PM	5:15PM	SEITZ 313	16M
12624	CS-4704	Software Engineering Capstone	L	3	15	KR Edmison	M W	4:00PM	5:15PM	NCB 170	16M
Comments for CRN 12624:		Prerequisite: C or better in CS 3704 OR CS 3714									
12625	CS-4784	Human-Computer Interact Capstn	L	3	30	AL Kavanaugh	F	1:00PM	3:45PM	MAC 253A	13F
Comments for CRN 12625:		Prerequisite: CS 3724 required; CS 3714 or 3744 recommended									
19924	CS-4784	Human-Computer Interact Capstn	L	3	0	DS McCrickard	F	12:30PM	3:15PM	MCB 655	12F

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- Input: Start and end time of each class.
- Constraint: Cannot schedule two overlapping classes to the same room.
- Output: Assign each class to a room and use smallest number of rooms possible.

Interval Partitioning

INTERVAL PARTITIONING

INSTANCE: Set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION: A partition of the jobs into k sets, where each set of jobs is mutually compatible, and k is minimised.

- This problem models the situation where you have a set of fixed jobs, and you want to schedule all jobs using as few resources as possible.

Depth of Intervals

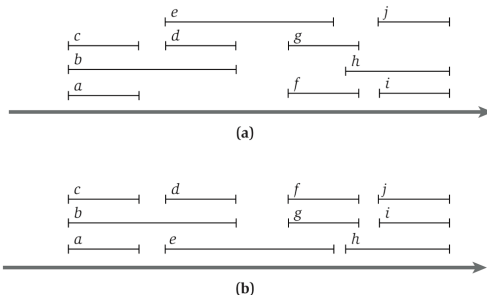


Figure 4.4 (a) An instance of the Interval Partitioning Problem with ten intervals (*a* through *j*). (b) A solution in which all intervals are scheduled using three resources: each row represents a set of intervals that can all be scheduled on a single resource.

- The *depth* of a set of intervals is the maximum number of intervals that contain any time point.

Depth of Intervals

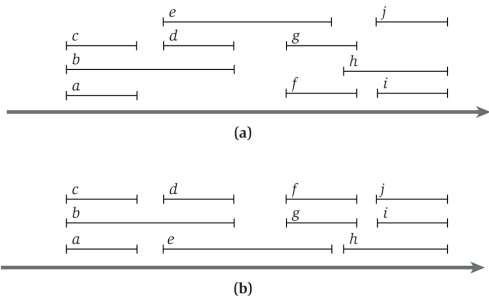


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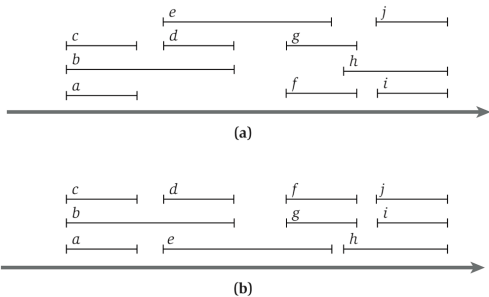


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- The *depth* of a set of intervals is the maximum number of intervals that contain any time point.
- Claim: In any instance of INTERVAL PARTITIONING, $k \geq \text{depth}$.
- Is it possible to compute the depth efficiently? Is $k = \text{depth}$?

Computing the Depth of the Intervals

- How efficiently can we compute the depth of a set of intervals?

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- How efficiently can we compute the depth of a set of intervals?
- ① Sort the start times and finish times of the jobs into a single list L .
- ② $d \leftarrow 0$.
- ③ For i ranging from 1 to $2n$
 - ① If L_i is a start time, increment d by 1.
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- Algorithm runs in $O(n \log n)$ time.

Interval Partitioning Algorithm

- First, compute the depth d of the intervals.

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Sort the intervals by their start times, breaking ties arbitrarily

Let I_1, I_2, \dots, I_n denote the intervals in this order

For $j = 1, 2, 3, \dots, n$

 For each interval I_i that precedes I_j in sorted order and overlaps it

 Exclude the label of I_i from consideration for I_j

 Endfor

 If there is any label from $\{1, 2, \dots, d\}$ that has not been excluded then

 Assign a nonexcluded label to I_j

 Else

 Leave I_j unlabeled

 Endif

Endfor

- Claim: Every interval gets a label and no pair of overlapping intervals get the same label.

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- Claim: The greedy algorithm is optimal.

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- The running time of the algorithm is $O(n \log n)$.

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- Claim: Every interval gets a label and no pair of overlapping intervals get the same label.
- Claim: The greedy algorithm is optimal.
- The running time of the algorithm is $O(n \log n)$. Can modify algorithm for computing depth to maintain set of available labels and to assign them efficiently.

Scheduling to Minimise Lateness

- Study different model: job i has a length $t(i)$ and a deadline $d(i)$.
- We want to schedule all n jobs on one resource.
- Our goal is to assign a starting time $s(i)$ to each job such that each job is delayed as little as possible.

Scheduling to Minimise Lateness

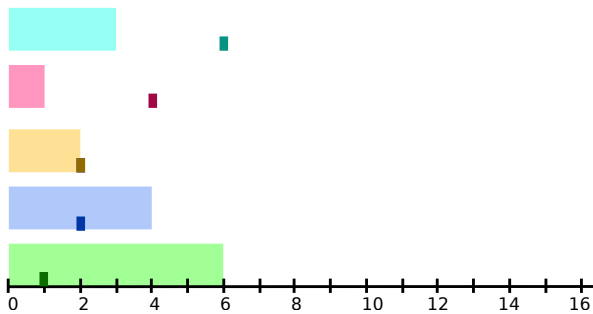
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- We want to schedule all n jobs on one resource.
- Our goal is to assign a starting time $s(i)$ to each job such that each job is delayed as little as possible.
- A job i is *delayed* if $f(i) > d(i)$; the *lateness of the job* is

$$\max(0, f(i) - d(i)).$$

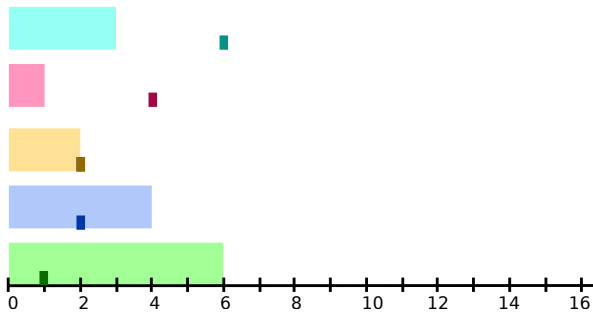
- The *lateness of a schedule* is

$$\max_{1 \leq i \leq n} (\max(0, f(i) - d(i))).$$

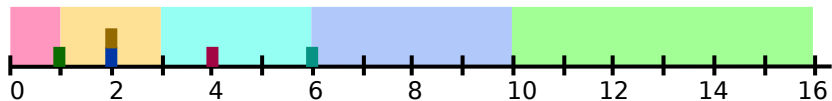
Examples of Lateness



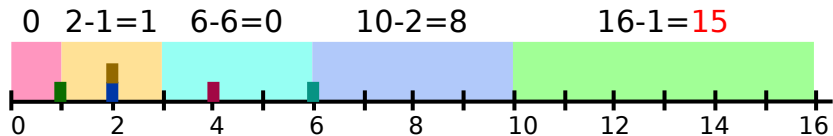
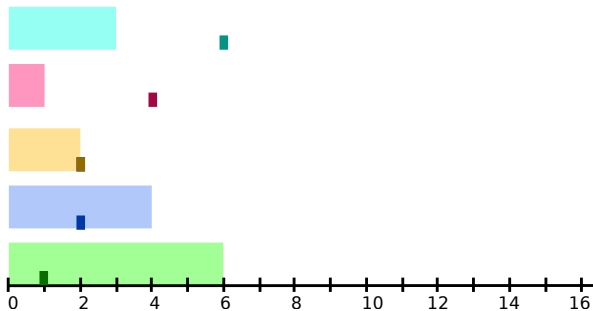
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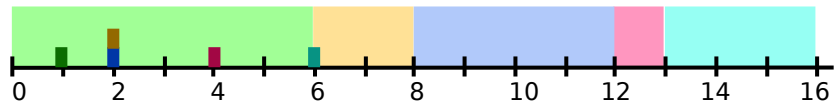
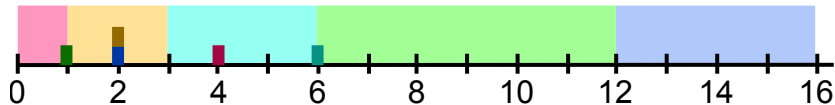
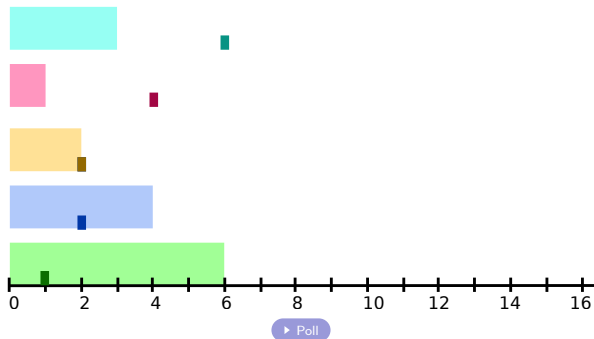
► Poll



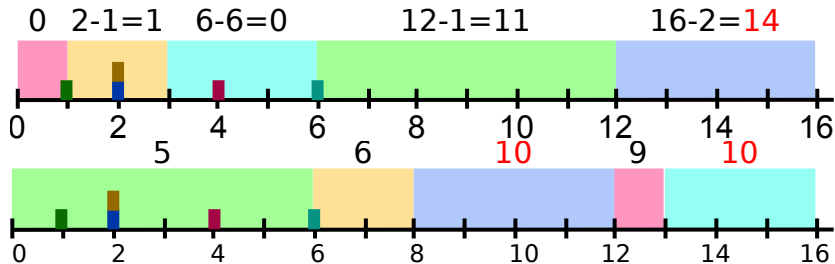
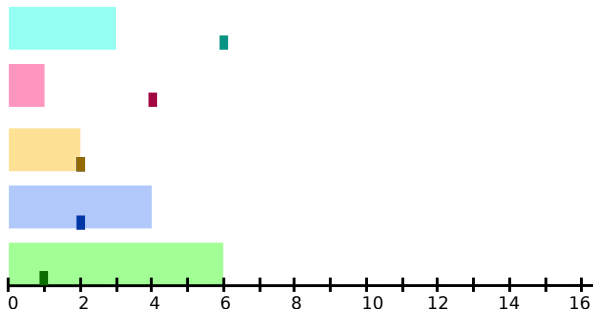
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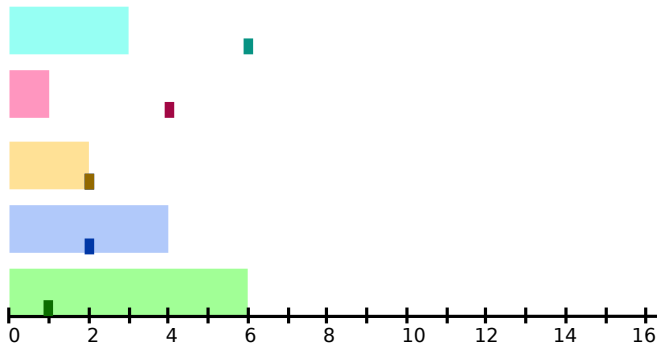


Scheduling to Minimise Lateness

MINIMISE LATENESS

INSTANCE: Set $\{(t(i), d(i)), 1 \leq i \leq n\}$ of lengths and deadlines of n jobs.

SOLUTION: Set $\{s(i), 1 \leq i \leq n\}$ of start times such that $\max_{1 \leq i \leq n} (\max(0, s(i) + t(i) - d(i)))$ is as small as possible.

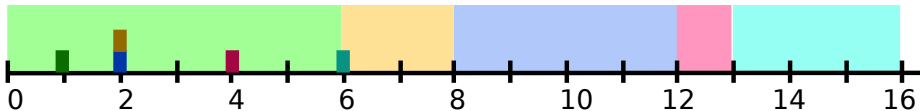
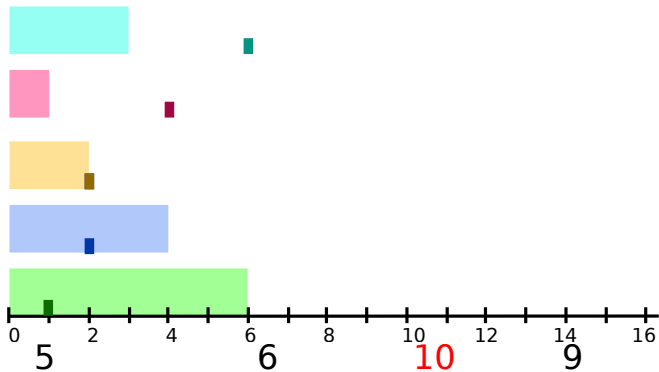


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Earliest deadline Increasing order of deadline $d(i)$. Correct? Does it make sense to tackle jobs with earliest deadlines first?

Minimising Lateness: Earliest Deadline First

Order the jobs in order of their deadlines

Assume for simplicity of notation that $d_1 \leq \dots \leq d_n$

Initially, $f = s$

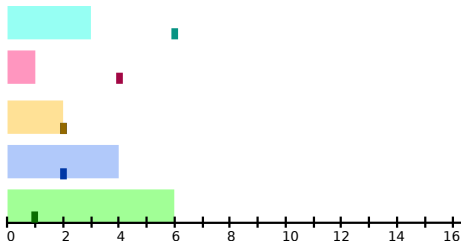
Consider the jobs $i = 1, \dots, n$ in this order

Assign job i to the time interval from $s(i) = f$ to $f(i) = f + t_i$

Let $f = f + t_i$

End

Return the set of scheduled intervals $[s(i), f(i)]$ for $i = 1, \dots, n$



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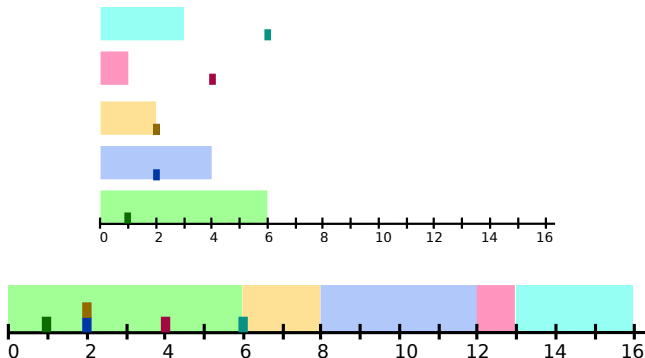
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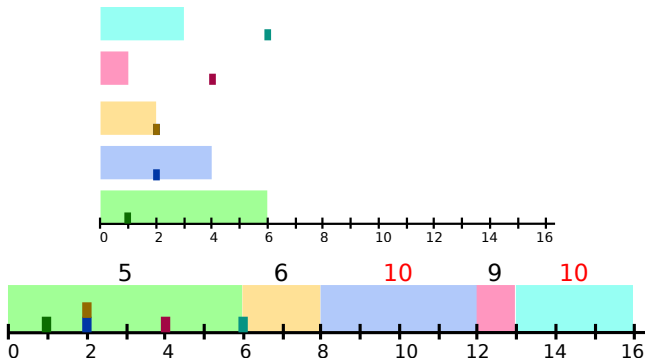
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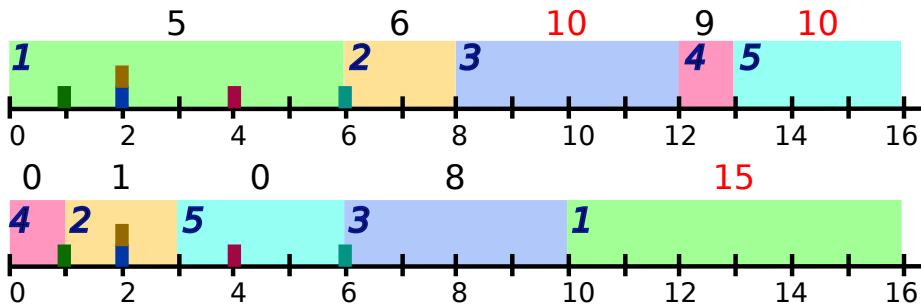
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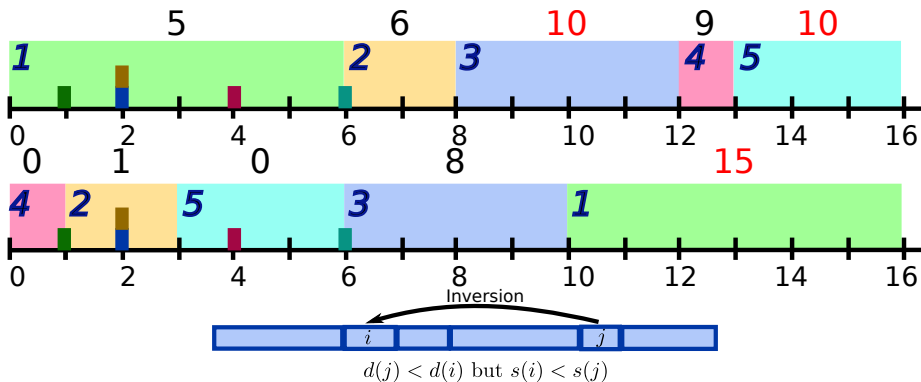
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Inversions

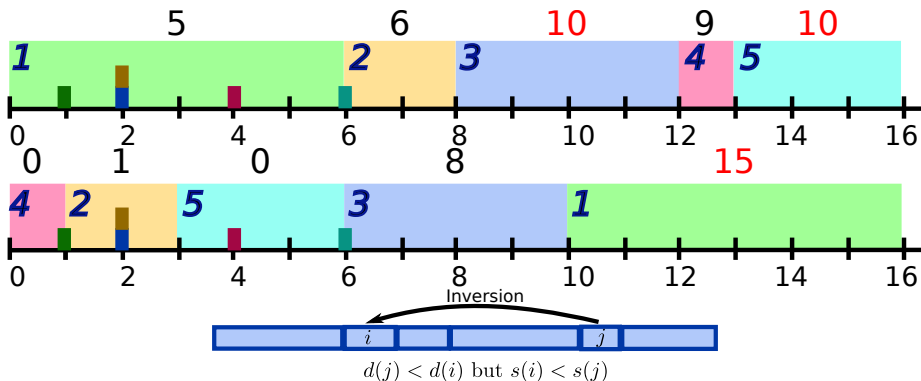


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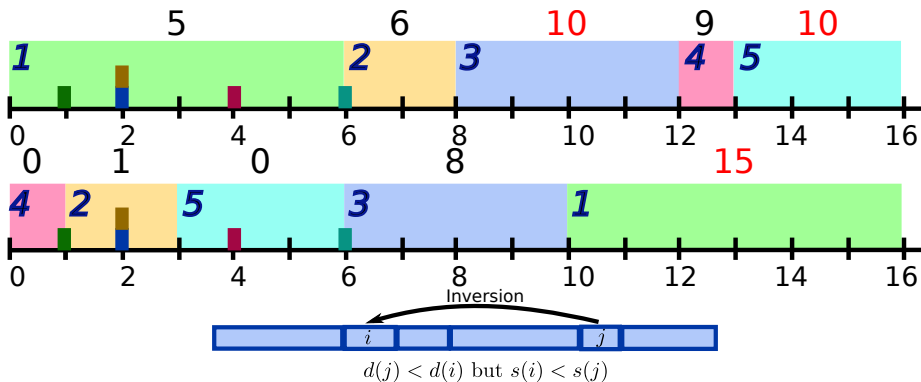
- A schedule has an *inversion* if a job i with deadline $d(i)$ is scheduled before a job j with an earlier deadline $d(j)$, i.e., $d(j) < d(i)$ and $s(i) < s(j)$.
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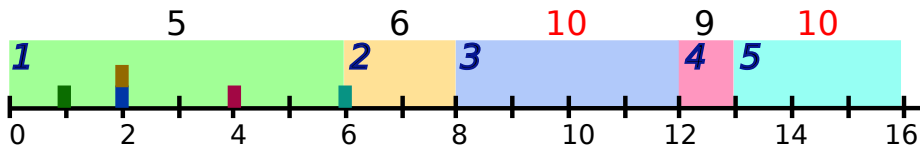
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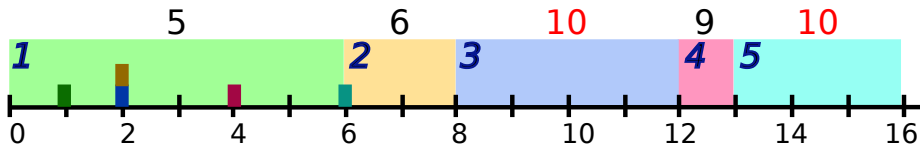
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 - ▶ Examples: 2 and 1, 3 and 1, 4 and 1, 5 and 1, 4 and 3.
- Claim: If a schedule has an inversion, then there is a pair of jobs i and j such that j is scheduled immediately after i and $d(j) < d(i)$.

Properties of Schedules



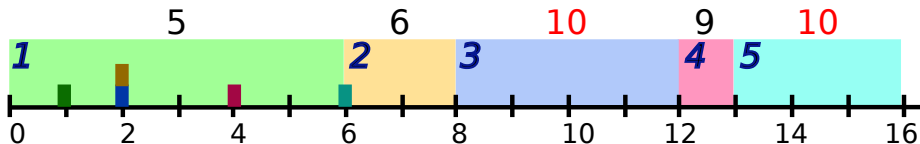
- Claim 1: The algorithm produces a schedule with no inversions and no idle time.

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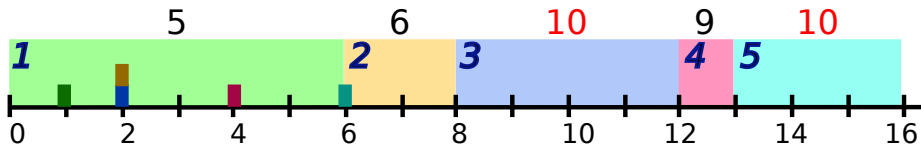
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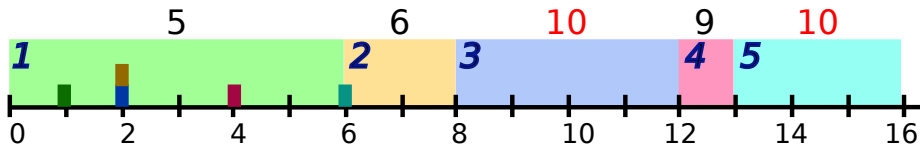
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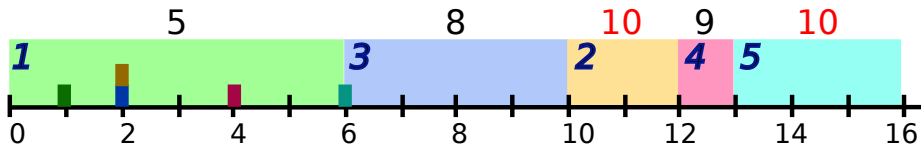
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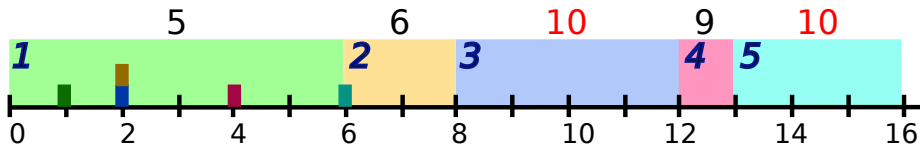
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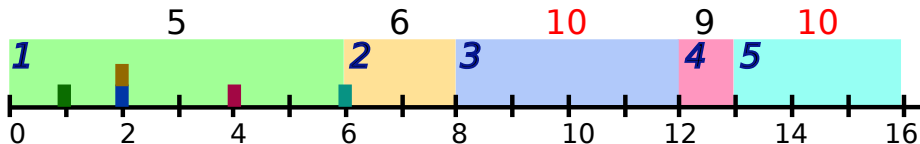
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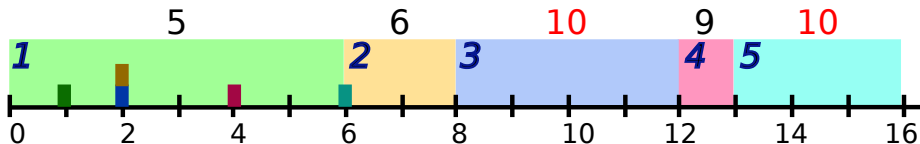
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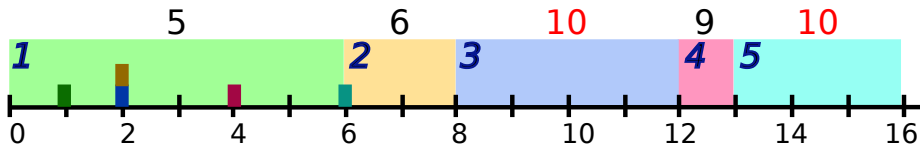
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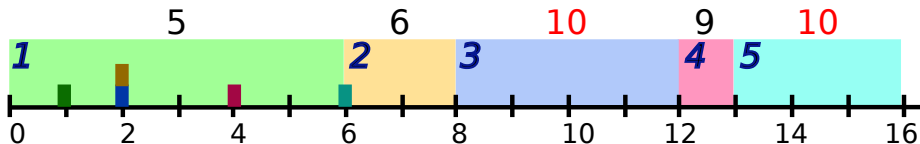
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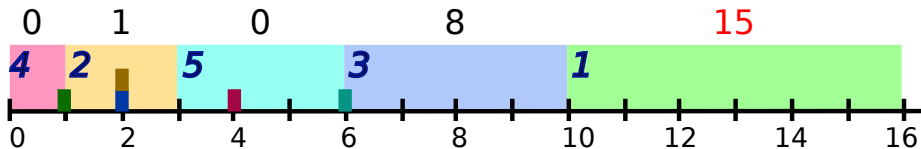
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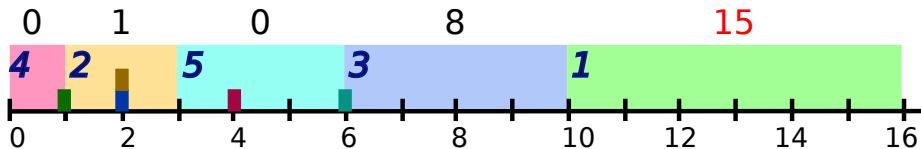
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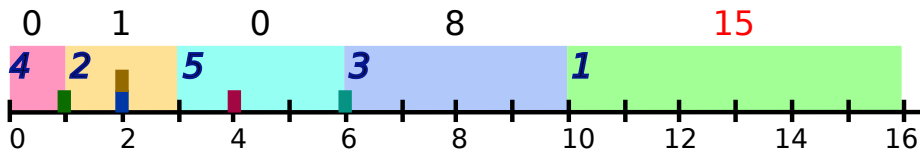
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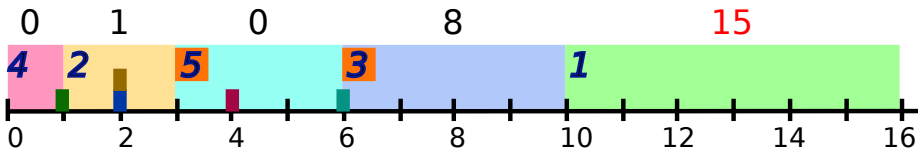
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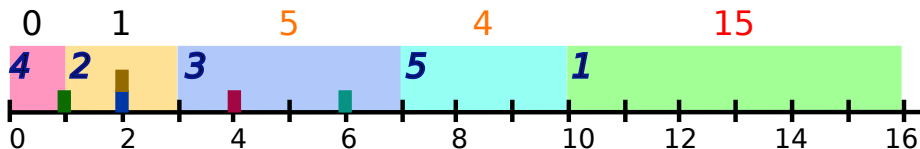
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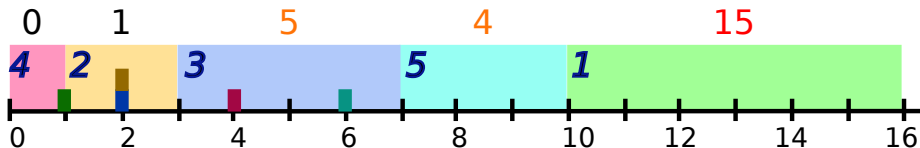
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- It is enough to prove the last item, since after $\binom{n}{2}$ swaps, we obtain a schedule with no inversions whose lateness is no larger than that of O .

Swapping Inverted Jobs

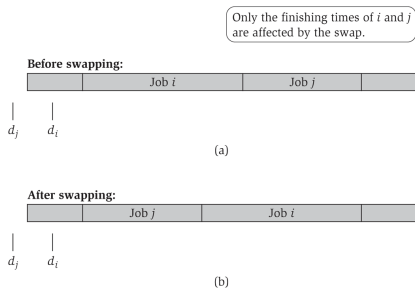


Figure 4.6 The effect of swapping two consecutive, inverted jobs.

- In O , assume each job r is scheduled for the interval $[s(r), f(r)]$ and has lateness $l(r)$. For O' , let the lateness of job r be $l'(r)$.

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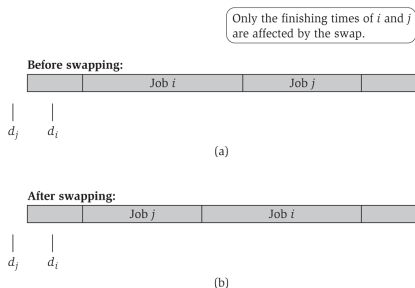


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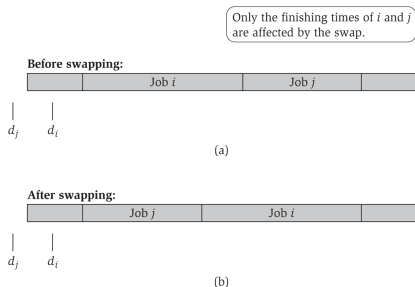


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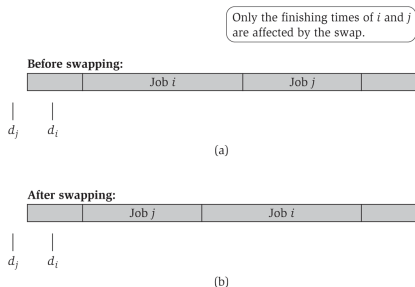


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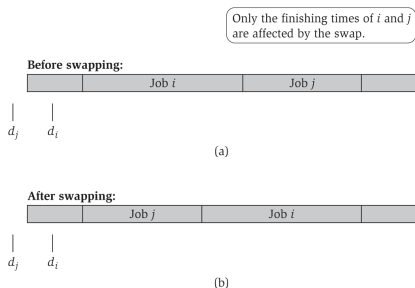


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- 6 Swap the jobs to get a new schedule X_{i-1} . Where does X_{i-1} lie?

Summary of Proof

- 1 Think of a schedule as a 2D point: x -coordinate is the number of inversions in the schedule and y -coordinate is the lateness of the schedule.
- 2 Where does the schedule A produced by the algorithm lie? Somewhere on the y -axis since it has no inversions, say $(0, l_A)$.
- 3 Where does some other schedule B with no inversions lie? Also at $(0, l_A)$ since all schedules with no inversions have the same lateness.
- 4 Let X be any schedule that is supposed to be optimal (and better than A). Where does X lie? At some point (i, l_X) , where $i > 0$ and l_X are the number of inversions in and lateness of X , respectively. $l_X < l_A$
- 5 Find an inversion in X and then isolate the inversion to be between consecutive jobs in X .
- 6 Swap the jobs to get a new schedule X_{i-1} . Where does X_{i-1} lie? X_{i-1} has one fewer inversion! Lateness cannot increase! So X_{i-1} is at $(i-1, l_X)$ or "below."
- 7 Repeat until we have X_1 with one inversion at

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- 7 Repeat until we have X_1 with one inversion at $(1, l_X)$ or "below", where $l_X < l_A$.
- 8 Repeat one more step: X_0 has no inversions. What is X_0 's location?

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- 8 Repeat one more step: X_0 has no inversions. What is X_0 's location? $(0, l_X)$ or "below" because of #7 and $(0, l_A)$ because of #3.

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- 9 We have a contradiction!
- 10 Lateness of A cannot be larger than that of O !

Common Mistakes with Exchange Arguments

- Wrong: start with algorithm's schedule A and argue that A cannot be improved by swapping two jobs.
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- Wrong: Swap two jobs that are not neighbouring in O . Pitfall is that the completion times of all intervening jobs changes.
- Correct: Show that an inversion exists between two neighbouring jobs and swap them.

Summary

- Greedy algorithms make local decisions.
- Three analysis strategies:

Greedy algorithm stays ahead Show that after each step in the greedy algorithm, its solution is at least as good as that produced by any other algorithm.

Structural bound First, discover a property that must be satisfied by every possible solution. Then show that the (greedy) algorithm produces a solution with this property.

Exchange argument Transform the optimal solution in steps into the solution by the greedy algorithm without worsening the quality of the optimal solution.