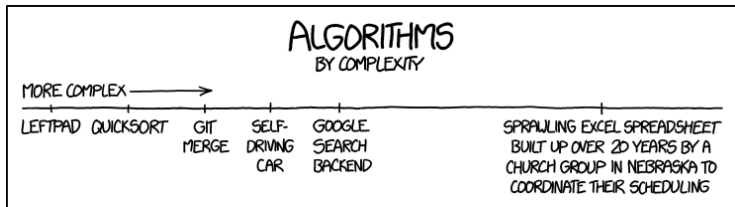


NP and Computational Intractability

T. M. Murali

April 14, 19, 21, 2021

Algorithm Design



• Patterns

- ▶ Greed.
- ▶ Divide-and-conquer.
- ▶ Dynamic programming.
- ▶ Duality.

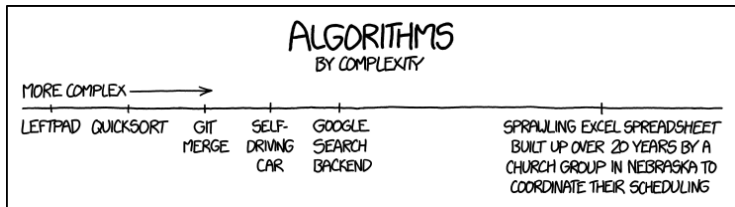
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$O(n \log n)$ counting inversions.

$O(n^3)$ RNA folding.

$O(n^2 m)$ maximum flow and minimum cuts.

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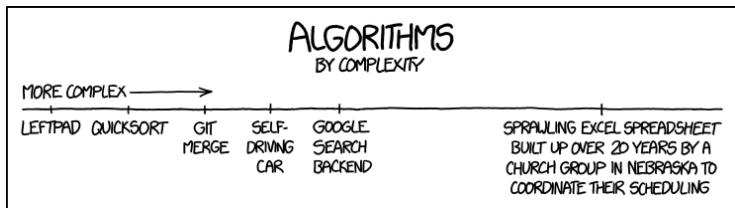
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IMAGE SEGMENTATION \leq_P MINIMUM s - t CUT

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IMAGE SEGMENTATION \leq_P MINIMUM s - t CUT

• “Anti-patterns”

- ▶ NP-completeness.
- ▶ PSPACE-completeness.
- ▶ Undecidability.

$O(n^k)$ algorithm unlikely.

$O(n^k)$ certification algorithm unlikely.

No algorithm possible.

Computational Tractability

- When is an algorithm an efficient solution to a problem?

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Polynomial time

Shortest path

Matching

Minimum cut

2-SAT

Planar four-colour

Bipartite vertex cover

Primality testing

Probably not

Longest path

3-D matching

Maximum cut

3-SAT

Planar three-colour

Vertex cover

Factoring

Problem Classification

- Classify problems based on whether they admit efficient solutions or not.
- Some extremely hard problems cannot be solved efficiently (e.g., chess on an n -by- n board).

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- Classify problems based on whether they admit efficient solutions or not.
- Some extremely hard problems cannot be solved efficiently (e.g., chess on an n -by- n board).
- However, classification is unclear for a very large number of discrete computational problems.
- We can prove that these problems are fundamentally equivalent and are manifestations of the same problem!

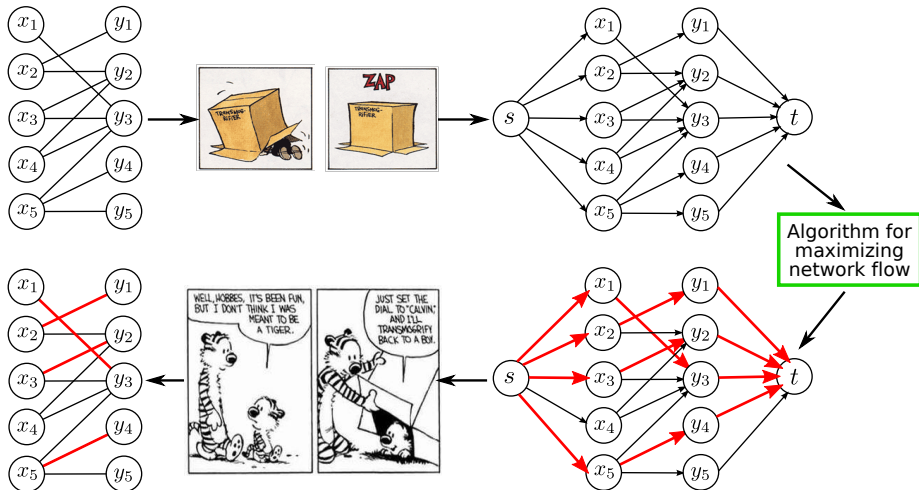
Polynomial-Time Reduction

- Goal is to express statements of the type “Problem X is at least as hard as problem Y .”
- Use the notion of *reductions*.
- Y is polynomial-time reducible to X ($Y \leq_P X$)

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Polynomial-Time Reduction



MAXIMUM BIPARTITE MATCHING \leq_P MAXIMUM s - t FLOW

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- $Y \leq_P X$ implies that “ X is at least as hard as Y .”
 - ▶ It is possible to solve Y using (potentially unknown) algorithm that solves X .
 - ▶ Not the reverse: we can solve X using an algorithm for Y .
- Such reductions are *Karp reductions*. *Cook reductions* allow a polynomial number of calls to the black box that solves X .

Usefulness of Reductions

- Claim: If $Y \leq_P X$ and X can be solved in polynomial time, then Y can be solved in polynomial time.

Usefulness of Reductions

- Claim: If $Y \leq_P X$ and X can be solved in polynomial time, then Y can be solved in polynomial time.
- Contrapositive: If $Y \leq_P X$ and Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.
- Informally: If Y is hard, and we can show that Y reduces to X , then the hardness “spreads” to X .

Reduction Strategies

- Simple equivalence.
- Special case to general case.
- Encoding with gadgets.

Optimisation versus Decision Problems

- So far, we have developed algorithms that solve optimisation problems.
 - ▶ Compute the *largest* flow.
 - ▶ Find the *closest* pair of points.
 - ▶ Find the schedule with the *least* completion time.

Optimisation versus Decision Problems

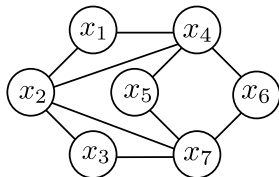
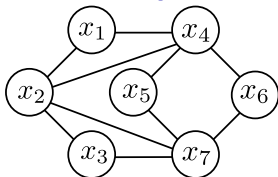
- So far, we have developed algorithms that solve optimisation problems.
 - ▶ Compute the *largest* flow.
 - ▶ Find the *closest* pair of points.
 - ▶ Find the schedule with the *least* completion time.
- Now, we will focus on *decision versions* of problems, e.g., is there a flow with value at least k , for a given value of k ?
- *Decision problem*: answer to every input is yes or no.

PRIMES

INSTANCE: A natural number n

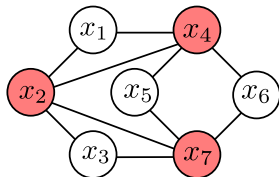
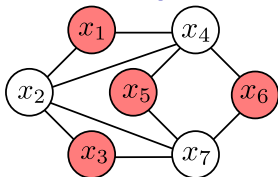
QUESTION: Is n prime?

Independent Set and Vertex Cover



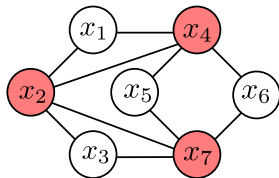
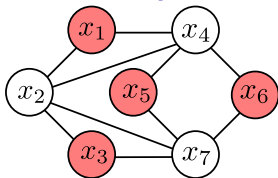
- Given an undirected graph $G(V, E)$, a subset $S \subseteq V$ is an *independent set* if no two vertices in S are connected by an edge.
- Given an undirected graph $G(V, E)$, a subset $S \subseteq V$ is a *vertex cover* if every edge in E is incident on at least one vertex in S . [▶ Poll](#)

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INDEPENDENT SET

INSTANCE: Undirected graph G and an integer k

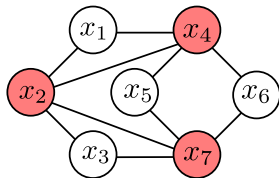
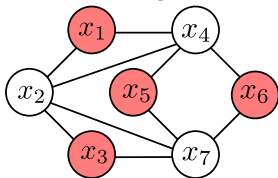
QUESTION: Does G contain an independent set of size $\geq k$?

VERTEX COVER

INSTANCE: Undirected graph G and an integer l

QUESTION: Does G contain a vertex cover of size $\leq l$?

Independent Set and Vertex Cover



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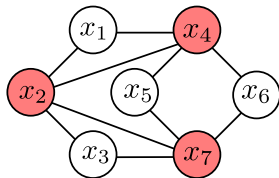
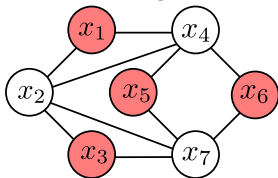
- Demonstrate simple equivalence between these two problems.

VERTEX COVER

INSTANCE: Undirected graph G and an integer l

QUESTION: Does G contain a vertex cover of size $\leq l$?

Independent Set and Vertex Cover



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- Demonstrate simple equivalence between these two problems.
- Claim: $\text{INDEPENDENT SET} \leq_P \text{VERTEX COVER}$ and $\text{VERTEX COVER} \leq_P \text{INDEPENDENT SET}$.

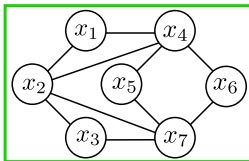
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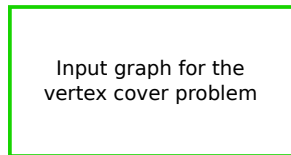
QUESTION: Does G contain a vertex cover of size $\leq l$?

Strategy for Proving Indep. Set \leq_P Vertex Cover

$k = 3$



$l = ?$



Yes, there is an independent set of size at least 3

No, every independent set is of size 3 or less



Yes

No

Black box algorithm for solving vertex cover

Strategy for Proving Indep. Set \leq_P Vertex Cover

- ① Start with an arbitrary input to INDEPENDENT SET: an undirected graph $G(V, E)$ and an integer k .
- ② From $G(V, E)$ and k , create an input to VERTEX COVER: an undirected graph $G'(V', E')$ and an integer l .
 - ▶ G' related to G in some way.
 - ▶ l can depend upon k and size of G .
- ③ Prove that $G(V, E)$ has an independent set of size $\geq k$ **if and only if** $G'(V', E')$ has a vertex cover of size $\leq l$.

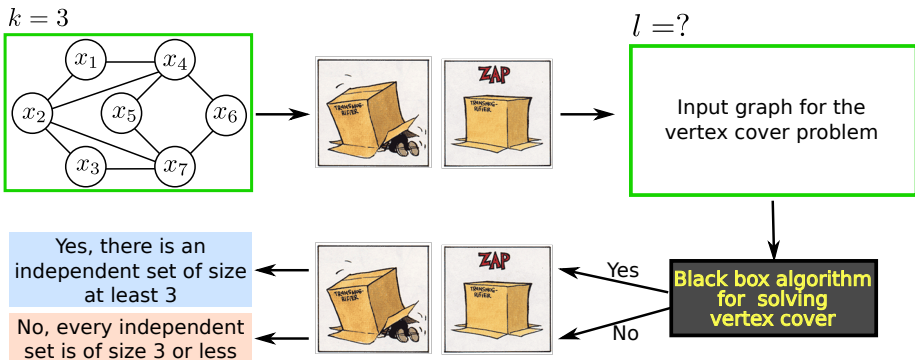


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 - Transformation and proof must be correct for all possible graphs $G(V, E)$ and all possible values of k .
 - Why is the proof an **iff** statement?

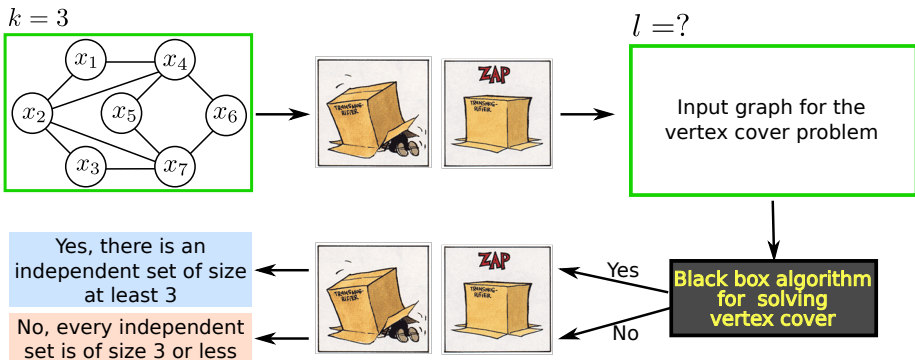


Reason for Two-Way Proof



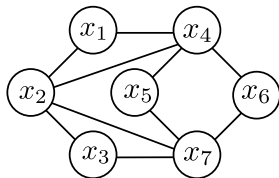
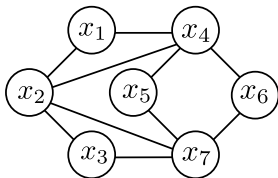
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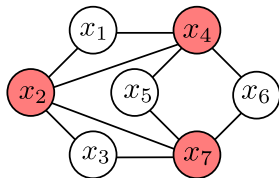
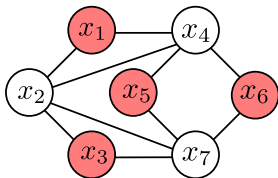
- Why is the proof an **iff** statement? In the reduction, we are using black box for VERTEX COVER to solve INDEPENDENT SET.
 - (i) If there is an independent set size $\geq k$, we must be sure that there is a vertex cover of size $\leq l$, so that we know that the black box will find this vertex cover.
 - (ii) If the black box finds a vertex cover of size $\leq l$, we must be sure we can construct an independent set of size $\geq k$ from this vertex cover.

Proof that Independent Set \leq_P Vertex Cover



- 1 Arbitrary input to INDEPENDENT SET: an undirected graph $G(V, E)$ and an integer k .
- 2 Let $|V| = n$.
- 3 Create an input to VERTEX COVER: same undirected graph $G(V, E)$ and integer $l = n - k$.

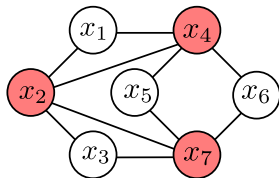
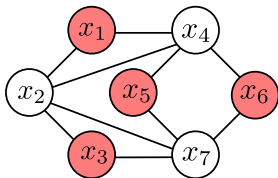
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- 4 Claim: $G(V, E)$ has an independent set of size $\geq k$ iff $G(V, E)$ has a vertex cover of size $\leq n - k$.

Proof: S is an independent set in G iff $V - S$ is a vertex cover in G .

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- Same idea proves that VERTEX COVER \leq_P INDEPENDENT SET

Vertex Cover and Set Cover

- INDEPENDENT SET is a “packing” problem: pack as many vertices as possible, subject to constraints (the edges).
- VERTEX COVER is a “covering” problem: cover all edges in the graph with as few vertices as possible.
- There are more general covering problems.

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MICROBE COVER

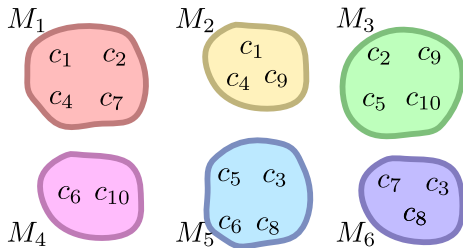
INSTANCE: A set U of n compounds, a collection M_1, M_2, \dots, M_l of microbes, where each microbe can make a subset of compounds in U , and an integer k .

QUESTION: Is there a subset of $\leq k$ microbes that can together make all the compounds in U ?

- Define a “microbe” to be the set of compounds it can make, e.g.,

$$M_1 = \{c_1, c_2, c_4, c_7\}.$$

► Poll



$$n = 10, l = 6, k = 3$$

Vertex Cover and Set Cover

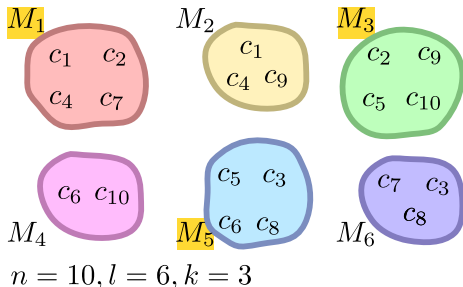
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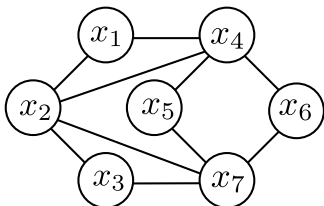
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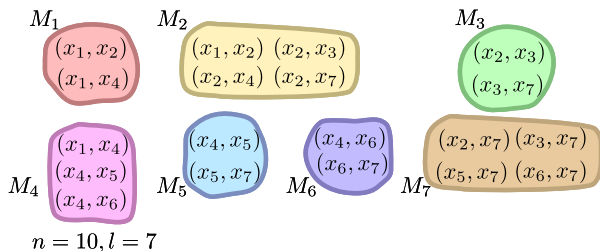
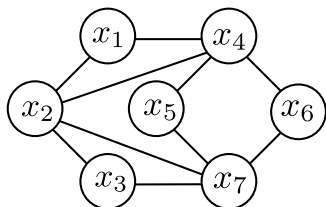


Vertex Cover \leq_P Microbe Cover



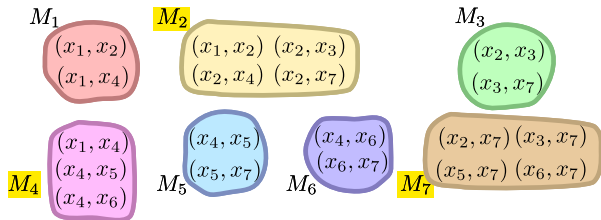
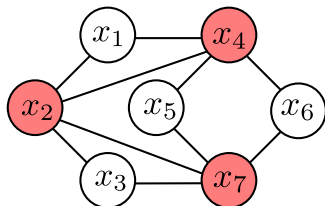
- Input to VERTEX COVER: an undirected graph $G(V, E)$ and an integer k .
- Let $|V| = l$.
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Vertex Cover \leq_P Microbe Cover



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 - ▶ $U = E$, i.e., each element of U is an edge of G , and
 - ▶ for each node $i \in V$, create a microbe M_i whose compounds are the set of edges incident on i .

Vertex Cover \leq_P Microbe Cover



$$n = 10, l = 7$$

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- Claim: U can be covered with $\leq k$ microbes iff G has a vertex cover with at $\leq k$ nodes.
- Proof strategy:
 - 1 If G has a vertex cover of size $\leq k$, then U can be covered with $\leq k$ microbes.
 - 2 If U can be covered with $\leq k$ microbes, then G has a vertex cover of size $\leq k$.

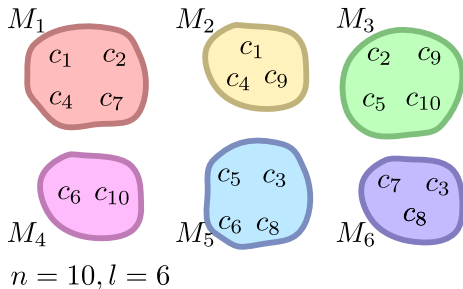
Microbe Cover and Set Cover

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- Purely combinatorial problem: a “microbe” is just a set of “compounds.”

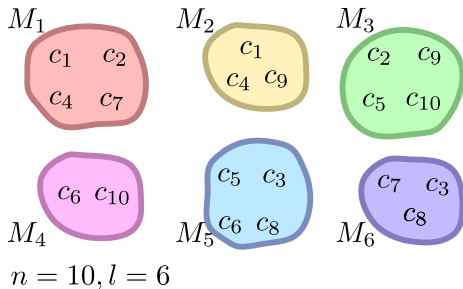


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SET COVER

INSTANCE: A set U of n elements, a collection S_1, S_2, \dots, S_m of subsets of U , and an integer k .

QUESTION: Is there a collection of $\leq k$ sets in the collection whose union is U ?

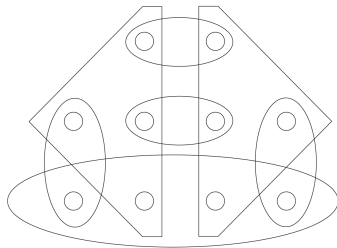


Figure 8.2 An instance of the Set Cover Problem.

Boolean Satisfiability

- Abstract problems formulated in Boolean notation.

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- Abstract problems formulated in Boolean notation.
- Given a set $X = \{x_1, x_2, \dots, x_n\}$ of n Boolean variables.
- Each variable can take the value 0 or 1.
- *Term*: a variable x_i or its negation $\overline{x_i}$.
- *Clause* of *length* l : (or) of l distinct terms $t_1 \vee t_2 \vee \dots \vee t_l$.
- *Truth assignment* for X : is a function $\nu : X \rightarrow \{0, 1\}$.
- An assignment ν *satisfies* a clause C if it causes at least one term in C to evaluate to 1 (since C is an or of terms).
- An assignment *satisfies* a collection of clauses C_1, C_2, \dots, C_k if it causes all clauses to evaluate to 1, i.e., $C_1 \wedge C_2 \wedge \dots \wedge C_k = 1$.
 - ▶ ν is a *satisfying assignment* with respect to C_1, C_2, \dots, C_k .
 - ▶ set of clauses C_1, C_2, \dots, C_k is *satisfiable*.

Example

- $X = \{x_1, x_2, x_3, x_4\}$
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- Clauses: [▶ Poll](#)

$$x_1 \vee \overline{x_2} \vee \overline{x_3}$$

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SAT and 3-SAT

SATISFIABILITY PROBLEM (SAT)

INSTANCE: A set of clauses C_1, C_2, \dots, C_k over a set $X = \{x_1, x_2, \dots, x_n\}$ of n variables.

QUESTION: Is there a satisfying truth assignment for X with respect to C ?

SAT and 3-SAT

3-SATISFIABILITY PROBLEM (3-SAT)

INSTANCE: A set of clauses C_1, C_2, \dots, C_k , each of length three, over a set $X = \{x_1, x_2, \dots, x_n\}$ of n variables.

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- SAT and 3-SAT are fundamental combinatorial search problems.
- We have to make n independent decisions (the assignments for each variable) while satisfying a set of constraints.
- Satisfying each constraint in isolation is easy, but we have to make our decisions so that all constraints are satisfied simultaneously.

Examples of 3-SAT

Example:

- ▶ $C_1 = x_1 \vee 0 \vee 0$
- ▶ $C_2 = x_2 \vee 0 \vee 0$
- ▶ $C_3 = \overline{x_1} \vee \overline{x_2} \vee 0$

▶ Poll

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❶ Is $C_1 \wedge C_2$ satisfiable?

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❶ Is $C_1 \wedge C_2$ satisfiable? Yes, by $x_1 = 1, x_2 = 1$.

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- 4 Is $C_1 \wedge C_2 \wedge C_3$ satisfiable? No.

3-SAT and Independent Set

$$C_1 = x_1 \vee \overline{x_2} \vee \overline{x_3}$$

$$C_2 = \overline{x_1} \vee x_2 \vee x_4$$

$$C_3 = \overline{x_1} \vee x_3 \vee \overline{x_4}$$

- We want to prove $3\text{-SAT} \leq_P \text{INDEPENDENT SET}$.

3-SAT and Independent Set

$$C_1 = x_1 \vee \overline{x_2} \vee \overline{x_3} \quad \textcircled{1} \text{ Select } x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1.$$

$$C_2 = \overline{x_1} \vee x_2 \vee x_4$$

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- We want to prove $3\text{-SAT} \leq_P \text{INDEPENDENT SET}$.
- Two ways to think about 3-SAT:
 - ① Make an independent 0/1 decision on each variable and succeed if we achieve one of three ways in which to satisfy each clause.

3-SAT and Independent Set

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 - ② Choose (at least) one term from each clause. Find a truth assignment that causes each chosen term to evaluate to 1. Ensure that no two terms selected *conflict*, e.g., select $\overline{x_2}$ in C_1 and x_2 in C_2 .

3-SAT and Independent Set

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Proving $3\text{-SAT} \leq_P \text{Independent Set}$

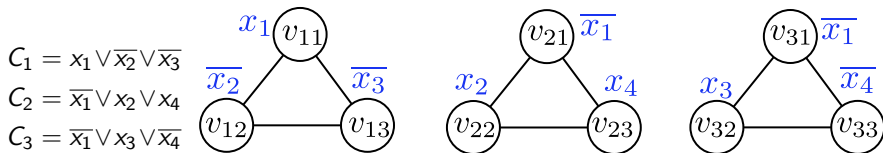
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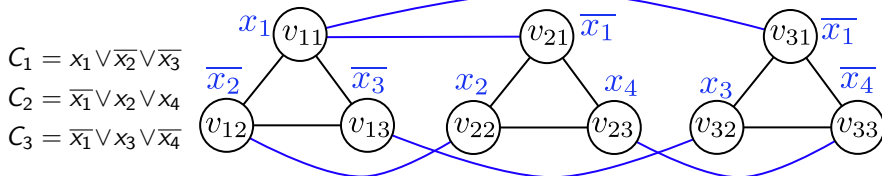
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 - ▶ Label each node v_{ij} , $1 \leq j \leq 3$ with the j th term in C_i . [▶ Poll](#)

Proving $3\text{-SAT} \leq_P \text{Independent Set}$



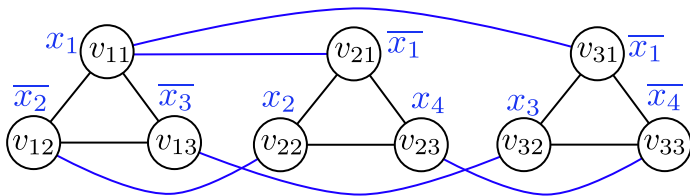
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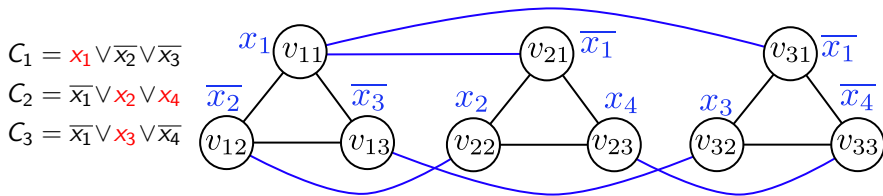
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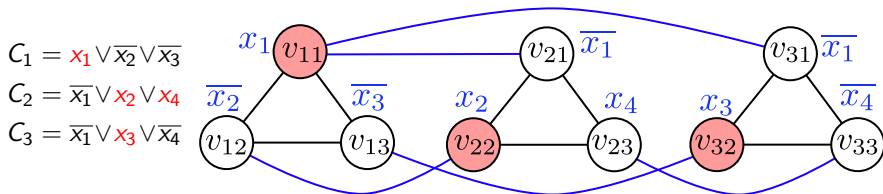
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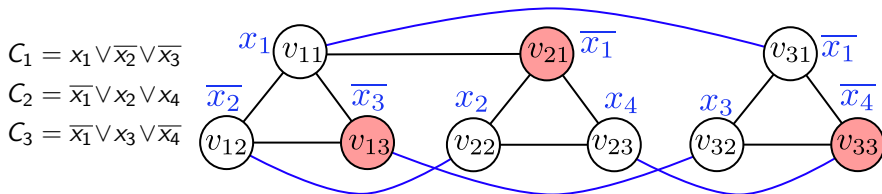
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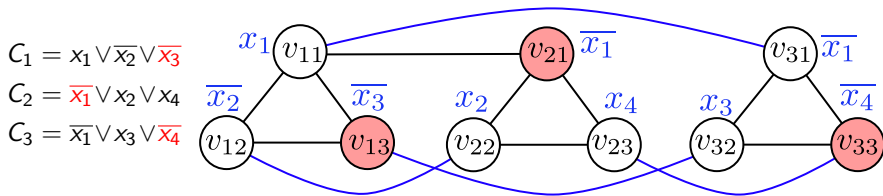
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Proving $3\text{-SAT} \leq_P \text{Independent Set}$



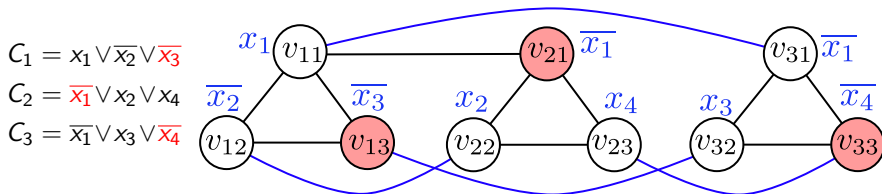
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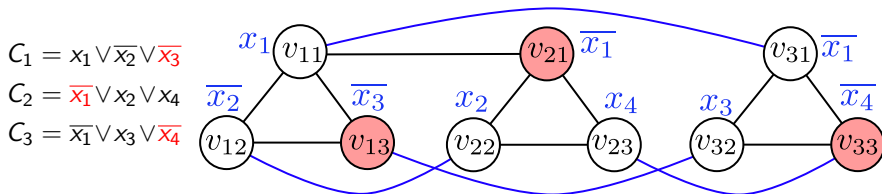
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 - ▶ If x_i is the label of a node in S , set $x_i = 1$; else set $x_i = 0$.
 - ▶ Why is each clause satisfied?

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- Claim: If $Z \leq_P Y$ and $Y \leq_P X$, then $Z \leq_P X$.

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- We have shown

$$3\text{-SAT} \leq_P \text{INDEPENDENT SET} \leq_P \text{VERTEX COVER} \leq_P \text{SET COVER}$$

Finding vs. Certifying

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- Is it easy to check if a given set of vertices in an undirected graph forms an independent set of size at least k ?
- Is it easy to check if a particular truth assignment satisfies a set of clauses?
- We draw a contrast between *finding* a solution and *checking* a solution (in polynomial time).
- Since we have not been able to develop efficient algorithms to solve many decision problems, let us turn our attention to whether we can check if a proposed solution is correct.

Problems and Algorithms

PRIMES

INSTANCE: A natural number n

QUESTION: Is n prime?

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A decision problem X is in \mathcal{P} iff there is an algorithm A with polynomial running time that solves X .

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- Certifier does not care about how to find these certificates.

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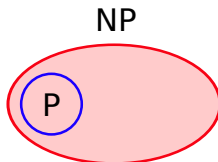
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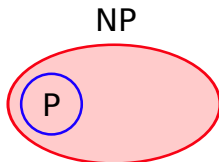
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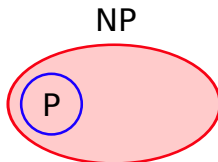
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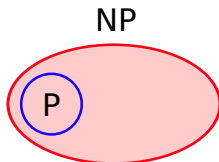
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P vs NP Problem



Suppose that you are organizing housing accommodations for a group of four hundred university students. Space is limited and only one hundred of the students will receive places in the dormitory. To complicate matters, the Dean has provided you with a list of pairs of incompatible students, and requested that no pair from this list appear in your final choice. This is an example of what computer scientists

call an NP-problem, since it is easy to check if a given choice of one hundred students proposed by a coworker is satisfactory (i.e., no pair taken from your coworker's list also appears on the list from the Dean's office), however the task of generating such a list from scratch seems to be so hard as to be completely impractical. Indeed, the total number of ways of choosing one hundred students from the four hundred applicants is greater than the number of atoms in the known universe! Thus no future civilization could ever hope to build a supercomputer capable of solving the problem by brute force; that is, by checking every possible combination of 100 students. However, this apparent difficulty may only reflect the lack of ingenuity of your programmer. In fact, one of the outstanding problems in computer science is determining whether questions exist whose answer can be quickly checked, but which require an impossibly long time to solve by any direct procedure. Problems like the one listed above certainly seem to be of this kind, but so far no one has managed to prove that any of them really are so hard as they appear, i.e., that there really is no feasible way to generate an answer with the help of a computer. Stephen Cook and Leonid Levin formulated the P (i.e., easy to find) versus NP (i.e., easy to check) problem independently in 1971.

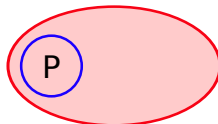
Image credit: on the left, Stephen Cook by Jiri Janiček (cropped). CC BY-SA 3.0

Rules:
Rules for the Millennium Prizes
Related Documents:
Official Problem Description
Minesweeper
Related Links:
Lecture by Vijaya Ramachandran

This problem is: Unsolved

Summary

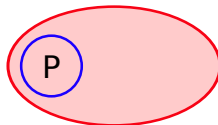
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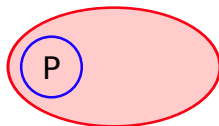
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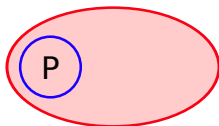
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- What are the hardest problems in \mathcal{NP} ?

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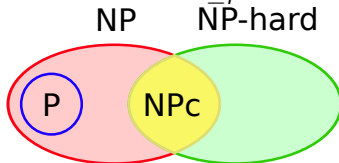
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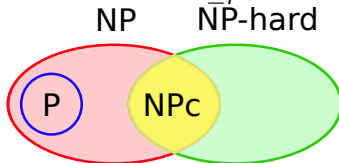
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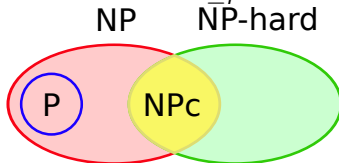
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- Does even one \mathcal{NP} -Complete problem exist?!** If it does, how can we prove that every problem in \mathcal{NP} reduces to this problem?

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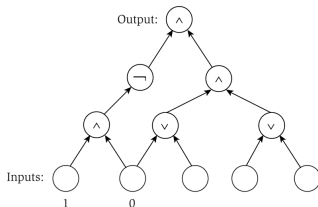


Figure 8.4 A circuit with three inputs, two additional sources that have assigned truth values, and one output.

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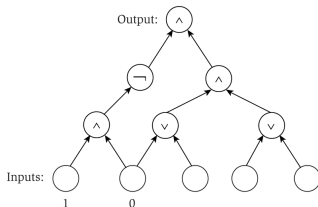


Figure 8.4 A circuit with three inputs, two additional sources that have assigned truth values, and one output.

► Skip proof; read textbook or Chapter 2.6 of Garey and Johnson.

CIRCUIT SATISFIABILITY

INSTANCE: A circuit K .

QUESTION: Is there a truth assignment to the inputs that causes the output to have value 1?

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- $s \in X$ iff there is an assignment of the input bits of K that makes K satisfiable.

Example of Transformation to Circuit Satisfiability

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Example of Transformation to Circuit Satisfiability

- Does a graph G on n nodes have a two-node independent set?
- s encodes the graph G with $\binom{n}{2}$ bits.
- t encodes the independent set with n bits.
- Certifier needs to check if
 - 1 at least two bits in t are set to 1 and
 - 2 no two bits in t are set to 1 if they form the ends of an edge (the corresponding bit in s is set to 1).

Example of Transformation to Circuit Satisfiability

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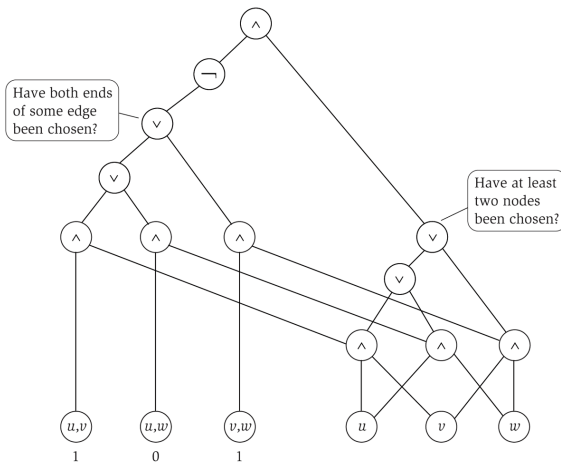


Figure 8.5 A circuit to verify whether a 3-node graph contains a 2-node independent set.

Asymmetry of Certification

- Definition of efficient certification and \mathcal{NP} is fundamentally asymmetric:
 - ▶ An input s is a “yes” instance iff there exists a short certificate t such that $B(s, t) = \text{yes}$.
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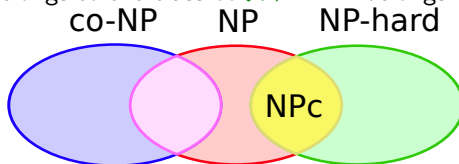
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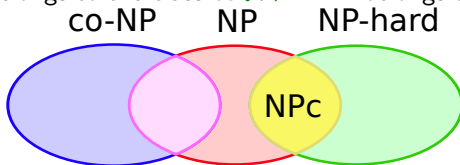
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- Open problem: Is $\mathcal{NP} = \text{co-}\mathcal{NP}$?

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- For a decision problem X , its *complementary problem* \bar{X} is the set of inputs s such that $s \in \bar{X}$ iff $s \notin X$.
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- If $X \in \mathcal{NP}$, then is $\bar{X} \in \mathcal{NP}$? Unclear in general.
- A problem X belongs to the class $\text{co-}\mathcal{NP}$ iff \bar{X} belongs to \mathcal{NP} .



- Open problem: Is $\mathcal{NP} = \text{co-}\mathcal{NP}$?
- Claim: If $\mathcal{NP} \neq \text{co-}\mathcal{NP}$ then $\mathcal{P} \neq \mathcal{NP}$.

Good Characterisations: the Class $\mathcal{NP} \cap \text{co-}\mathcal{NP}$

- If a problem belongs to both \mathcal{NP} and $\text{co-}\mathcal{NP}$, then
 - ▶ When the answer is yes, there is a short proof.
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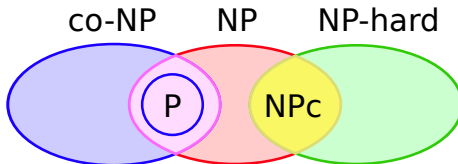
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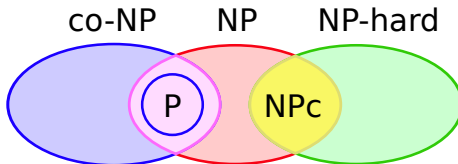
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