Greedy Graph Algorithms

T. M. Murali

February 17, 22, 24, 2021
Algorithm Design

- Start discussion of different ways of designing algorithms.
- Greedy algorithms, divide and conquer, dynamic programming, flow-based approaches.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.
Algorithm Design

- Start discussion of different ways of designing algorithms.
- Greedy algorithms, divide and conquer, dynamic programming, flow-based approaches.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.
- Greedy algorithms: make the current best choice.
  - First discussed greedy algorithms for scheduling (Chapters 4.1 to 4.3).
  - Now we will discuss greedy graph algorithms.
G(V, E) is a connected directed graph. Each edge e has a length \( l(e) \geq 0 \).

- **Length of a path** \( P \) is the sum of the lengths of the edges in \( P \).
Shortest Paths Problem

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- *Length of a path* $P$ is the sum of the lengths of the edges in $P$. 
Shortest Paths Problem

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- *Length of a path* $P$ is the sum of the lengths of the edges in $P$.
- Goal: compute the shortest path from a specified start node $s$ to each node in $V$.

**Shortest Paths**

**INSTANCE:** A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$

**SOLUTION:** A set $\{P_u, u \in V\}$ of paths, where $P_u$ is the shortest path in $G$ from $s$ to $u$. 
Shortest Paths Problem

- $G(V, E)$ is a connected directed graph. Each edge $e$ has a length $l(e) \geq 0$.
- **Length of a path** $P$ is the sum of the lengths of the edges in $P$.
- Goal: compute the shortest path from a specified start node $s$ to each node in $V$.
- Aside: If $G$ is undirected, convert to a directed graph by replacing each edge in $G$ by two directed edges.

**Shortest Paths**

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**SOLUTION:** A set $\{P_u, u \in V\}$ of paths, where $P_u$ is the shortest path in $G$ from $s$ to $u$. 

- **S** **T** **R** **O** **K** **EST** **P** **A** **T** **H** **S**
Shortest Paths Problem Instance

Graph representation with nodes s, e, f, b, a, c and edges with weights 1, 2, 3, 4.
Generalizing BFS
Generalizing BFS

Unweighted graph: Use BFS. Process nodes in non-decreasing order of distance.
Weighted graph: Edge weights are integers. Can we make the graph unweighted?
Generalizing BFS

Add dummy nodes: Edge of weight $w$ gets $w - 1$ nodes.
Generalizing BFS

Dummy nodes: BFS computes shortest paths correctly. Running time is
Generalizing BFS

Dummy nodes: BFS computes shortest paths correctly. Running time is $O(m + n + \sum_{e \in E} l(e))$. Pseudo-polynomial time: depends on input values.
Generalizing BFS to Dijkstra’s Algorithm

Like BFS: explore nodes in non-increasing order of distance from s. Once a node is explored, its distance is fixed.
Generalizing BFS to Dijkstra’s Algorithm

Unlike BFS: Layers are not uniform. Which node to process next? Candidates are nodes with an edge from a explored node.
Generalizing BFS to Dijkstra’s Algorithm

For each unexplored node, determine “best” preceding explored node.
Generalizing BFS to Dijkstra’s Algorithm

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Generalizing BFS to Dijkstra’s Algorithm

For each unexplored node, determine “best” preceding explored node. Record shortest path length only through explored nodes.
Generalizing BFS to Dijkstra’s Algorithm

For each unexplored node, determine “best” preceding explored node.
Generalizing BFS to Dijkstra’s Algorithm

Explore node with smallest path length only through explored nodes.
Like BFS: Record previous node in the computed path.
Generalizing BFS to Dijkstra’s Algorithm

Follow previous nodes to compute shortest path. Like BFS: these edges form a tree.
Maintain a set $S$ of explored nodes.

- For each node $u \in S$, compute a value $d(u)$, which (we will prove) is the length of the shortest path from $s$ to $u$.
- For each node $x \not\in S$, maintain a value $d'(x)$, which is the length of the shortest path from $s$ to $x$ using only the nodes in $S$ (and $x$, of course).
**Idea Underlying Dijkstra’s Algorithm**

- Maintain a set $S$ of explored nodes.
  - For each node $u \in S$, compute a value $d(u)$, which (we will prove) is the length of the shortest path from $s$ to $u$.
  - For each node $x \not\in S$, maintain a value $d'(x)$, which is the length of the shortest path from $s$ to $x$ using only the nodes in $S$ (and $x$, of course).
- “Greedily” add a node $v$ to $S$ that has the smallest value of $d'(v)$ (is closest to $s$ using only nodes in $S$).
Dijkstra’s Algorithm

**Dijkstra’s Algorithm** ($G, l, s$)

1. $S = \{s\}$ and $d(s) = 0$
2. while $S \neq V$ do
3. for every node $x \in V - S$ do
4. Set $d'(x) = \min_{u \in S} (d(u) + l(u, x))$
5. Set $v = \arg \min_{x \in V - S} d'(x)$
6. Add $v$ to $S$ and set $d(v) = d'(v)$

---

![Dijkstra's Algorithm Diagram](image-url)
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- How do we parse \(d'(x) = \min_{u, x \in S}(d(u) + l(u, x))\)?
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- How do we parse $d'(x) = \min_{u:x} (d(u) + l(u, x))$?
  - The algorithm is examining a particular (unexplored) node $x$ in $V - S$. 

![Diagram of a graph with nodes and edges labeled with distances and colors indicating unexplored and explored nodes.](image)
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- How do we parse \(d'(x) = \min_{(u,x):u \in S}(d(u) + l(u, x))\)?
  - The algorithm is examining a particular (unexplored) node \(x\) in \(V - S\).
  - Argument of min runs over all edges of the type \((u, x)\), where \(u\) is in \(S\) (i.e., \(u\) is explored).
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1. \( S = \{ s \} \) and \( d(s) = 0 \)
2. \textbf{while} \( S \neq V \) \textbf{do}
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- How do we parse \( d'(x) = \min_{(u,x):u \in S} (d(u) + l(u,x)) \)?
  - The algorithm is examining a particular (unexplored) node \( x \) in \( V - S \).
  - Argument of min runs over all edges of the type \((u,x)\), where \( u \) is in \( S \) (i.e., \( u \) is explored).
  - For each such edge, we compute the length of the shortest path from \( s \) to \( x \) via \( u \), which is \( d(u) + l(u,x) \).

![Diagram showing the Dijkstra's Algorithm process on a graph]

**Candidates**

<table>
<thead>
<tr>
<th>Unexplored</th>
<th>Explored</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td>[1]</td>
</tr>
<tr>
<td>[2]</td>
<td>[3]</td>
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<table>
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<tr>
<th>Min distance d</th>
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<tbody>
<tr>
<td>[0]</td>
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<tr>
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  - Argument of min runs over all edges of the type \((u, x)\), where \(u\) is in \(S\) (i.e., \(u\) is explored).
  - For each such edge, we compute the length of the shortest path from \(s\) to \(x\) via \(u\), which is \(d(u) + l(u, x)\).
  - We store the smallest of these values in \(d'(x)\).
**Dijkstra’s Algorithm**

Dijkstra’s Algorithm($G, l, s$)

1: $S = \{s\}$ and $d(s) = 0$
2: while $S \neq V$ do
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- How do we parse \(v = \arg \min_{x \in V - S} d'(x)\)?
  - Run over all (unexplored) nodes \(x\) in \(V - S\).
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**How do we parse** \(v = \arg \min_{x \in V - S} d'(x)\)?

- Run over all (unexplored) nodes \(x\) in \(V - S\).
- Examine the \(d'\) values for these nodes.
Dijkstra’s Algorithm

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1: $S = \{s\}$ and $d(s) = 0$
2: while $S \neq V$ do
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- Return the argument (i.e., the node) that has the smallest value of $d'(x)$.
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  - Examine the \( d' \) values for these nodes.
  - Return the argument (i.e., the node) that has the smallest value of \( d'(x) \).

- To compute the shortest paths: when adding a node \( v \) to \( S \), store the predecessor \( u \) that minimises \( d'(v) \).
Proof of Correctness

- Let $P_u$ be the path computed by the algorithm for an arbitrary node $u$.
- Claim: $P_u$ is the shortest path from $s$ to $u$.
- Prove by induction on the size of $S$. 

Base case: $|S| = 1$. The only node in $S$ is $s$.

Inductive hypothesis: $|S| = k$, for some $k \geq 1$.

The algorithm has correctly computed $P_u$ for every node $u \in S$. Strong induction.

Inductive step: $|S| = k + 1$ because we add the node $v$ to $S$. Could there be a shorter path $P$ from $s$ to $v$? We must prove this cannot be the case.
Proof of Correctness

- Let $P_u$ be the path computed by the algorithm for an arbitrary node $u$.
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  - Inductive step: $|S| = k + 1$ because we add the node $v$ to $S$. Could there be a shorter path $P$ from $s$ to $v$? We must prove this cannot be the case.

![Diagram](Image)

**Figure 4.8** The shortest path $P_v$ and an alternate $s$-$v$ path $P$ through the node $y$. 
Comments about Dijkstra’s Algorithm

- Algorithm cannot handle negative edge lengths. We will discuss the Bellman-Ford algorithm in a few weeks.
- Union of shortest paths output by Dijkstra’s algorithm forms a tree. Why?
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- Algorithm cannot handle negative edge lengths. We will discuss the Bellman-Ford algorithm in a few weeks.
- Union of shortest paths output by Dijkstra’s algorithm forms a tree. Why?
- Union of shortest paths from a fixed source $s$ forms a tree; paths not necessarily computed by Dijkstra’s algorithm.
Dijkstra’s Algorithm ($G, l, s$)

1. $S = \{s\}$ and $d(s) = 0$
2. **while** $S \neq V$ **do**
3. **for** every node $x \in V - S$ **do**
4. Set $d'(x) = \min_{(u,x): u \in S}(d(u) + l(u, x))$
5. Set $v = \arg \min_{x \in V - S} d'(x)$
6. Add $v$ to $S$ and set $d(v) = d'(v)$

- $V$ has $n$ nodes and $E$ has $m$ edges.
- How many iterations are there of the while loop?
Running time of Dijkstra’s Algorithm

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- \( V \) has \( n \) nodes and \( E \) has \( m \) edges.
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- \(V\) has \(n\) nodes and \(E\) has \(m\) edges.
- How many iterations are there of the while loop? \(n - 1\).
- In each iteration, for each node \(x \in V - S\), compute
  \[
  d'(x) = \min_{(u, x), u \in S} (d(u) + l(u, x))
  \]
Running time of Dijkstra’s Algorithm

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- $V$ has $n$ nodes and $E$ has $m$ edges.
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- In each iteration, for each node $x \in V - S$, compute
  
  $$d'(x) = \min_{(u, x), u \in S} (d(u) + l(u, x))$$

- Running time per iteration is
Running time of Dijkstra’s Algorithm

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- $V$ has $n$ nodes and $E$ has $m$ edges.
- How many iterations are there of the while loop? $n - 1$.
- In each iteration, for each node $x \in V - S$, compute
  $$d'(x) = \min_{(u, x), u \in S} (d(u) + l(u, x))$$
- Running time per iteration is $O(m)$, since the algorithm processes each edge $(u, x)$ in the graph exactly once (when computing $d'(x)$).
- The overall running time is $O(nm)$. 
A Faster implementation of Dijkstra’s Algorithm

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1: \( S = \{s\} \) and \( d(s) = 0 \)
2: \textbf{while} \( S \neq V \) \textbf{do}
3: \hspace{1em} \text{Set} \( v = \text{arg min}_{x \in V - S} d'(x) \)
4: \hspace{1em} \text{Add} \( v \) to \( S \) and set \( d(v) = d'(v) \)
5: \hspace{1em} \textbf{for} \text{ every node} \( x \in V - S \) \textbf{do}
6: \hspace{2em} \text{Set} \( d'(x) = \min_{(u, x): u \in S} (d(u) + l(u, x)) \)

- Observation: If we add \( v \) to \( S \), \( d'(x) \) changes only when \((v, x)\) is an edge in \( G \) and \( x \) is not in \( S \).

\( d'(w) \) does not change\n\( d'(x) \) can change
A Faster implementation of Dijkstra’s Algorithm

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1: $S = \{s\}$ and $d(s) = 0$
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6:       Set $d'(x) = \min_{(u, x)}: u \in S (d(u) + l(u, x))$

- Observation: If we add $v$ to $S$, $d'(x)$ changes only if $(v, x)$ is an edge in $G$ and $x$ is not in $S$. 
A Faster implementation of Dijkstra’s Algorithm

Dijkstra’s Algorithm \((G, l, s)\)

1. \(S = \{s\}\) and \(d(s) = 0\)
2. \textbf{while } \(S \neq V\) \textbf{ do}
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4. \quad Add \(v\) to \(S\) and set \(d(v) = d'(v)\)
5. \quad \textbf{for every node } \(x \in V - S\) \textbf{ do}
6. \quad \quad Set \(d'(x) = \min_{(u, x)}: u \in S (d(u) + l(u, x))\)

- Observation: If we add \(v\) to \(S\), \(d'(x)\) changes only if \((v, x)\) is an edge in \(G\) and \(x\) is not in \(S\).

- Idea: For each node \(x \in V - S\), store the current value of \(d'(x)\). Upon adding a node \(v\) to \(S\), update \(d'(\cdot)\) only for neighbours of \(v\) that are not in \(S\).
A Faster implementation of Dijkstra’s Algorithm

\textbf{Dijkstra’s Algorithm}(G, l, s)

1: \( S = \{s\} \) and \( d(s) = 0 \)
2: \textbf{while} \( S \neq V \) \textbf{do}
3: \hspace{1em} Set \( v = \arg\min_{x \in V - S} d'(x) \)
4: \hspace{1em} Add \( v \) to \( S \) and set \( d(v) = d'(v) \)
5: \hspace{1em} for every node \( x \in V - S \) \textbf{do}
6: \hspace{2em} Set \( d'(x) = \min_{(u,x)}:u \in S (d(u) + l(u,x)) \)

- Observation: If we add \( v \) to \( S \), \( d'(x) \) changes only if \((v,x)\) is an edge in \( G \) and \( x \) is not in \( S \).
- Idea: For each node \( x \in V - S \), store the current value of \( d'(x) \). Upon adding a node \( v \) to \( S \), update \( d'(x) \) only for neighbours of \( v \) that are not in \( S \).
- How do we efficiently compute \( v = \arg\min_{x \in V - S} d'(x) \)?
A Faster implementation of Dijkstra’s Algorithm

Dijkstra’s Algorithm \((G, l, s)\)

1. \(S = \{s\}\) and \(d(s) = 0\)
2. \textbf{while} \(S \neq V\) \textbf{do}
3. \hspace{1em} Set \(v = \arg \min_{x \in V - S} d'(x)\)
4. \hspace{1em} Add \(v\) to \(S\) and set \(d(v) = d'(v)\)
5. \hspace{1em} \textbf{for} \ every \ node \(x \in V - S\) \textbf{do}
6. \hspace{2em} Set \(d'(x) = \min_{(u,x): u \in S}(d(u) + l(u,x))\)

- Observation: If we add \(v\) to \(S\), \(d'(x)\) changes only if \((v, x)\) is an edge in \(G\) and \(x\) is not in \(S\).
- Idea: For each node \(x \in V - S\), store the current value of \(d'(x)\). Upon adding a node \(v\) to \(S\), update \(d'()\) only for neighbours of \(v\) that are not in \(S\).
- How do we efficiently compute \(v = \arg \min_{x \in V - S} d'(x)\)?
- Use a priority queue!
Faster Dijkstra’s Algorithm

Dijkstra’s Algorithm \((G, l, s)\)

1: \textbf{Insert} \((Q, s, 0)\).
2: \textbf{while} \(S \neq V\) \textbf{do}
3: \qquad \((v, d'(v)) = \text{ExtractMin}(Q)\)
4: \qquad \text{Add} \(v\) to \(S\) and set \(d(v) = d'(v)\)
5: \qquad \textbf{for} every node \(x \in V - S\) such that \((v, x)\) is an edge in \(G\) \textbf{do}
6: \qquad \qquad \textbf{if} \(d(v) + l(v, x) < d'(x)\) \textbf{then}
7: \qquad \qquad \quad d'(x) = d(v) + l(v, x)
8: \qquad \textbf{ChangeKey}(Q, x, d'(x))

- For each node \(x \in V - S\), store the pair \((x, d'(x))\) in a priority queue \(Q\) with \(d'(x)\) as the key.
- Determine the next node \(v\) to add to \(S\) using \text{ExtractMin} \textbf{ (line 3).}
- After adding \(v\) to \(S\), for each node \(x \in V - S\) such that there is an edge from \(v\) to \(x\), check if \(d'(x)\) should be updated, i.e., if there is a shortest path from \(s\) to \(x\) via \(v\) \textbf{ (lines 5–8).}
- In line 8, if \(x\) is not in \(Q\), simply insert it.
Running Time of Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm** \((G, l, s)\)

1: \text{Insert} \((Q, s, 0)\).
2: \textbf{while} \(S \neq V\) do
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6: \hspace{1em} \textbf{if} \(d(v) + l(v, x) < d'(x)\) then
7: \hspace{2em} \(d'(x) = d(v) + l(v, x)\)
8: \hspace{2em} \text{ChangeKey} \((Q, x, d'(x))\)

How many times does the algorithm invoke \text{ExtractMin}?
Running Time of Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm** \((G, l, s)\)

1: **Insert** \((Q, s, 0)\).
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7: \(d'(x) = d(v) + l(v, x)\)
8: \(\text{ChangeKey}(Q, x, d'(x))\)

- How many times does the algorithm invoke \(\text{ExtractMin}\)? \(n - 1\) times.
Dijkstra’s Algorithm \((G, l, s)\)

1. \textsc{Insert} \((Q, s, 0)\).
2. \textbf{while} \(S \neq V\) \textbf{do}
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5. \hspace{1em} \textbf{for} every node \(x \in V - S\) such that \((v, x)\) is an edge in \(G\) \textbf{do}
6. \hspace{2.5em} \textbf{if} \(d(v) + l(v, x) < d'(x)\) \textbf{then}
7. \hspace{3.5em} \(d'(x) = d(v) + l(v, x)\)
8. \hspace{3.5em} \textsc{ChangeKey}(Q, x, d'(x))

- How many times does the algorithm invoke \textsc{ExtractMin}? \(n - 1\) times.
- For every node \(v\), what is the running time of step 5?
Running Time of Faster Dijkstra’s Algorithm

Dijkstra’s Algorithm\((G, l, s)\)

1: \text{Insert}(Q, s, 0).
2: \textbf{while } S \neq V \textbf{ do}
3: \quad (v, d'(v)) = \text{ExtractMin}(Q)
4: \quad \text{Add } v \text{ to } S \text{ and set } d(v) = d'(v)
5: \quad \textbf{for} \text{ every node } x \in V - S \text{ such that } (v, x) \text{ is an edge in } G \text{ do}
6: \quad \quad \textbf{if } d(v) + l(v, x) < d'(x) \text{ then}
7: \quad \quad \quad d'(x) = d(v) + l(v, x)
8: \quad \quad \text{CHANGEKEY}(Q, x, d'(x))

- How many times does the algorithm invoke \text{EXTRACTMIN}? \(n - 1\) times.
- For every node \(v\), what is the running time of step 5? \(O(d_v)\), the number of outgoing neighbours of \(v\).
Running Time of Faster Dijkstra’s Algorithm

Dijkstra’s Algorithm($G, l, s$)

1: Insert($Q, s, 0$).
2: while $S \neq V$ do
3:   $(v, d'(v)) = \text{ExtractMin}(Q)$
4:   Add $v$ to $S$ and set $d(v) = d'(v)$
5:   for every node $x \in V - S$ such that $(v, x)$ is an edge in $G$ do
6:     if $d(v) + l(v, x) < d'(x)$ then
7:        $d'(x) = d(v) + l(v, x)$
8:        ChangeKey($Q, x, d'(x)$)

- How many times does the algorithm invoke ExtractMin? $n - 1$ times.
- For every node $v$, what is the running time of step 5? $O(d_v)$, the number of outgoing neighbours of $v$.
- What is the total running time of step 5?
# Running Time of Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm** \((G, l, s)\)

1: **Insert** \((Q, s, 0)\).  
2: **while** \(S \neq V\) **do**  
3: \((v, d'(v)) = \text{ExtractMin}(Q)\)  
4: Add \(v\) to \(S\) and set \(d(v) = d'(v)\)  
5: **for** every node \(x \in V - S\) such that \((v, x)\) is an edge in \(G\) **do**  
6: **if** \(d(v) + l(v, x) < d'(x)\) **then**  
7: \(d'(x) = d(v) + l(v, x)\)  
8: **ChangeKey** \((Q, x, d'(x))\)

- How many times does the algorithm invoke \(\text{ExtractMin}\)? \(n - 1\) times.  
- For every node \(v\), what is the running time of step 5? \(O(d_v)\), the number of *outgoing* neighbours of \(v\).  
- What is the total running time of step 5? \(\sum_{v \in V} O(d_v) = O(m)\).
Running Time of Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm** \((G, l, s)\)

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Running Time of Faster Dijkstra’s Algorithm

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Running Time of Faster Dijkstra’s Algorithm

\textbf{Dijkstra’s Algorithm}(G, l, s)

1: \textsc{Insert}(Q, s, 0).
2: \textbf{while} \ S \neq V \ \textbf{do}
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- How many times does the algorithm invoke \textsc{ExtractMin}? \( n - 1 \) times.
- For every node \( v \), what is the running time of step 5? \( O(d_v) \), the number of \textit{outgoing} neighbours of \( v \).
- What is the total running time of step 5? \( \sum_{v \in V} O(d_v) = O(m) \).
- How many times does the algorithm invoke \textsc{ChangeKey}? At most \( m \) times.
- What is total running time of the algorithm?
### Running Time of Faster Dijkstra’s Algorithm

**Dijkstra’s Algorithm**\((G, l, s)\)

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Running Time of Faster Dijkstra’s Algorithm

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- For every node \(v\), what is the running time of step 5? \(O(d_v)\), the number of \textit{outgoing} neighbours of \(v\).
- What is the total running time of step 5? \(\sum_{v \in V} O(d_v) = O(m)\).
- How many times does the algorithm invoke \text{ChangeKey}? At most \(m\) times.
- What is total running time of the algorithm? \(O(m \log n)\).
- State of the art: Fibonacci heaps achieve a running time of \(O(m)\) for all \text{ChangeKey} operations, for a running time of \(O(n \log n + m)\).
Network Design

- Connect a set of nodes using a set of edges with certain properties.
- Input is usually a graph and the desired network (the output) should use subset of edges in the graph.
- Example: connect all nodes using a cycle of shortest total length.
Network Design

- Connect a set of nodes using a set of edges with certain properties.
- Input is usually a graph and the desired network (the output) should use subset of edges in the graph.
- Example: connect all nodes using a cycle of shortest total length. This problem is the NP-complete traveling salesman problem.
Minimum Spanning Tree (MST)

- Given an undirected graph $G(V, E)$ with a cost $c(e) > 0$ associated with each edge $e \in E$.
- Find a subset $T$ of edges such that the graph $(V, T)$ is connected and the cost $\sum_{e \in T} c(e)$ is as small as possible.
Shortest Paths

Minimum Spanning Trees

Implementation

Minimum Spanning Tree (MST)

- Given an undirected graph \( G(V, E) \) with a cost \( c(e) > 0 \) associated with each edge \( e \in E \).
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(a) (b) (c)

(d) (e) (f)

T. M. Murali February 17, 22, 24, 2021 Greedy Graph Algorithms
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(a) Not connected
(b) Not smallest cost
(c) Not smallest cost
(d) Not smallest cost
(e) Not smallest cost
(f) Smallest cost
**Minimum Spanning Tree (MST)**

**Minimum Spanning Tree**

**INSTANCE:** An undirected graph $G(V, E)$ and a function $c : E \rightarrow \mathbb{R}^+$

**SOLUTION:** A set $T \subseteq E$ of edges such that $(V, T)$ is connected and the cost $\sum_{e \in T} c(e)$ is as small as possible.

Claim: If $T$ is a minimum-cost solution to this problem then $(V, T)$ is a tree.

A subset $T$ of $E$ is a spanning tree of $G$ if $(V, T)$ is a tree.
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A subset $T$ of $E$ is a *spanning tree* of $G$ if $(V, T)$ is a tree.
Greedy Algorithm for the MST Problem

- Template: process edges in some order. Add an edge to $T$ if tree property is not violated.
Greedy Algorithm for the MST Problem

- **Template**: process edges in some order. Add an edge to $T$ if tree property is not violated.

**Increasing cost order**  Process edges in increasing order of cost. Discard an edge if it creates a cycle.

**Dijkstra-like**  Start from a node $s$ and grow $T$ outward from $s$: add the node that can be attached most cheaply to current tree.

**Decreasing cost order**  Delete edges in order of decreasing cost as long as graph remains connected.
Greedy Algorithm for the MST Problem

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- Which of these algorithms works?
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- **Template**: process edges in some order. Add an edge to $T$ if tree property is not violated.

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  - **Decreasing cost order**  Delete edges in order of decreasing cost as long as graph remains connected. **Reverse-Delete algorithm**

- Which of these algorithms works? All of them!
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  Decreasing cost order: Delete edges in order of decreasing cost as long as graph remains connected. **Reverse-Delete algorithm**

- Which of these algorithms works? All of them!

- Simplifying assumption: all edge costs are distinct.
Characterising MSTs

Does the edge of smallest cost belong to an MST?
Characterising MSTs

- Does the edge of smallest cost belong to an MST? Yes. Why?

- Wrong proof: because Kruskal's algorithm adds it. We have not yet proved correctness of Kruskal's algorithm!

- Correct proof: will work it out soon.

- Which edges must belong to an MST?

- What happens when we delete an edge from an MST?
  - MST breaks up into sub-trees.

- Which edge should we add to join them?

- Which edges cannot belong to an MST?

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  - Which edge in the cycle can we be sure does not belong to an MST?
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T. M. Murali February 17, 22, 24, 2021 Greedy Graph Algorithms
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A *cut* in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).
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A cut in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).

Every set $S \subseteq V$ ($S$ cannot be empty or the entire set $V$) has a corresponding cut: $\text{cut}(S)$ is the set of edges $(v, w)$ such that $v \in S$ and $w \in V - S$.

$\text{cut}(S)$ is a “cut” because deleting the edges in $\text{cut}(S)$ disconnects $S$ from $V - S$. 
Graph Cuts

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Graph Cuts

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- cut($S$) is a “cut” because deleting the edges in cut($S$) disconnects $S$ from $V - S$. 

(a) Not cut({a, b, d}): (c, g)  
(b) Is cut({a, b, d})  
(c) Not cut({a, b, d}): (a, b)  
(d) Not cut({a, b, d}): (b, e)
A cut in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).

Every set $S \subset V$ ($S$ cannot be empty or the entire set $V$) has a corresponding cut: $\text{cut}(S)$ is the set of edges $(v, w)$ such that $v \in S$ and $w \in V - S$.

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![Graph Cuts Diagram](image-url)
Graph Cuts

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(a) Not cut({a, e, g}): (b, c)
(b) Not cut({a, g, e}): (f, h)
(c) Is cut({a, e, g})
(d) Not cut({a, e, g}): (d, e)
**Cut Property**

- When is it safe to include an edge in an MST?

---

Claim: For every $S \subset V$, $S \neq \emptyset$, every MST contains the cheapest edge in $\text{cut}(S)$.

Proof by contradiction using exchange argument.

How do you state the contradiction to the claim?

There is a set $S \subset V$ and an MST $T$ such that $T$ does not contain the cheapest edge in $\text{cut}(S)$.

$\text{▶}$ Let $e = (u, v)$ be the cheapest edge in $\text{cut}(S)$.

Proof strategy: If $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$. 

---

![Graph](image)

4 5 15 2
11 7
12 3 1
8 20 6
Cut Property

- When is it safe to include an edge in an MST?
- Claim: For every $S \subset V, S \neq \emptyset$, every MST contains the cheapest edge in $\text{cut}(S)$.
Cut Property

- When is it safe to include an edge in an MST?
- Claim: For every $S \subset V, S \neq \emptyset$, every MST contains the cheapest edge in cut($S$).
- Proof by contradiction using exchange argument.
Cut Property

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- Proof by contradiction using exchange argument.
- How do you state the contradiction to the claim?
Cut Property

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- **Proof by contradiction using exchange argument.**
- How do you state the contradiction to the claim?
  There is a set $S \subset V$ and an MST $T$ such that $T$ does not contain the cheapest edge in $\text{cut}(S)$.

```
<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>15</td>
<td>2</td>
<td>12</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>7</td>
<td>1</td>
<td>12</td>
<td>20</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>
```
Cut Property

- When is it safe to include an edge in an MST?
- Claim: For every \( S \subset V, S \neq \emptyset \), every MST contains the cheapest edge in cut(\( S \)).
- Proof by contradiction using exchange argument.
- How do you state the contradiction to the claim? There is a set \( S \subset V \) and an MST \( T \) such that \( T \) does not contain the cheapest edge in cut(\( S \)).
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Cut Property

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![Graph Image]
Proof of Cut Property

- There is a set $S \subseteq V$ and an MST $T$ such that $T$ does not contain the cheapest edge in cut$(S)$.
- Proof strategy: If $T$ does not contain $e$, show that there is a tree with smaller cost than $T$ that contains $e$.

Wrong proof:
- Since $T$ is spanning, it must contain some edge, e.g., $f$.
- $T - \{f\} \cup \{e\}$ has smaller cost than $T$ but may not be a spanning tree.

Correct proof:
- Add $e$ to $T$ forming a cycle.
- This cycle must contain an edge $e'$ in cut$(S)$.
- Poll $T - \{e'\} \cup \{e\}$ has smaller cost than $T$ and is a spanning tree.
**Proof of Cut Property**

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Proof of Cut Property

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Prim’s Algorithm

- Maintain a tree \((S, T)\), i.e. a set of nodes and a set of edges, which we will show will always be a tree.
- Start with an arbitrary node \(s \in S\).
Prim’s Algorithm

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**Prim’s Algorithm** \((G, c, s)\)

1. \(S = \{s\}\) and \(T = \emptyset\)
2. while \(S \neq V\) do
3. Compute \((u, v) = \arg \min_{u \in S, v \in V - S} c(u, v)\)
4. Add the node \(v\) to \(S\) and add the edge \((u, v)\) to \(T\).
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1. \(S = \{s\}\) and \(T = \emptyset\)
2. **while** \(S \neq V\) **do**
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4. Add the node \(v\) to \(S\) and add the edge \((u, v)\) to \(T\).

Note that

\[
\arg \min_{(u, v), u \in S, v \in V - S} c(u, v) = \arg \min_{(u, v) \in \text{cut}(S)} c(u, v).
\]

In other words, in each step, Prim’s algorithm computes and adds the cheapest edge in the current value of \(\text{cut}(S)\).
Example of Prim’s Algorithm

![Graph with edge weights]

- a to b: 4
- a to c: 5
- a to g: 15
- b to c: 11
- b to g: 2
- c to h: 7
- d to e: 8
- e to g: 20
Example of Prim’s Algorithm
Example of Prim’s Algorithm
Example of Prim’s Algorithm

The graph below illustrates an example of Prim’s algorithm.

- **Nodes:** a, b, c, d, e, f, g, h
- **Edges and Weights:**
  - a-b: 4
  - a-c: 5
  - a-g: 11
  - b-c: 12
  - b-h: 3
  - c-h: 7
  - c-e: 1
  - d-e: 8
  - e-f: 20
  - g-f: 6

Prim’s algorithm starts with an arbitrary node (e.g., a) and then iteratively adds the next cheapest edge that connects a new node to the growing tree until all nodes are included.
Example of Prim’s Algorithm
Example of Prim’s Algorithm

Graph with edges and weights: a-b (4), a-c (5), a-d (11), b-c (12), b-e (3), c-e (1), c-f (20), d-e (8), e-h (7), g-h (2), g-f (6).
Example of Prim’s Algorithm

Diagram of a graph with vertices a, b, c, d, e, f, g, and h, and edges with weights 4, 5, 15, 2, 11, 7, 12, 3, 1, 8, 20, and 6.
Example of Prim’s Algorithm
Example of Prim’s Algorithm
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Optimality of Prim’s Algorithm

**Prim’s Algorithm** \((G, c, s)\)

1: \(S = \{s\}\) and and \(T = \emptyset\)
2: while \(S \neq V\) do
3: \(\text{Compute } (u, v) = \arg \min_{(u, v) \in \text{cut}(S)} c(u, v)\)
4: \(\text{Add the node } v \text{ to } S \text{ and add the edge } (u, v) \text{ to } T.\)

- Claim: Prim’s algorithm outputs an MST.
Optimality of Prim’s Algorithm

**Prim’s Algorithm** \((G, c, s)\)

1: \(S = \{s\}\) and \(T = \emptyset\)
2: **while** \(S \neq V\) **do**
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Claim: Prim’s algorithm outputs an MST.

1. Prove that every edge inserted satisfies the cut property.

2. Prove that the graph constructed is a spanning tree.
Optimality of Prim’s Algorithm

**Prim’s Algorithm** \((G, c, s)\)

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2. \(\text{while } S \neq V \text{ do}\)
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4. \(\text{Add the node } v \text{ to } S \text{ and add the edge } (u, v) \text{ to } T.\)

**Claim:** Prim’s algorithm outputs an MST.
1. Prove that every edge inserted satisfies the cut property.
2. By construction, in each iteration \((u, v)\) is the cheapest edge in \(\text{cut}(S)\) for the current value of \(S\).
3. Prove that the graph constructed is a spanning tree.
Optimality of Prim’s Algorithm

**Prim’s Algorithm** \( (G, c, s) \)

1. \( S = \{ s \} \) and \( T = \emptyset \)
2. \textbf{while} \( S \neq V \) \textbf{do}
3. \hspace{1em} Compute \( (u, v) = \arg \min_{(u, v) \in \text{cut}(S)} c(u, v) \)
4. \hspace{1em} Add the node \( v \) to \( S \) and add the edge \( (u, v) \) to \( T \).

- **Claim**: Prim’s algorithm outputs an MST.
  1. Prove that every edge inserted satisfies the cut property.
     - By construction, in each iteration \( (u, v) \) is the cheapest edge in \( \text{cut}(S) \) for the current value of \( S \).
  2. Prove that the graph constructed is a spanning tree.
     - Why are there no cycles in \( (V, T) \)?

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February 17, 22, 24, 2021
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Optimality of Prim’s Algorithm

**Prim’s Algorithm**\((G, c, s)\)

1: \(S = \{s\}\) and and \(T = \emptyset\)
2: while \(S \neq V\) do
3: Compute \((u, v) = \arg \min_{(u, v) \in \text{cut}(S)} c(u, v)\)
4: Add the node \(v\) to \(S\) and add the edge \((u, v)\) to \(T\).

- **Claim:** Prim’s algorithm outputs an MST.
  1. Prove that every edge inserted satisfies the cut property.
     - By construction, in each iteration \((u, v)\) is the cheapest edge in \(\text{cut}(S)\) for the current value of \(S\).
  2. Prove that the graph constructed is a spanning tree.
     - Why are there no cycles in \((V, T)\)?
     - Why is \((V, T)\) a spanning tree (edges in \(T\) connect all nodes in \(V\))?
Kruskal’s Algorithm

- Start with an empty set $T$ of edges.
- Process edges in $E$ in increasing order of cost.
- Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle. Discard $e$ if it creates a cycle.
Example of Kruskal’s Algorithm

```
    a
   /  \
  4   5
 /     \n b      c
 /  11  3 \
 12     1
 /  8  20 \
 d       e
       /  \
       6   
    f
```

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Example of Kruskal’s Algorithm

Graph with weighted edges:
- a to b: 4
- a to c: 1
- a to g: 2
- b to c: 11
- c to d: 12
- c to e: 3
- c to h: 7
- d to e: 8
- e to f: 20
- g to h: 6

The minimum spanning tree includes:
- a to b
- c to d
- c to e
- c to h
- e to f

Total weight of the minimum spanning tree: 4 + 11 + 3 + 7 + 20 = 45
Example of Kruskal’s Algorithm

Graph representation:
- Nodes: a, b, c, d, e, f, g, h
- Edges with weights:
  - a to b: 4
  - b to c: 11
  - c to d: 12
  - c to e: 3
  - d to e: 8
  - e to f: 20
  - f to g: 6
  - g to h: 2
  - h to c: 7

Kruskal’s Algorithm proceeds by selecting the minimum weight edge at each step until a spanning tree is formed.
Example of Kruskal’s Algorithm
Example of Kruskal’s Algorithm

Graph with weighted edges:
- a to b: 4
- a to c: 5
- b to c: 11
- b to d: 12
- c to e: 3
- d to e: 8
- e to f: 20
- h to g: 2
- g to e: 15
- h to c: 7
- h to f: 6

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Example of Kruskal’s Algorithm

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Example of Kruskal’s Algorithm

The diagram shows a weighted graph with vertices labeled from 'a' to 'f' and edges connecting them with weights indicated on the edges.
Example of Kruskal’s Algorithm

Graph:

- Vertices: a, b, c, d, e, f, g, h
- Edges with weights:
  - (a, b) = 4
  - (b, c) = 11
  - (c, d) = 8
  - (d, e) = 12
  - (e, f) = 20
  - (b, e) = 3
  - (c, g) = 5
  - (c, h) = 15
  - (g, h) = 2

The algorithm constructs a minimum spanning tree by iteratively adding the least weight edges that do not create a cycle.
Example of Kruskal’s Algorithm
Example of Kruskal’s Algorithm

\begin{center}
\begin{tikzpicture}
\node[fill=blue!20] (a) at (0,6) {a};
\node[fill=blue!20] (b) at (2,4) {b};
\node[fill=blue!20] (c) at (6,4) {c};
\node[fill=blue!20] (d) at (4,0) {d};
\node[fill=blue!20] (e) at (2,-4) {e};
\node[fill=blue!20] (f) at (6,-4) {f};
\node[fill=blue!20] (g) at (10,3) {g};
\node[fill=blue!20] (h) at (10,-3) {h};

\draw[blue, ultra thick] (a) -- (b) node[midway, above] {4};
\draw[blue, ultra thick] (b) -- (c) node[midway, above] {5};
\draw[blue, ultra thick] (c) -- (d) node[midway, left] {11};
\draw[blue, ultra thick] (c) -- (e) node[midway, left] {15};
\draw[blue, ultra thick] (c) -- (h) node[midway, right] {7};
\draw[blue, ultra thick] (h) -- (f) node[midway, above] {2};
\draw[blue, ultra thick] (b) -- (e) node[midway, left] {12};
\draw[blue, ultra thick] (e) -- (f) node[midway, left] {20};
\draw[blue, ultra thick] (d) -- (e) node[midway, above] {8};
\end{tikzpicture}
\end{center}
Example of Kruskal’s Algorithm

```
a
  b c
d e f
g
h
4
1
25
11
12 3
8 20
7
15
6
```
Example of Kruskal’s Algorithm

![Graph depiction with nodes a, b, c, d, e, f, g, h and edges with weights 4, 5, 15, 2, 11, 7, 6, 20, 8, 3, 1, 12]
Example of Kruskal’s Algorithm

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Example of Kruskal’s Algorithm
Optimality of Kruskal’s Algorithm

- Kruskal’s algorithm:
  - Start with an empty set $T$ of edges.
  - Process edges in $E$ in increasing order of cost.
  - Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle. Discard $e$ if it creates a cycle.

- Note: at any iteration, $T$ may contain several connected components and each node in $V$ is in some component.

- Claim: Kruskal’s algorithm outputs an MST.
Optimality of Kruskal’s Algorithm

Kruskal’s algorithm:

- Start with an empty set $T$ of edges.
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Claim: Kruskal’s algorithm outputs an MST.

1. For every edge $e$ added, demonstrate the existence of a set $S \subset V$ (and $V - S$) such that $e$ and $S$ satisfy the cut property, i.e., $e$ is the cheapest edge in $\text{cut}(S)$.

2. Prove that the algorithm computes a spanning tree.
Optimality of Kruskal’s Algorithm

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   * If $e = (u, v)$, let $S$ be the set of nodes connected to $u$ in the current graph $T$.

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Optimality of Kruskal’s Algorithm

Kruskal’s algorithm:

▶ Start with an empty set \( T \) of edges.
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   - If \( e = (u, v) \), let \( S \) be the set of nodes connected to \( u \) in the current graph \( T \).
   - Why is \( e \) the cheapest edge in \( \text{cut}(S) \)?

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   - If $e = (u, v)$, let $S$ be the set of nodes connected to $u$ in the current graph $T$.
   - Why is $e$ the cheapest edge in cut($S$)?

2. Prove that the algorithm computes a spanning tree.
   - $(V, T)$ contains no cycles by construction.
Optimality of Kruskal’s Algorithm

Kruskal’s algorithm:
- Start with an empty set \( T \) of edges.
- Process edges in \( E \) in increasing order of cost.
- Add the next edge \( e \) to \( T \) only if adding \( e \) does not create a cycle. Discard \( e \) if it creates a cycle.

Note: at any iteration, \( T \) may contain several connected components and each node in \( V \) is in some component.

Claim: Kruskal’s algorithm outputs an MST.

1. For every edge \( e \) added, demonstrate the existence of a set \( S \subset V \) (and \( V - S \)) such that \( e \) and \( S \) satisfy the cut property, i.e., \( e \) is the cheapest edge in \( \text{cut}(S) \).
   - If \( e = (u, v) \), let \( S \) be the set of nodes connected to \( u \) in the current graph \( T \).
   - Why is \( e \) the cheapest edge in \( \text{cut}(S) \)\

2. Prove that the algorithm computes a spanning tree.
   - \((V, T)\) contains no cycles by construction.
   - If \((V, T)\) is not connected, there exists a subset \( S \) of nodes not connected to \( V - S \). What is the contradiction?

T. M. Murali
Cycle Property

- When can we be sure that an edge cannot be in *any* MST?

Let $C$ be any cycle in $G$ and let $e = (v, w)$ be the most expensive edge in $C$. Claim: $e$ does not belong to any MST of $G$.

Proof: exchange argument. If a supposed MST $T$ contains $e$, show that there is a tree with smaller cost than $T$ that does not contain $e$. 

T. M. Murali February 17, 22, 24, 2021 Greedy Graph Algorithms
Cycle Property

- When can we be sure that an edge cannot be in any MST?
- Let $C$ be any cycle in $G$ and let $e = (v, w)$ be the most expensive edge in $C$.
- Claim: $e$ does not belong to any MST of $G$. 

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T. M. Murali February 17, 22, 24, 2021 Greedy Graph Algorithms
Cycle Property

• When can we be sure that an edge cannot be in any MST?
• Let \( C \) be any cycle in \( G \) and let \( e = (v, w) \) be the most expensive edge in \( C \).
• Claim: \( e \) does not belong to any MST of \( G \).
• Proof: exchange argument. If a supposed MST \( T \) contains \( e \), show that there is a tree with smaller cost than \( T \) that does not contain \( e \).

![Figure 4.11](image-url) Swapping the edge \( e' \) for the edge \( e \) in the spanning tree \( T \), as described in the proof of (4.20).
Optimality of the Reverse-Delete Algorithm

- Reverse-Delete algorithm: Maintain a set $E'$ of edges.
  - Start with $E' = E$.
  - Process edges in decreasing order of cost.
  - Delete the next edge $e$ from $E'$ only if $(V, E')$ is connected after deletion.
  - Stop after processing all the edges.

- Claim: the Reverse-Delete algorithm outputs an MST.

1. Show that every edge deleted belongs to no MST.
   ⋆ A deleted edge must belong to some cycle $C$.
   ⋆ Since the edge is the first encountered by the algorithm, it is the most expensive edge in $C$.

2. Prove that the graph remaining at the end is a spanning tree.
   ⋆ $(V, E')$ is connected at the end, by construction.
   ⋆ If $(V, E')$ contains a cycle, consider the costliest edge in that cycle. The algorithm would have deleted that edge.
Optimality of the Reverse-Delete Algorithm

Reverse-Delete algorithm: Maintain a set $E'$ of edges.
- Start with $E' = E$.
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  - Delete the next edge $e$ from $E'$ only if $(V, E')$ is connected after deletion.
  - Stop after processing all the edges.

- Claim: the Reverse-Delete algorithm outputs an MST.
  1. Show that every edge deleted belongs to no MST.
     - A deleted edge must belong to some cycle $C$.
     - Since the edge is the first encountered by the algorithm, it is the most expensive edge in $C$.
  2. Prove that the graph remaining at the end is a spanning tree.
Optimality of the Reverse-Delete Algorithm

- **Reverse-Delete algorithm:** Maintain a set $E'$ of edges.
  - Start with $E' = E$.
  - Process edges in decreasing order of cost.
  - Delete the next edge $e$ from $E'$ only if $(V, E')$ is connected after deletion.
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     - Since the edge is the first encountered by the algorithm, it is the most expensive edge in $C$.
  2. Prove that the graph remaining at the end is a spanning tree.
     - $(V, E')$ is connected at the end, by construction.
     - If $(V, E')$ contains a cycle, consider the costliest edge in that cycle. The algorithm would have deleted that edge.
Comments on MST Algorithms

- To handle multiple edges with the same length, perturb each length by a random infinitesimal amount. Read the textbook.

- Any algorithm that constructs a spanning tree by including edges that satisfy the cut property and deleting edges that satisfy the cycle property will yield an MST!
Implementing Prim’s Algorithm

**Prim’s Algorithm** $(G, c, s)$

1. $S = \{s\}$ and $T = \emptyset$
2. while $S \neq V$ do
3. Compute $(u, v) = \arg\min_{u \in S, v \in V - S} c(u, v)$
4. Add the node $v$ to $S$ and add the edge $(u, v)$ to $T$.

- Implementation and analysis are very similar to Dijkstra’s algorithm.
- Maintain $S$ and store attachment costs $a(v) = \min_{e \in \text{cut}(S)} c(e)$ for every node $v \in V - S$ in a priority queue.
- At each step, extract the node $v$ with the minimum attachment cost from the priority queue and update the attachment costs of the neighbours of $v$. 
Final Version of Prim’s Algorithm

**Prim’s Algorithm** \((G, c, s)\)

1. **Insert** \((Q, s, 0, \emptyset)\)
2. **while** \(S \neq V\) **do**
3. \((v, a(v), u) = \text{ExtractMin}(Q)\)
4. Add node \(v\) to \(S\) and edge \((u, v)\) to \(T\).
5. **for** every node \(x \in V - S\) such that \((v, x)\) is an edge in \(G\) **do**
6. **if** \(c(v, x) < a(x)\) **then**
7. \(a(x) = c(v, x)\)
8. **ChangeKey** \((Q, x, a(x), v)\)

- \(Q\) is a priority queue.
- Each element in \(Q\) is a triple: the node, its attachment cost, and its predecessor in the MST.
- In Step 8, if \(x\) is not already in \(Q\), simply **Insert** \((x, a(x), v)\) into \(Q\).
Final Version of Prim’s Algorithm

**Prim’s Algorithm**$(G, c, s)$

1: **Insert**$(Q, s, 0, \emptyset)$
2: while $S \neq V$ do
3: \quad $(v, a(v), u) =$ **ExtractMin**$(Q)$
4: \quad Add node $v$ to $S$ and edge $(u, v)$ to $T$.
5: \quad for every node $x \in V - S$ such that $(v, x)$ is an edge in $G$ do
6: \quad \quad if $c(v, x) < a(x)$ then
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- $Q$ is a priority queue.
- Each element in $Q$ is a triple: the node, its attachment cost, and its predecessor in the MST.
- In Step 8, if $x$ is not already in $Q$, simply **Insert** $(x, a(x), v)$ into $Q$.
- Total of $n - 1$ **ExtractMin** and $m$ **ChangeKey**/**Insert** operations, yielding a running time of $O(m \log n)$. 
Implementing Kruskal’s Algorithm

- Start with an empty set $T$ of edges.
- Process edges in $E$ in increasing order of cost.
- Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle.
Implementing Kruskal’s Algorithm

- Start with an empty set $T$ of edges.
- Process edges in $E$ in increasing order of cost.
- Add the next edge $e$ to $T$ only if adding $e$ does not create a cycle.
- Sorting edges takes $O(m \log n)$ time.
- Key question: “Does adding $e = (u, v)$ to $T$ create a cycle?”
  - Maintain set of connected components of $T$.
  - $\text{FIND}(u)$: return the name of the connected component of $T$ that $u$ belongs to.
  - $\text{UNION}(A, B)$: merge connected components $A$ and $B$. 

T. M. Murali February 17, 22, 24, 2021 Greedy Graph Algorithms
Analysing Kruskal’s Algorithm

- How many \texttt{FIND} invocations does Kruskal’s algorithm need?
Analysing Kruskal’s Algorithm

- How many \texttt{FIND} invocations does Kruskal’s algorithm need? $2m$.
- How many \texttt{UNION} invocations does Kruskal’s algorithm need?
Analysing Kruskal’s Algorithm

- How many `FIND` invocations does Kruskal’s algorithm need? $2m$.
- How many `UNION` invocations does Kruskal’s algorithm need? $n - 1$. 
Analysing Kruskal’s Algorithm

- How many $\text{FIND}$ invocations does Kruskal’s algorithm need? $2m$.
- How many $\text{UNION}$ invocations does Kruskal’s algorithm need? $n - 1$.
- Textbook describes two implementations of $\text{UNION-FIND}$: (see appendix to this set of slides)
  - Each $\text{FIND}$ takes $O(1)$ time, $k$ invocations of $\text{UNION}$ take $O(k \log k)$ time in total.
  - Each $\text{FIND}$ takes $O(\log n)$ time and each invocation of $\text{UNION}$ takes $O(1)$ time.

Total running time of Kruskal’s algorithm is $O(m \log n)$. 
Analysing Kruskal’s Algorithm

- How many `FIND` invocations does Kruskal’s algorithm need? 2m.
- How many `UNION` invocations does Kruskal’s algorithm need? n − 1.
- Textbook describes two implementations of `UNION-FIND`: (see appendix to this set of slides)
  - Each `FIND` takes $O(1)$ time, $k$ invocations of `UNION` take $O(k \log k)$ time in total.
  - Each `FIND` takes $O(\log n)$ time and each invocation of `UNION` takes $O(1)$ time.
- Total running time of Kruskal’s algorithm is $O(m \log n)$. 
Comments on Union-Find and MST

- The **Union-Find** data structure is useful to maintain the connected components of a graph as edges are added to the graph.
- The data structure does not support edge **deletion** efficiently.
- Current best algorithm for MST runs in $O(m\alpha(m,n))$ time (Chazelle 2000) and $O(m)$ randomised time (Karger, Klein, and Tarjan, 1995).
- Holy grail: $O(m)$ deterministic algorithm for MST.
Union-Find Data Structure

- Abstraction of the data structure needed by Kruskal’s algorithm.
- Maintain disjoint subsets of elements from a universe $U$ of $n$ elements.
- Each subset has an name. We will set a set’s name to be the identity of some element in it.
- Support three operations:
  1. \text{MAKE\_UNION\_FIND}(U): initialise the data structure with elements in $U$.
  2. \text{FIND}(u): return the identity of the subset that contains $u$.
  3. \text{UNION}(A, B): merge the sets named $A$ and $B$ into one set.
Union-Find Data Structure: Implementation 1

- Store all the elements of $U$ in an array $\text{COMPONENT}$.
  - Assume identities of elements are integers from 1 to $n$.
  - $\text{COMPONENT}[s]$ is the name of the set containing $s$.
- Implementing the operations:
Union-Find Data Structure: Implementation 1

- Store all the elements of $U$ in an array $\text{COMPONENT}$.  
  - Assume identities of elements are integers from 1 to $n$.  
  - $\text{COMPONENT}[s]$ is the name of the set containing $s$.

- Implementing the operations:
  1. $\text{MAKE\textsc{UnionFind}}(U)$: For each $s \in U$, set $\text{COMPONENT}[s] = s$ in $O(n)$ time.  
  2. $\text{Find}(s)$: return $\text{COMPONENT}[s]$ in $O(1)$ time.  
  3. $\text{Union}(A, B)$: merge $B$ into $A$ by scanning $\text{COMPONENT}$ and updating each index whose value is $B$ to the value $A$. Takes $O(n)$ time.
Appendix: Union-Find

Union-Find Data Structure: Implementation 1

- Store all the elements of \( U \) in an array \( \text{Component} \).
  - Assume identities of elements are integers from 1 to \( n \).
  - \( \text{Component}[s] \) is the name of the set containing \( s \).

Implementing the operations:

1. \text{MAKEUNIONFIND}(U): For each \( s \in U \), set \( \text{Component}[s] = s \) in \( O(n) \) time.
2. \text{FIND}(s): return \( \text{Component}[s] \) in \( O(1) \) time.
3. \text{UNION}(A, B): merge \( B \) into \( A \) by scanning \( \text{Component} \) and updating each index whose value is \( B \) to the value \( A \). Takes \( O(n) \) time.

\text{UNION} is very slow because
Appendix: Union-Find

Union-Find Data Structure: Implementation 1

- Store all the elements of $U$ in an array $\text{Component}$.
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- Implementing the operations:
  1. $\text{MakeUnionFind}(U)$: For each $s \in U$, set $\text{Component}[s] = s$ in $O(n)$ time.
  2. $\text{Find}(s)$: return $\text{Component}[s]$ in $O(1)$ time.
  3. $\text{Union}(A, B)$: merge $B$ into $A$ by scanning $\text{Component}$ and updating each index whose value is $B$ to the value $A$. Takes $O(n)$ time.

- $\text{Union}$ is very slow because we cannot efficiently find the elements that belong to a set.
Appendix: Union-Find

Union-Find Data Structure: Implementation 2

- Optimisation 1: Use an array $\text{ELEMENTS}$
  - Indices of $\text{ELEMENTS}$ range from 1 to $n$.
  - $\text{ELEMENTS}[s]$ stores the elements in the subset named $s$ in a list.

- Execute $\text{UNION}(A, B)$ by merging $B$ into $A$ in two steps:
  1. Updating $\text{COMPONENT}$ for elements of $B$ in $O(|B|)$ time.
  2. Append $\text{ELEMENTS}[B]$ to $\text{ELEMENTS}[A]$ in $O(1)$ time.

- $\text{UNION}$ takes $\Omega(n)$ in the worst-case.
Union-Find Data Structure: Implementation 2

- Optimisation 1: Use an array \texttt{ELEMENTS}
  - Indices of \texttt{ELEMENTS} range from 1 to \(n\).
  - \texttt{ELEMENTS}[s] stores the elements in the subset named \(s\) in a list.

- Execute \texttt{UNION}(\(A, B\)) by merging \(B\) into \(A\) in two steps:
  1. Updating \texttt{COMPONENT} for elements of \(B\) in \(O(|B|)\) time.
  2. Append \texttt{ELEMENTS}[\(B\)] to \texttt{ELEMENTS}[\(A\)] in \(O(1)\) time.

- \texttt{UNION} takes \(\Omega(n)\) in the worst-case.

- Optimisation 2: Store size of each set in an array (say, \texttt{SIZE}). If \(\texttt{SIZE}[B] \leq \texttt{SIZE}[A]\), merge \(B\) into \(A\). Otherwise merge \(A\) into \(B\). Update \texttt{SIZE}.
Union-Find Data Structure: Analysis of Implementation

- **MAKEUNIONFIND**(S) and **FIND**(u) are as before.

Union(\(A, B\)): Running time is proportional to the size of the smaller set, which may be \(\Omega(n)\).

Any sequence of \(k\) Union operations takes \(O(k \log k)\) time.

- \(k\) Union operations touch at most \(2k\) elements.
- Intuition: running time of Union is dominated by updates to Component.
  - Charge each update to the element being updated and bound number of charges per element.
  - Consider any element \(s\). Every time \(s\)'s set identity is updated, the size of the set containing \(s\) at least doubles \(\Rightarrow s\)'s set can change at most \(\log(2k)\) times \(\Rightarrow\) the total work done in \(k\) Union operations is \(O(k \log k)\).

Find is fast in the worst case, Union is fast in an amortised sense. Can we make both operations worst-case efficient?
Union-Find Data Structure: Analysis of Implementation

- `MAKEUNIONFIND(S)` and `FIND(u)` are as before.
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Appendix: Union-Find

Union-Find Data Structure: Analysis of Implementation

- `MakeUnionFind(S)` and `Find(u)` are as before.
- `Union(A, B)`: Running time is proportional to the size of the smaller set, which may be $\Omega(n)$.
- Any sequence of $k$ `Union` operations takes $O(k \log k)$ time.
Union-Find Data Structure: Analysis of Implementation

- **MAKEUNIONFIND**(*S*) and **FIND**(*u*) are as before.
- **UNION**(*A*, *B*): Running time is proportional to the size of the smaller set, which may be \(\Omega(n)\).
- Any sequence of *k* **UNION** operations takes \(O(k \log k)\) time.
  - *k* **UNION** operations touch at most 2\(k\) elements.
Union-Find Data Structure: Analysis of Implementation

- **MAKEUNIONFIND(S)** and **FIND(u)** are as before.
- **UNION(A, B)**: Running time is proportional to the size of the smaller set, which may be $\Omega(n)$.
- Any sequence of $k$ **UNION** operations takes $O(k \log k)$ time.
  - $k$ **UNION** operations touch at most $2k$ elements.
  - Intuition: running time of **UNION** is dominated by updates to **COMPONENT**. Charge each update to the element being updated and bound number of charges per element.
Appendix: Union-Find

Union-Find Data Structure: Analysis of Implementation

- \textsc{MakeUnionFind}(S) and \textsc{Find}(u) are as before.
- \textsc{Union}(A, B): Running time is proportional to the size of the smaller set, which may be \(\Omega(n)\).
- Any sequence of \(k\) \textsc{Union} operations takes \(O(k \log k)\) time.
  - \(k\) \textsc{Union} operations touch at most \(2k\) elements.
  - Intuition: running time of \textsc{Union} is dominated by updates to \textsc{Component}. Charge each update to the element being updated and bound number of charges per element.
  - Consider any element \(s\). Every time \(s\)'s set identity is updated, the size of the set containing \(s\) at least doubles \(\Rightarrow\) \(s\)'s set can change at most \(\log(2k)\) times \(\Rightarrow\) the total work done in \(k\) \textsc{Union} operations is \(O(k \log k)\).

Find is fast in the worst case, Union is fast in an amortised sense. Can we make both operations worst-case efficient?
MakeUnionFind\((S)\) and Find\((u)\) are as before.

Union\((A, B)\): Running time is proportional to the size of the smaller set, which may be \(\Omega(n)\).

Any sequence of \(k\) Union operations takes \(O(k \log k)\) time.

\begin{itemize}
  \item \(k\) Union operations touch at most \(2k\) elements.
  \item Intuition: running time of Union is dominated by updates to Component. Charge each update to the element being updated and bound number of charges per element.
  \item Consider any element \(s\). Every time \(s\)’s set identity is updated, the size of the set containing \(s\) at least doubles \(\Rightarrow\) \(s\)’s set can change at most \(\log(2k)\) times \(\Rightarrow\) the total work done in \(k\) Union operations is \(O(k \log k)\).
\end{itemize}

Find is fast in the worst case, Union is fast in an amortised sense. Can we make both operations worst-case efficient?
Union-Find Data Structure: Implementation 3

- Goal: Implement $\text{FIND}$ in $O(\log n)$ and $\text{UNION}$ in $O(1)$ worst-case time.
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Represent each subset in a tree using pointers:

- Each tree node contains an element and a pointer to a parent.
- The identity of the set is the identity of the element at the root.

**Figure 4.12** A Union–Find data structure using pointers. The data structure has only two sets at the moment, named after nodes $v$ and $j$. The dashed arrow from $u$ to $v$ is the result of the last Union operation. To answer a Find query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query Find($i$) would involve following the arrows $i$ to $x$, and then $x$ to $j$. 
**Union-Find Data Structure: Implementation 3**

- **Goal**: Implement `FIND` in $O(\log n)$ and `UNION` in $O(1)$ worst-case time.
- **Represent each subset in a tree using pointers:**
  - Each tree node contains an element and a pointer to a parent.
  - The identity of the set is the identity of the element at the root.
- **Implementing `FIND(u)`**: follow pointers from $u$ to the root of $u$'s tree.

![Diagram of Union-Find data structure](image)

**Figure 4.12** A Union–Find data structure using pointers. The data structure has only two sets at the moment, named after nodes $v$ and $j$. The dashed arrow from $u$ to $v$ is the result of the last `Union` operation. To answer a `Find` query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query `Find(i)` would involve following the arrows $i$ to $x$, and then $x$ to $j$. 
Union-Find Data Structure: Implementation 3

- **Goal:** Implement \( \text{FIND} \) in \( O(\log n) \) and \( \text{UNION} \) in \( O(1) \) worst-case time.
- **Represent each subset in a tree using pointers:**
  - Each tree node contains an element and a pointer to a parent.
  - The identity of the set is the identity of the element at the root.
- **Implementing \( \text{FIND}(u) \):** follow pointers from \( u \) to the root of \( u \)'s tree.
- **Implementing \( \text{UNION}(A, B) \):** make smaller tree's root a child of the larger tree's root. Takes \( O(1) \) time.

![Diagram](Image)

**Figure 4.12** A Union-Find data structure using pointers. The data structure has only two sets at the moment, named after nodes \( v \) and \( j \). The dashed arrow from \( u \) to \( v \) is the result of the last Union operation. To answer a Find query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query \( \text{Find}(i) \) would involve following the arrows \( i \) to \( x \), and then \( x \) to \( j \).
Why does \texttt{Find}(u) take \(O(\log n)\) time?

\textbf{Figure 4.12} A Union-Find data structure using pointers. The data structure has only two sets at the moment, named after nodes \(v\) and \(j\). The dashed arrow from \(u\) to \(v\) is the result of the last \texttt{Union} operation. To answer a \texttt{Find} query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query \texttt{Find}(i) would involve following the arrows \(i\) to \(x\), and then \(x\) to \(j\).
Appendix: Union-Find

Union-Find Data Structure: Find in Implementation 3

Why does $\text{Find}(u)$ take $O(\log n)$ time?

- Number of pointers followed equals the number of times the identity of the set containing $u$ changed.
- Every time $u$’s set’s identity changes, the set at least doubles in size $\Rightarrow$ there are $O(\log n)$ pointers followed.

Figure 4.12 A Union-Find data structure using pointers. The data structure has only two sets at the moment, named after nodes $v$ and $j$. The dashed arrow from $u$ to $v$ is the result of the last Union operation. To answer a Find query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query $\text{Find}(i)$ would involve following the arrows $i$ to $x$, and then $x$ to $j$. 

The set $\{s, u, w\}$ was merged into $\{t, v, z\}$.
Every time we invoke $\text{FIND}(u)$, we follow the same set of pointers.
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Path compression: make all nodes visited by \texttt{FIND}(u) children of the root.
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Path compression: make all nodes visited by `FIND(u)` children of the root.
Every time we invoke $\text{Find}(u)$, we follow the same set of pointers.

Path compression: make all nodes visited by $\text{Find}(u)$ children of the root.

Can prove that total time taken by $n$ $\text{Find}$ operations is $O(n\alpha(n))$, where $\alpha(n)$ is the inverse of the Ackermann function, and grows extremely slowly with $n$. 

![Diagram showing path compression in Union-Find](image)