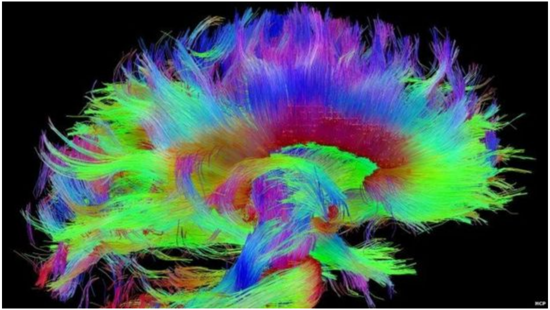
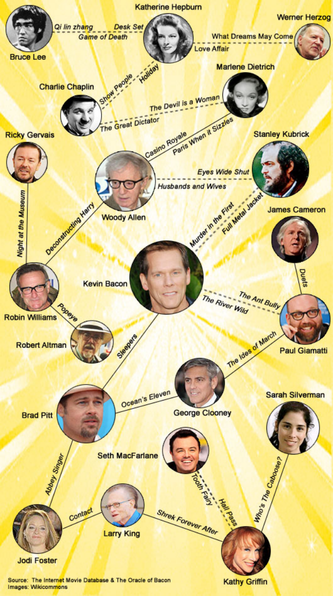
CS 4884: Introduction to Graphs

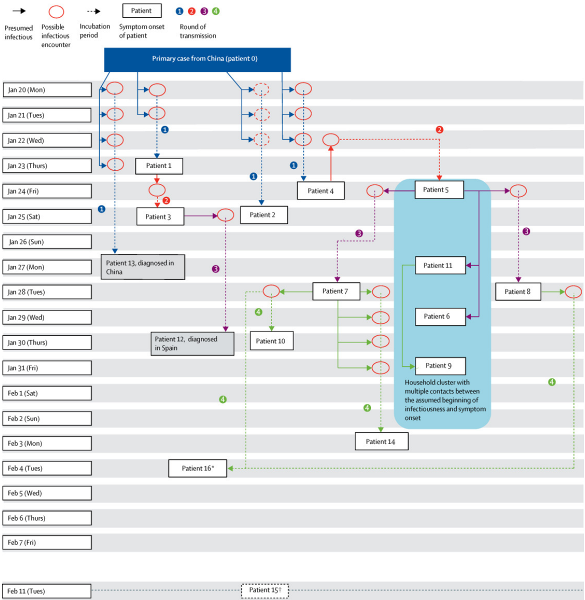
T. M. Murali

January 25, 2022

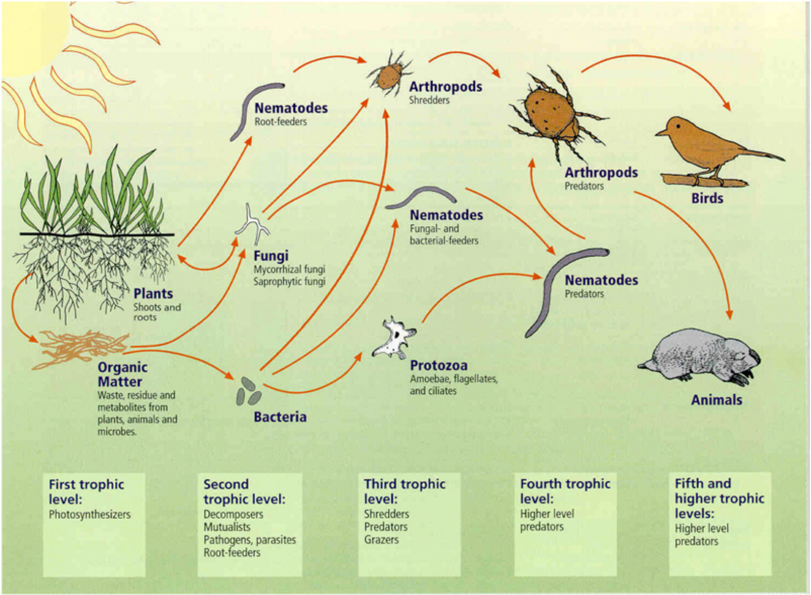


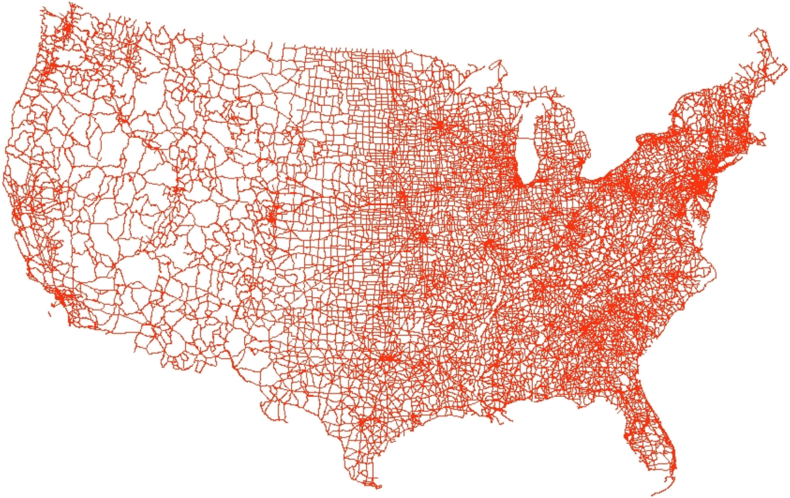


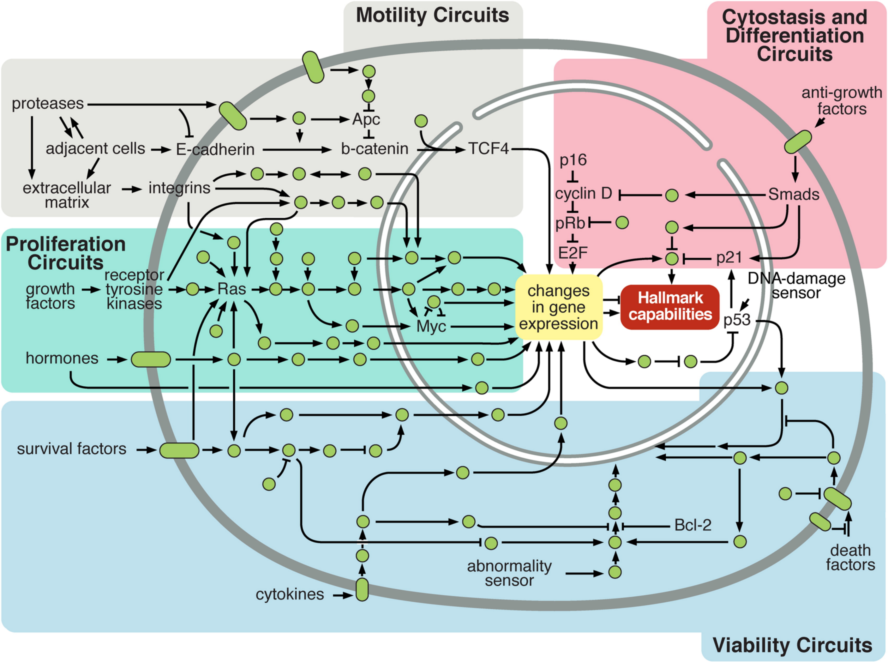
The Oracle of Bacon



(Böhmer *et al.*, *The Lancet*, May 15, 2020)


Nematodes
Root-feeders
Arthropods
Shredders
Arthropods
Predators
Birds
Fungi
Mycorrhizal fungi
Saprophytic fungi
Nematodes
Fungal- and bacterial-feeders
Plants
Shoots and roots
Nematodes
Predators
Organic Matter
Waste, residue and metabolites from plants, animals and microbes.
Protozoa
Amoebae, flagellates, and ciliates
Bacteria
Animals
First trophic level:
Photosynthesizers
Second trophic level:
Decomposers
Mutualists
Pathogens, parasites
Root-feeders
Third trophic level:
Shredders
Predators
Grazers
Fourth trophic level:
Higher level predators
Fifth and higher trophic levels:
Higher level predators




Motility Circuits
Cytostasis and Differentiation Circuits
Proliferation Circuits
Viability Circuits
proteases
adjacent cells
E-cadherin
extracellular matrix
integrins
Apc
b-catenin
TCF4
anti-growth factors
Smads
p16
cyclin D
pRb
E2F
p21
DNA-damage sensor
p53
growth factors
receptor tyrosine kinases
Ras
Myc
hormones
changes in gene expression
Hallmark capabilities
survival factors
Bcl-2
abnormality sensor
death factors
cytokines

Graphs

Graph $\equiv$ Network

- Model pairwise relationships (edges) between objects (nodes).

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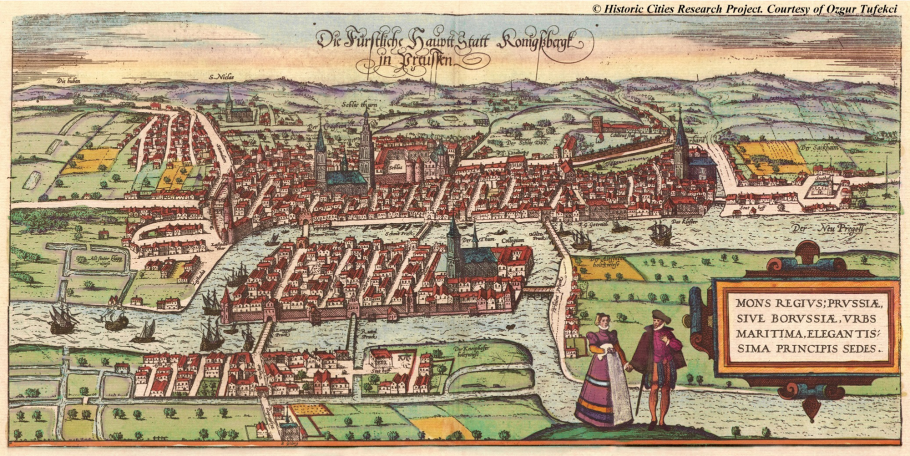
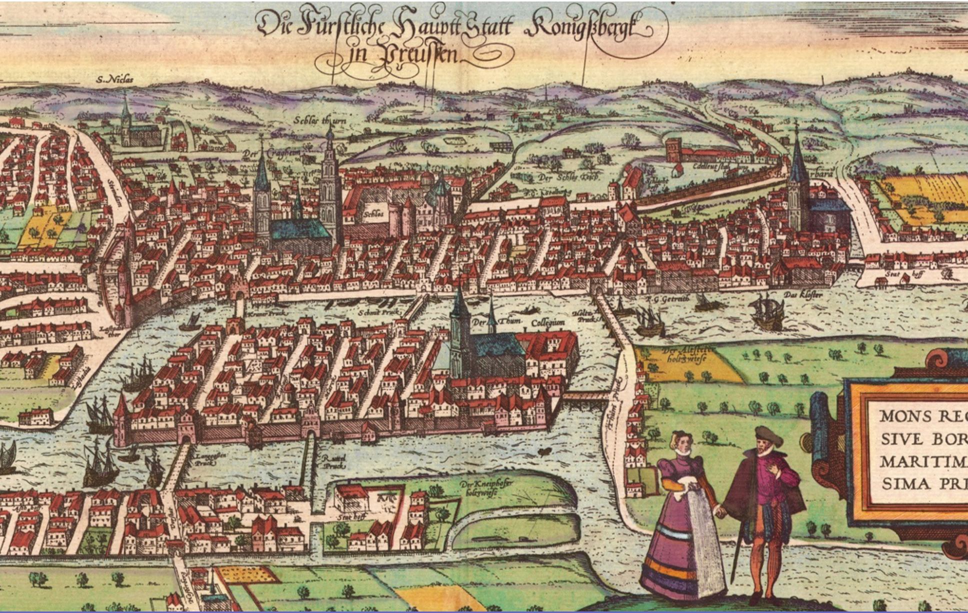
- Other examples: computer networks, the World Wide Web, ecology (food webs), social networks, software systems, job scheduling, VLSI circuits, cellular networks, transportation networks, ...
- Problems involving graphs have a rich history dating back to Euler.

Euler and Graphs



Devise a walk through the city that
crosses each of the seven bridges exactly once.

Euler and Graphs



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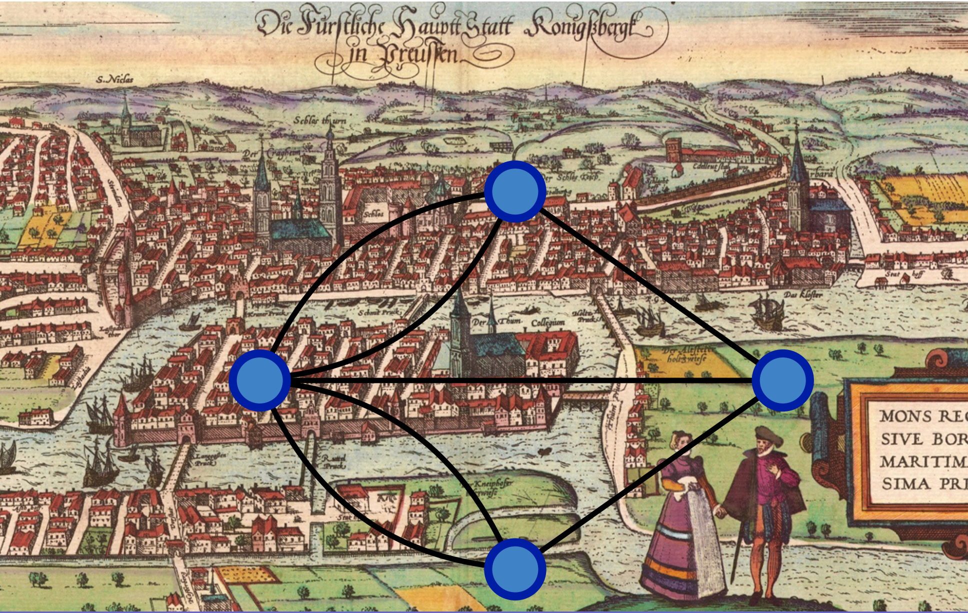
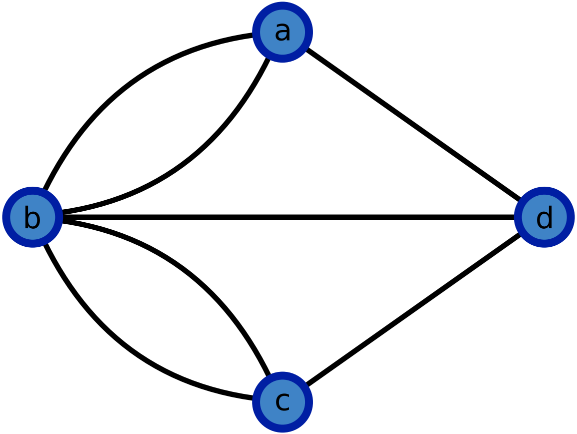


Definition of an Undirected Graph

- *Undirected graph* $G = (V, E)$: set V of nodes and set E of edges.
  - Each element of E is an unordered pair of nodes.
  - Edge (u, v) is *incident* on u, v; u and v are *neighbours* of each other.
  - G contains no self loops.



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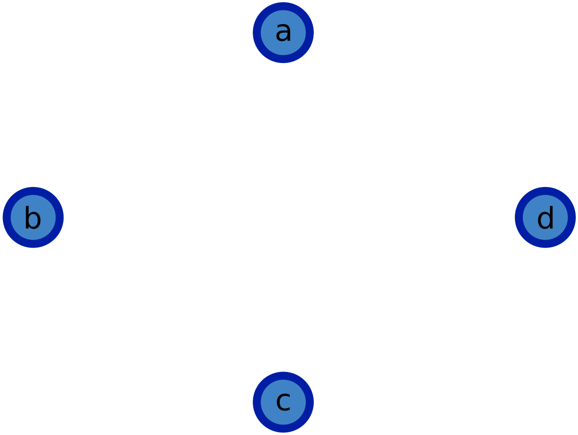

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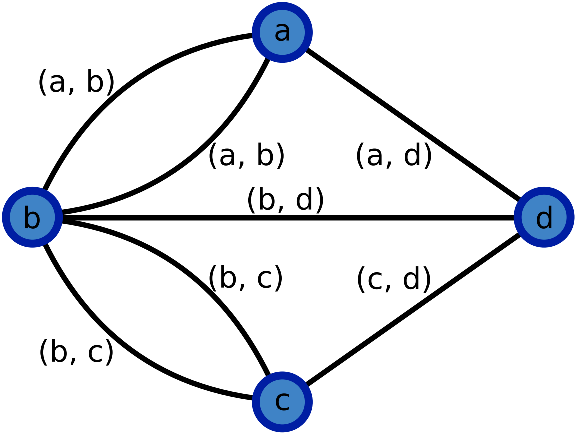
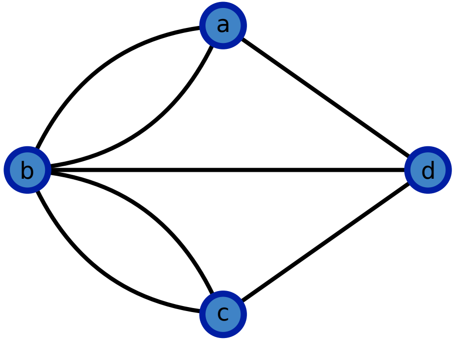


Paths and Cycles in Graphs



- A *v_1-v_k path* in an undirected graph $G = (V, E)$ is a sequence of nodes $v_1, v_2, \ldots, v_{k-1}, v_k \in V$ such that for every $i, 1 \leq i < k$, (v_i, v_{i+1}) is an edge in E.

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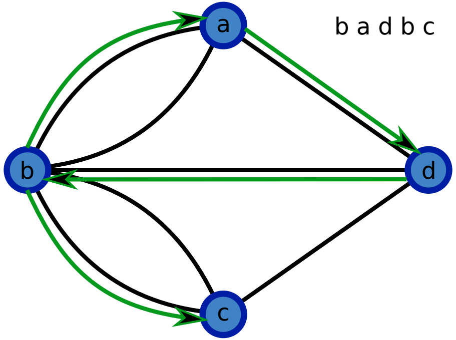

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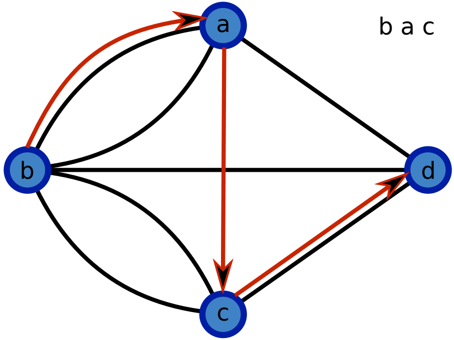

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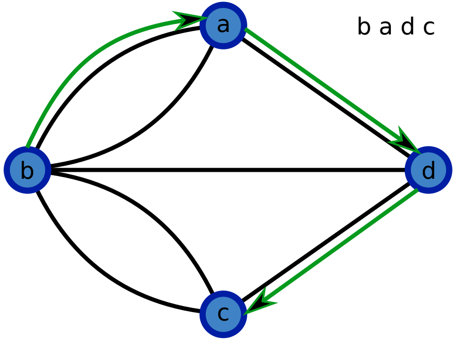

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- A path is *simple* if all its nodes are distinct.

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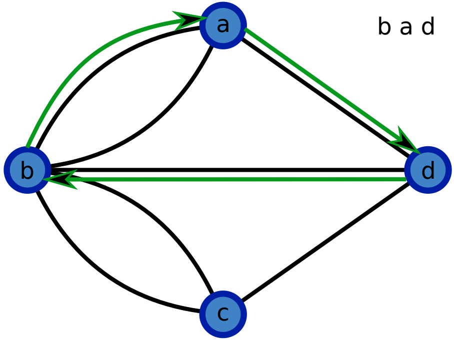

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- A *cycle* is a path where the first $k - 1$ nodes are distinct and $v_1 = v_k$.

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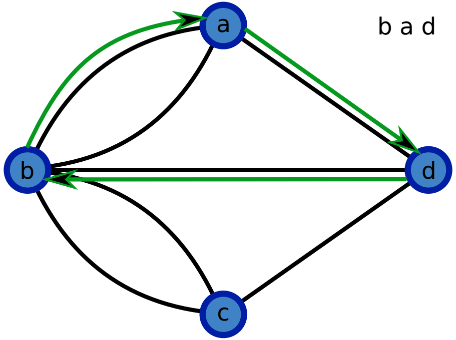

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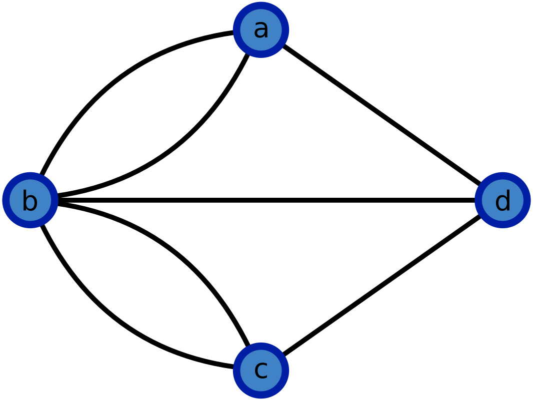
- A path is *simple* if all its nodes are distinct.
- A *cycle* is a path where the first $k-1$ nodes are distinct and $v_1 = v_k$.
- An undirected graph G is *connected* if for every pair of nodes $u, v \in V$, there is a u-v path in G.

Bridges to Graphs



EULERIAN TOUR

Given an undirected graph $G(V, E)$,

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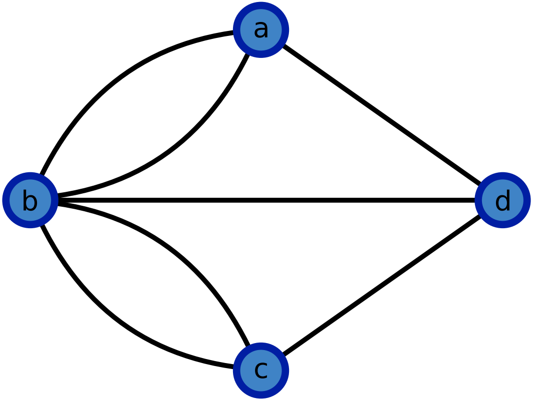
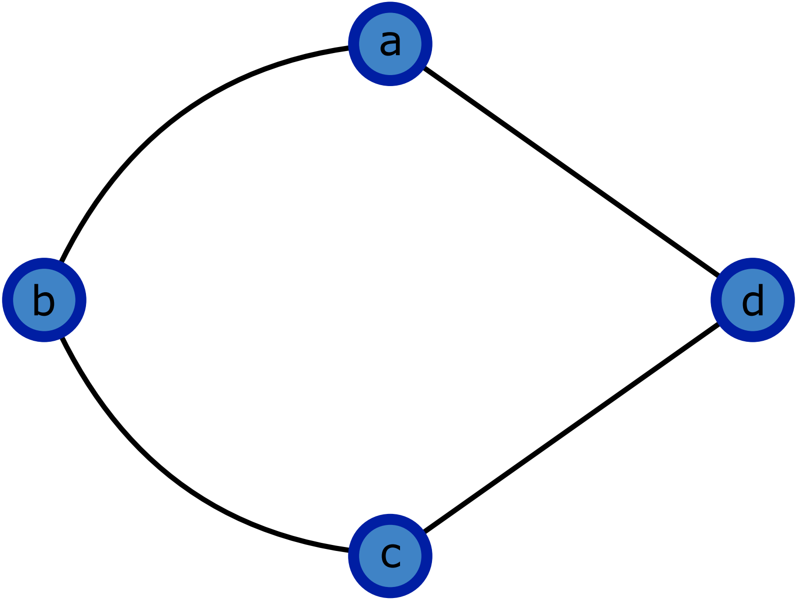


Eulerian tour

Given an undirected graph $G(V, E)$,

construct an *Eulerian tour*, i.e., a path in G that traverses each edge in E exactly once, if such a tour exists.

Examples of Euler Tours



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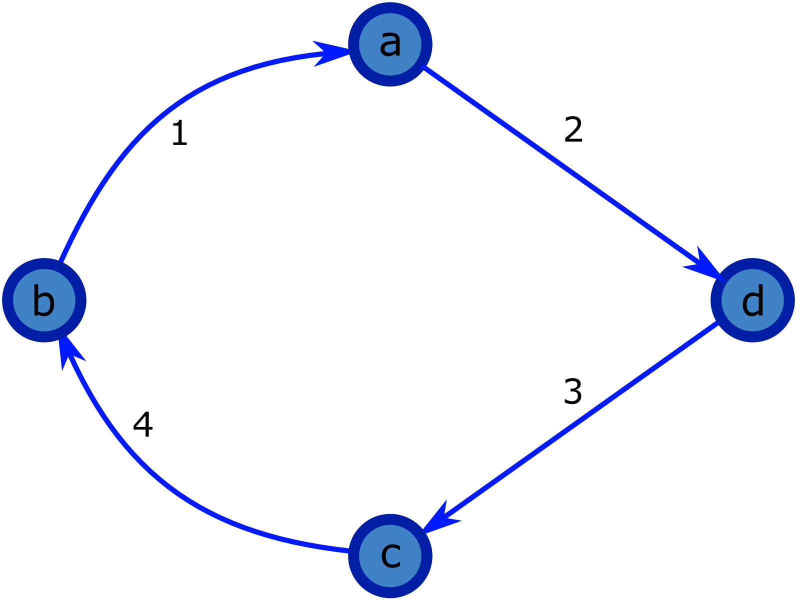

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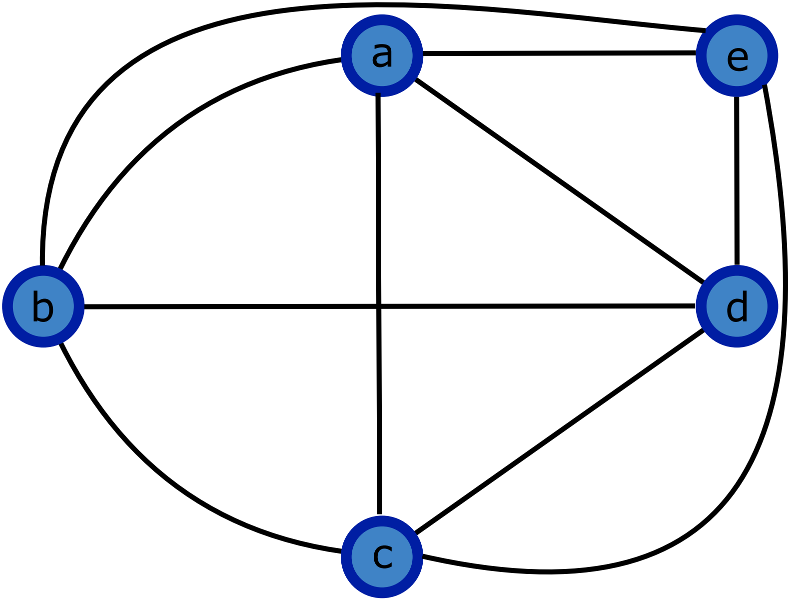

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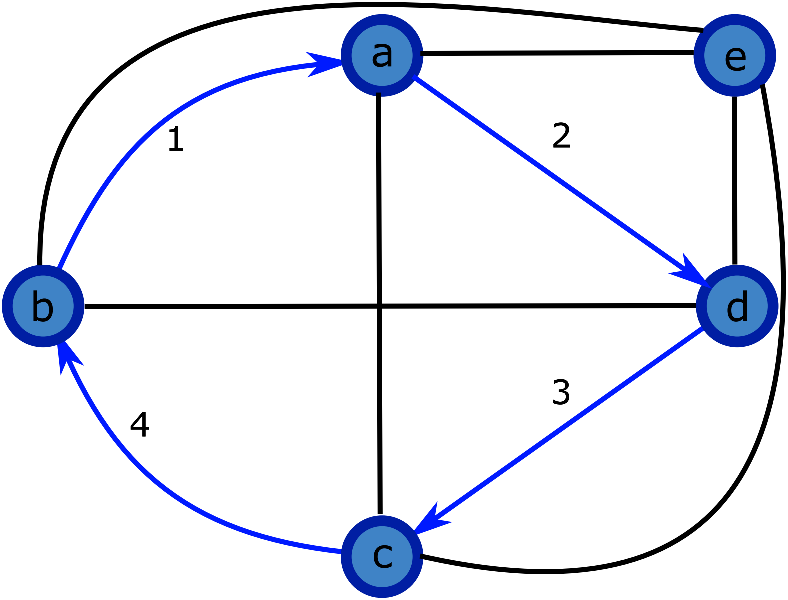

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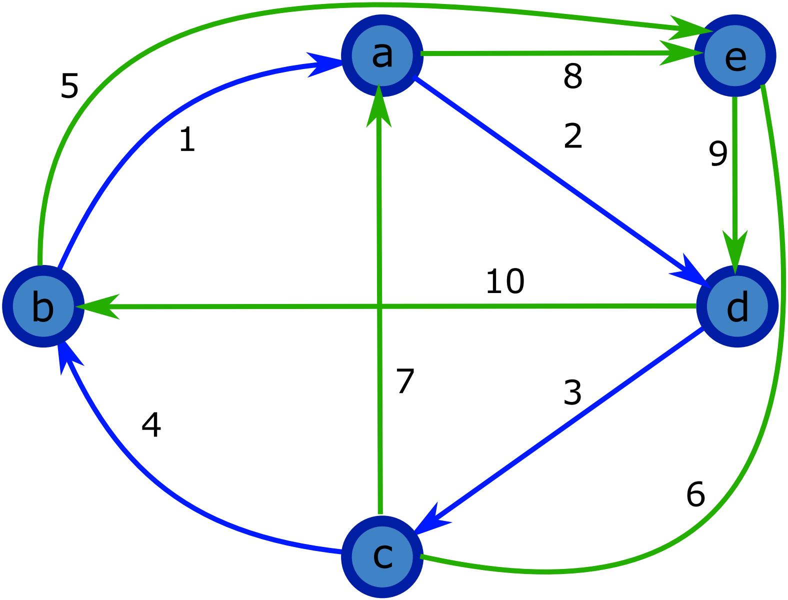

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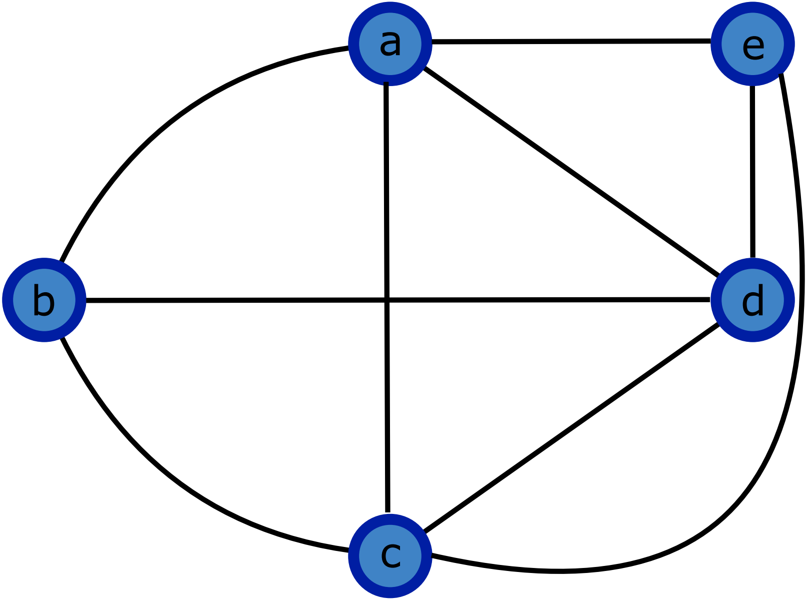

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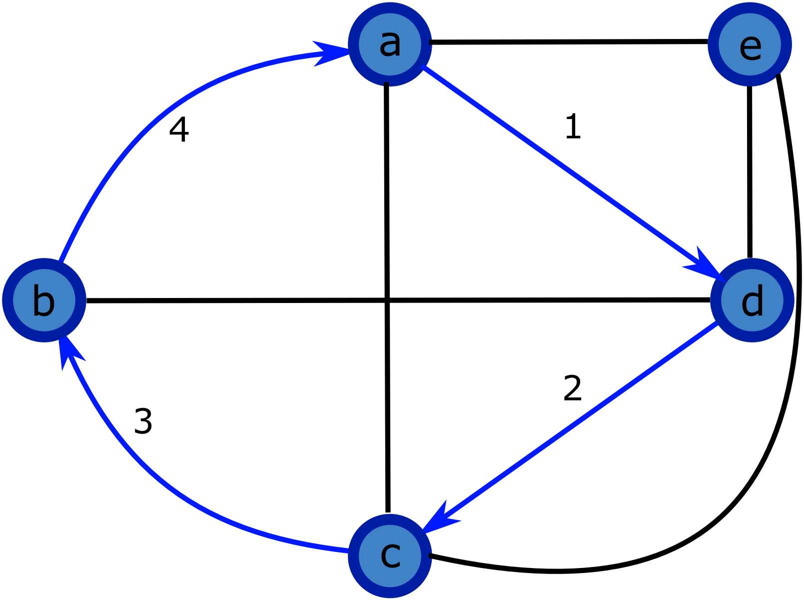

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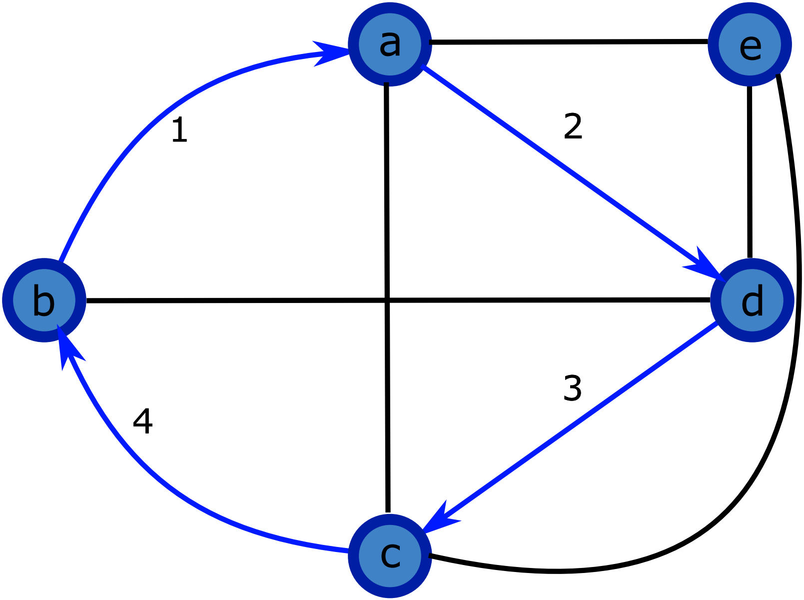

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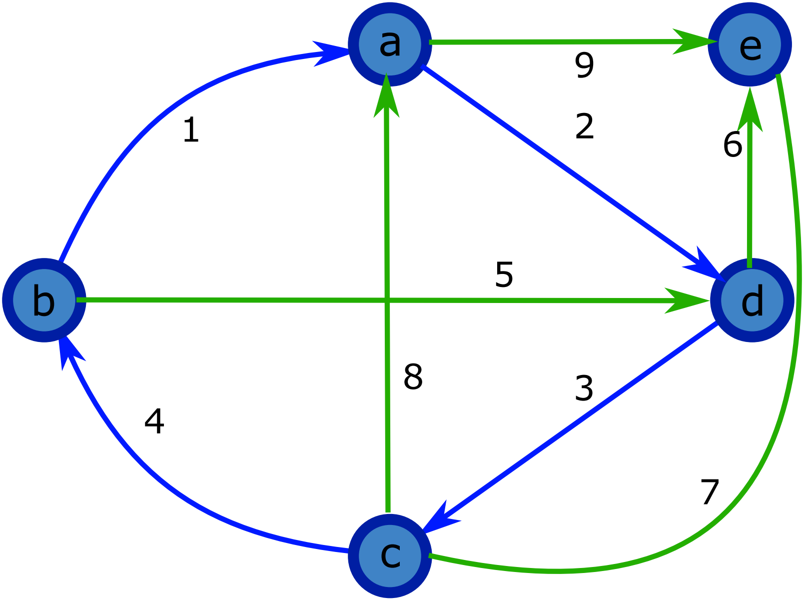

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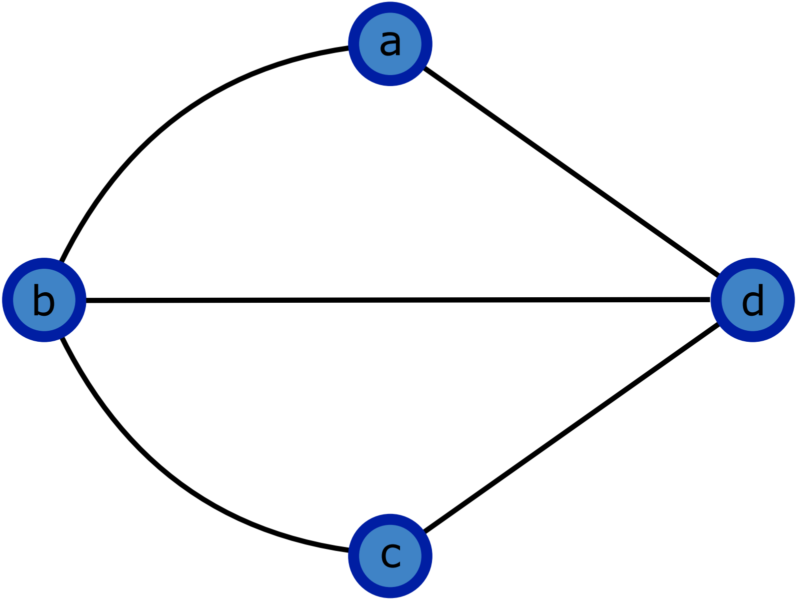

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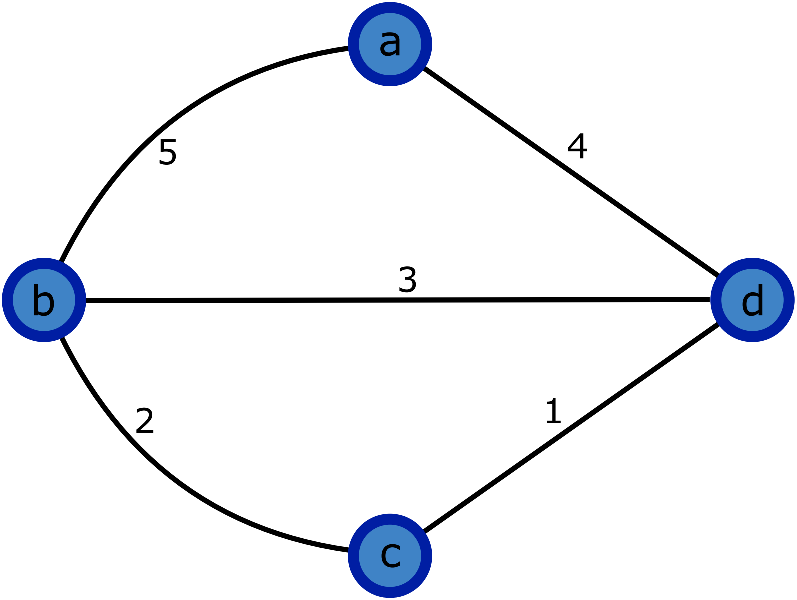

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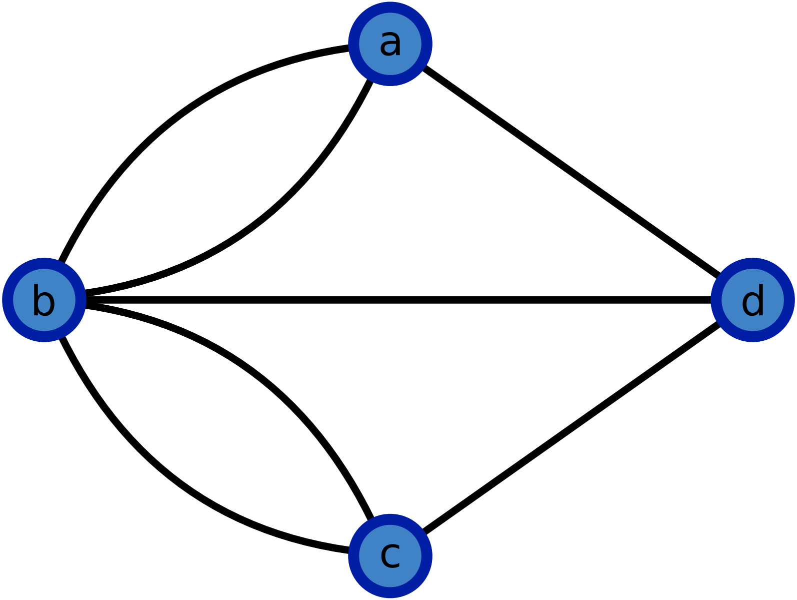


What Euler Proved

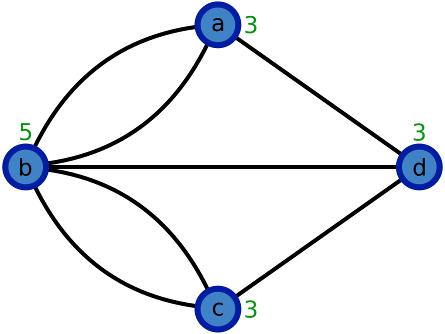
AD GEOMETRIAM SITVS PERTINENTIS. 139

§. 19. Praeterea ſi duo tantum numeri litteris A, B, C etc. adſcripti fuerint impares, reliqui vero omnes pares, tum ſemper deſideratus tranſitus ſuccedet, ſi modo curſus ex regione ad quam pontium impar numerus tendit incipiatur. Si enim pares numeri biſecentur atque etiam impares vnitate aucti, vti praeceptum eſt, ſumma harum medietatum vnitate erit maior quam numerus pontium, ideoque aequalis ipſi numero praefixo. Ex hocque porro perſpicitur, ſi quatuor vel ſex vel octo etc. fuerint numeri impares in ſecunda columna, tum ſummam numerorum tertiae columnae maiorem fore numero praefixo, eumque excedere vel vnitate, vel binario vel ternario etc. et idcirco tranſitus fieri nequit.

§. 20. Caſu ergo quocunque propoſito ſtatim facillime poterit cognoſci, vtrum tranſitus per omnes pontes ſemel inſtitui queat an non, ope huius regulae. Si fuerint plures duabus regiones, ad quas ducentium pontium numerus eſt impar, tum certo affirmari poteſt, talem tranſitum non dari. Si autem ad duas tantum regiones ducentium pontium numerus eſt impar, tunc tranſitus fieri poterit, ſi modo curſus in altera harum regionum incipiatur. Si denique nulla omnino fuerit regio, ad quam pontes numero impares conducant, tum tranſitus deſiderato modo inſtitui poterit, in quacunque regione ambulandi initium ponatur. Hac igitur data regula problemati propoſito pleniſſime ſatisfit.

S 2 §. 21.

What Euler Proved (in English)



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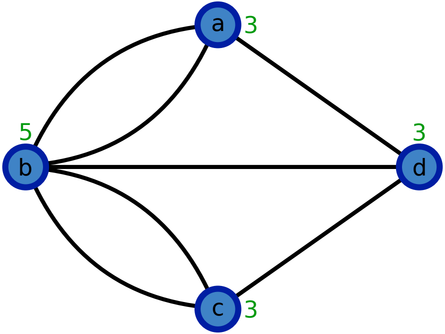

- *Degree $d(v)$ of a node v* is the number of edges incident on it.
- Euler's conclusion:
  1. If there are more than two nodes with odd degree, then the graph has no Eulerian tour.
  2. If exactly two nodes have odd degree, then there is tour that starts at one of these nodes and ends at the other node.
  3. If all nodes have even degree, then there exists a tour starting at any node.

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- What about constructing such a tour if it exists?

What Didn't Euler Prove?

140 *SOLVTIO PROBLEMATIS AD GEOM. &c.*

§. 21. Quando autem inuentum fuerit talem tranſitum inſtitui poſſe, quaeſtio ſupereſt quomodo curſus ſit dirigendus. Pro hoc ſequenti vtor regula; tollantur cogitatione quoties fieri poteſt, bini pontes, qui ex vna regione in aliam ducunt, quo pacto pontium numerus vehementer plerumque diminuetur, tum quaeratur, quod facile fiet, curſus deſideratus per pontes reliquos, quo inuento pontes cogitatione ſublati hunc ipſum curſum non multum turbabunt, id quod paululum attendenti ſtatim patebit; neque opus eſſe iudico plura ad curſus reipſa formandos praecipere.

THEO-

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- What about constructing such a tour if it exists?
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  - Method to accomplish this was trivial, and Euler did not want to spend a great deal of time on it.

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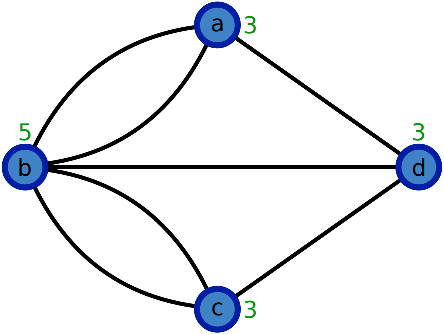
- Hierholzer provided an algorithm.

Hierholzer's Algorithm



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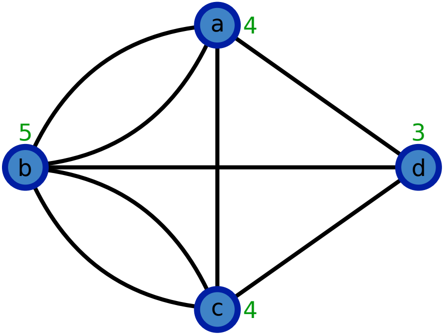

- If there are two nodes in G with odd degree, call them s and t.
- Otherwise, let s be any node in G.

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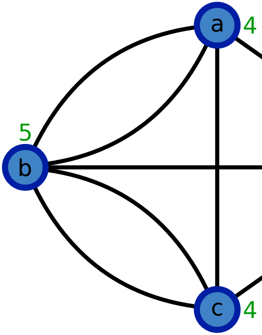

- If there are two nodes in G with odd degree, call them s and t.
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$u \leftarrow s$ #u denotes the currently-visited node.
while $d(u) > 0$ **do**
  Output u.
  Let v be a neighbour of u.
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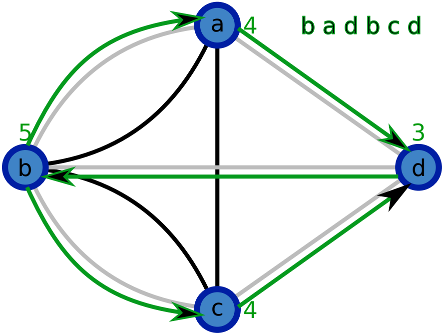
end while

Hierholzer's Algorithm



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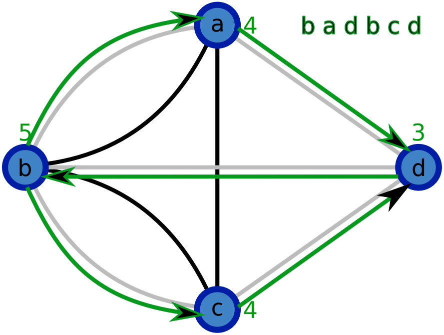
Properties of Heilholzer's Algorithm



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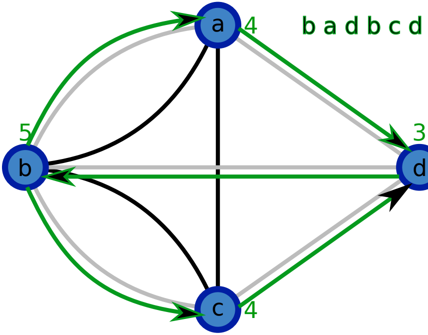

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- If it terminates, what can we say about node u at termination?
- Will all edges of G have been traversed upon termination?

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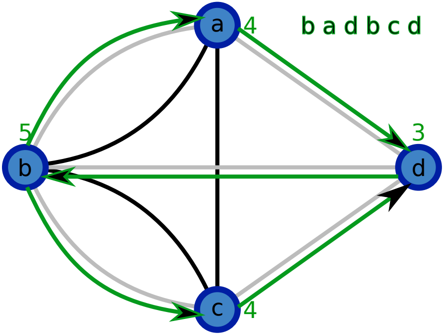

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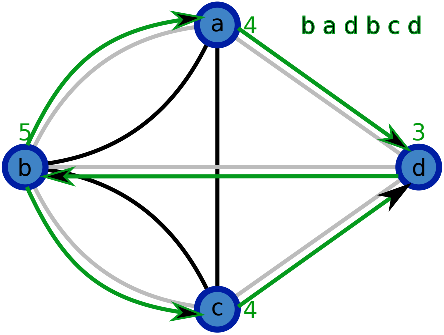
Properties of Heilholzer's Algorithm



$u \leftarrow s$
while $d(u) > 0$ **do**
  Output u.
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end while

- Will the algorithm terminate? Yes, because we traverse a new edge in each iteration.
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  - If G had no nodes of odd degree, then $u = s$.
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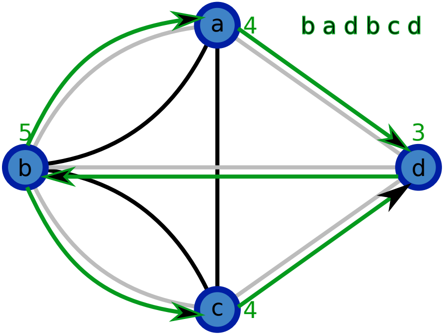

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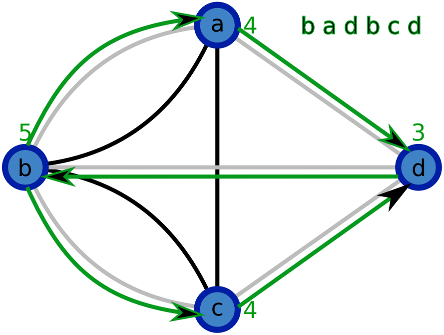

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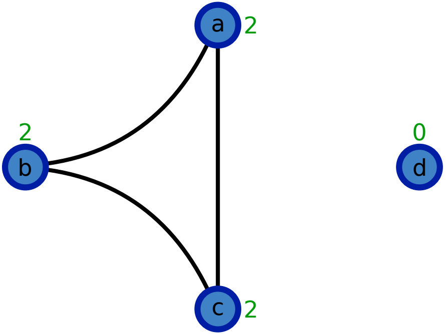

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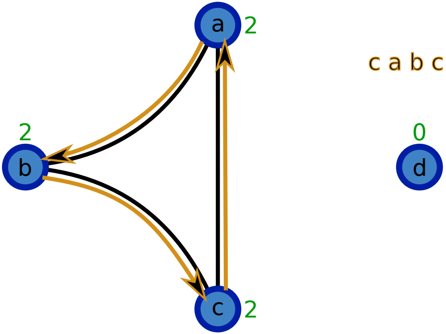

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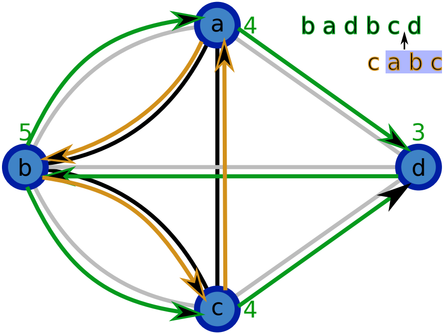

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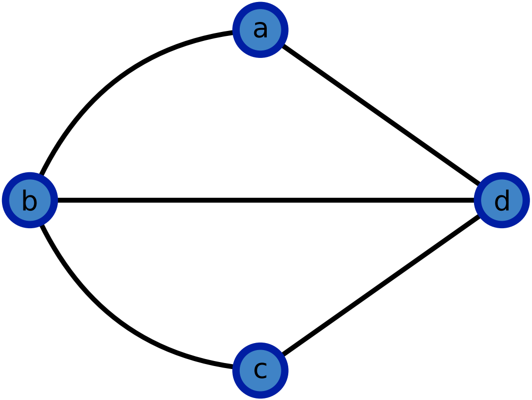
- Algorithm's running time is $O(|V| + |E|)$, i.e., linear in the size of G.

Visiting Nodes Rather than Edges



EULERIAN TOUR

Given an undirected graph $G(V, E)$,

construct an *Eulerian tour*, i.e., a path in G that traverses each edge in E exactly once, if such a tour exists.

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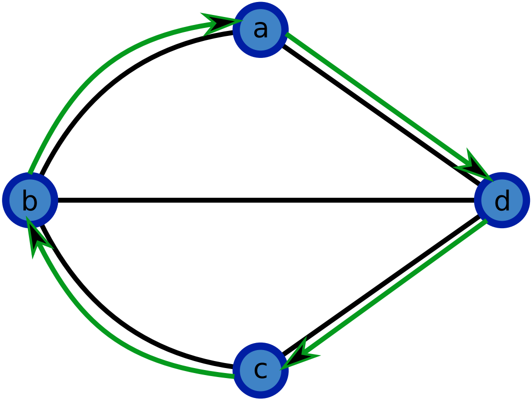

HAMILTONIAN CYCLE

Given an undirected graph $G(V, E)$,

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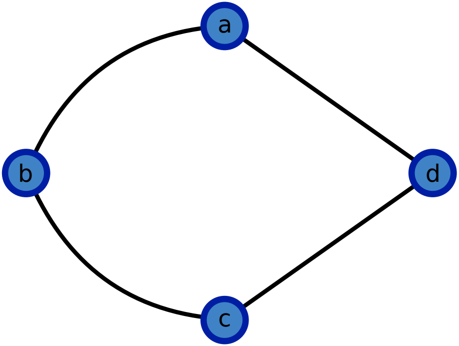
Conditions for Existence of Hamiltonian Cycle



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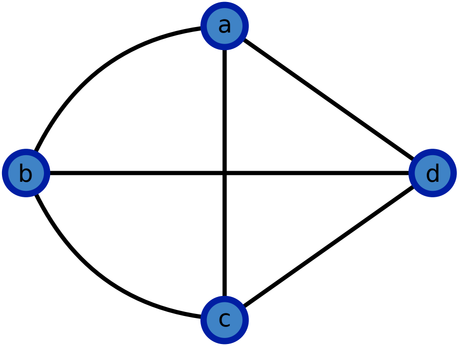

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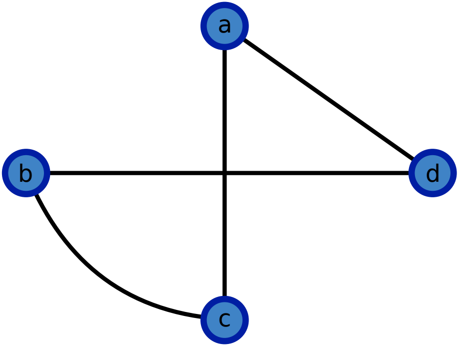

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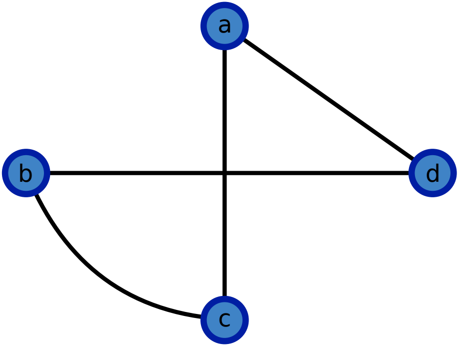

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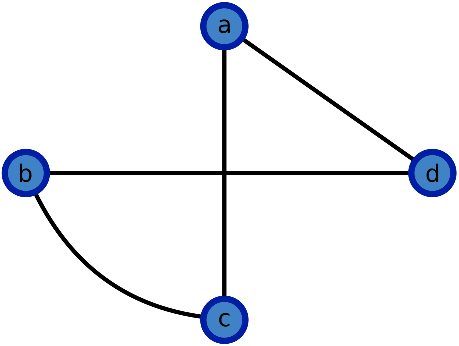

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  - two disconnected nodes with sum of degrees $\geq n$ (Ore, 1952).

Status of Hamiltonian Cycle Problem

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  - Dynamic programming: running time of $O(n^2 2^n)$ (Held and Karp 1962).
  - Fastest known algorithm runs in time $O(1.657^n)$ (Björklund 2010).