Applications of Minimum Spanning Trees

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Minimum Spanning Trees

- We motivated MSTs through the problem of finding a low-cost network connecting a set of nodes.
- MSTs are useful in a number of seemingly disparate applications.
- We will consider two problems: minimum bottleneck graphs (problem 9 in Chapter 4) and clustering (Chapter 4.7).
Minimum Bottleneck Spanning Tree (MBST)

- The MST minimises the total cost of a spanning network.
- Consider another network design criterion:
  - Build a network of roads to connect all cities in a mountainous region but ensure road with highest elevation is as low as possible.
  - Total road length is not a criterion.
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- In an undirected graph $G(V, E)$, let $(V, T)$ be a spanning tree. The *bottleneck edge* in $T$ is the edge with largest cost in $T$. 
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**Minimum Bottleneck Spanning Tree (MBST)**

**INSTANCE:** An undirected graph $G(V, E)$ and a function $c : E \rightarrow \mathbb{R}^+$

**SOLUTION:** A set $T \subseteq E$ of edges such that $(V, T)$ is a spanning tree and there is no spanning tree in $G$ with a cheaper bottleneck edge.
Minimum Bottleneck Spanning Trees Clustering

**MBST Examples**

![MBST Examples Diagram](image-url)
Two Questions on MBSTs

1. Assume edge costs are distinct.
2. Is every MBST tree an MST?
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2. Is every MBST tree an MST? No. It is easy to create a counterexample.
3. Is every MST an MBST? Yes. Use the cycle property.

- Let $T$ be the MST and let $T'$ be a spanning tree with a cheaper bottleneck edge. Let $e$ be the bottleneck edge in $T$. 
  - Every edge in $T'$ is cheaper than $e$.
  - Adding $e$ to $T'$ creates a cycle consisting only of edges in $T'$ and $e$.
  - Since $e$ is the costliest edge in this cycle, by the cycle property, $e$ cannot belong to any MST, which contradicts the fact that $T$ is an MST.
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Motivation for Clustering

- Given a set of objects and distances between them.
- Objects can be images, web pages, people, species . . . .
- Distance function: increasing distance corresponds to decreasing similarity.
- Goal: group objects into clusters, where each cluster is a set of similar objects.
Example of Clustering
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Formalising the Clustering Problem

- Let $U$ be the set of $n$ objects labelled $p_1, p_2, \ldots, p_n$.
- For every pair $p_i$ and $p_j$, we have a distance $d(p_i, p_j)$.
- We require $d(p_i, p_i) = 0$, $d(p_i, p_j) > 0$, if $i \neq j$, and $d(p_i, p_j) = d(p_j, p_i)$.
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- Given a positive integer $k$, a $k$-clustering of $U$ is a partition of $U$ into $k$ non-empty subsets or “clusters” $C_1, C_2, \ldots, C_k$. 
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- Given a positive integer $k$, a $k$-clustering of $U$ is a partition of $U$ into $k$ non-empty subsets or “clusters” $C_1, C_2, \ldots C_k$.
- The spacing of a clustering is the smallest distance between objects in two different subsets:
  \[
  \text{spacing}(C_1, C_2, \ldots C_k) = \min_{1 \leq i, j \leq k, \ i \neq j, \ p \in C_i, q \in C_j} d(p, q)
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Clustering of Maximum Spacing

\textbf{INSTANCE:} A set $U$ of objects, a distance function $d : U \times U \to \mathbb{R}^+$, and a positive integer $k$

\textbf{SOLUTION:} A $k$-clustering of $U$ whose spacing is the largest over all possible $k$-clusterings.
Example of Clustering
Algorithm for Clustering of Maximum Spacing

Intuition: greedily cluster objects in increasing order of distance.

Let \( C \) be a set of \( n \) clusters, with each object in \( U \) in its own cluster.

Process pairs of objects in increasing order of distance.

- Let \((p, q)\) be the next pair with \( p \in C_p \) and \( q \in C_q \).
- If \( C_p \neq C_q \), add new cluster \( C_p \cup C_q \) to \( C \), delete \( C_p \) and \( C_q \) from \( C \).

Stop when there are \( k \) clusters in \( C \).

Same as Kruskal's algorithm but do not add last \( k - 1 \) edges in MST.
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**Algorithm for Clustering of Maximum Spacing**

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- Stop when there are $k$ clusters in $C$.
- Same as Kruskal’s algorithm but do not add last $k - 1$ edges in MST.
What is the Spacing of the Algorithm’s Clustering?

- Let $C$ be the clustering produced by the algorithm.
- What is spacing($C$)?
What is the Spacing of the Algorithm’s Clustering?

- Let $C$ be the clustering produced by the algorithm.
- What is spacing($C$)? It is the cost of the $(k - 1)$st most expensive edge in the MST. Let this cost be $d^*$. 
Why Does the Algorithm Compute the Clustering of Largest Spacing?

Let $C'$ be any other clustering (with $k$ clusters).

We will prove that $\text{spacing}(C') \leq d^*$. 

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Let $C'$ be any other clustering (with $k$ clusters).
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Poll
spacing($C'$) $\leq d^*$: Intuition

There is a pair of objects in the same cluster in $C'$ but in different clusters in $C$. Not useful for proof.
\[ \text{spacing}(C') \leq d^* : \text{Intuition} \]

There is a pair of objects in the same cluster in \( C \) but in different clusters in \( C' \). Can use in proof since they are connected by edges in the tree containing them.
$\text{spacing}(C') \leq d^*$: Intuition

There is a pair of objects in the same cluster in $C$ but in different clusters in $C'$. An MST edge that the algorithm has already added connects these objects.
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\[ \text{spacing}(C') \leq \text{distance between these objects} \leq d^* \]
\[ \text{spacing}(C') \leq d^* \]

- There must be two objects \( p_i \) and \( p_j \) that are in the same cluster \( C_r \) in \( C \) but in different clusters in \( C' \):
Minimum Bottleneck Spanning Trees Clustering

\[ \text{spacing}(C') \leq d^* \]

There must be two objects \( p_i \) and \( p_j \) that are in the same cluster \( C_r \) in \( C \) but in different clusters in \( C' \): \( \text{spacing}(C') \leq d(p_i, p_j) \).
There must be two objects $p_i$ and $p_j$ that are in the same cluster $C_r$ in $C$ but in different clusters in $C'$: $\text{spacing}(C') \leq d(p_i, p_j)$. But $d(p_i, p_j)$ could be $> d^*$. 

Suppose $p_i \in C'_s$ and $p_j \in C'_t$ in $C'$. 

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There must be two objects $p_i$ and $p_j$ that are in the same cluster $C_r$ in $C$ but in different clusters in $C'$: $\text{spacing}(C') \leq d(p_i, p_j)$. But $d(p_i, p_j)$ could be $> d^*$. 

Suppose $p_i \in C'_s$ and $p_j \in C'_t$ in $C'$.

All edges in the path $Q$ connecting $p_i$ and $p_j$ in the MST have length $\leq d^*$. 

In particular, there is an object $p \in C'_s$ and an object $p' \notin C'_s$ such that $p$ and $p'$ are adjacent in $Q$.

$$d(p, p') \leq d^* \Rightarrow \text{spacing}(C') \leq d(p, p') \leq d^*.$$

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**Figure 4.15** An illustration of the proof of (4.26), showing that the spacing of any other clustering can be no larger than that of the clustering found by the single-linkage algorithm.