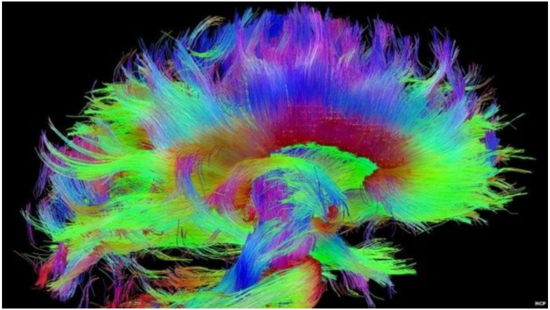
CS 4884: Connectivity Matrices and Node Degrees

T. M. Murali

February 1, 2022



Definition of an Undirected Graph

- *Weighted, undirected graph* $G = (V, E, w)$:
  - set V of nodes.
  - set E of edges.
    - ★ Each element of E is an unordered pair of nodes.
    - ★ Exactly one edge between any pair of nodes (G is not a multigraph).
    - ★ G contains no self loops, i.e., edges of the form (u, u).
  - Each edge (u, v) in E has a weight $w(u, v) \in \mathbb{R}$
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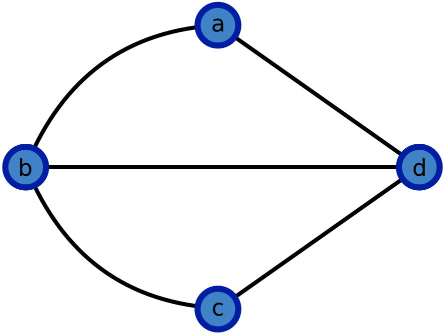
    - ★ G is *unweighted* if all edges have weight 1.



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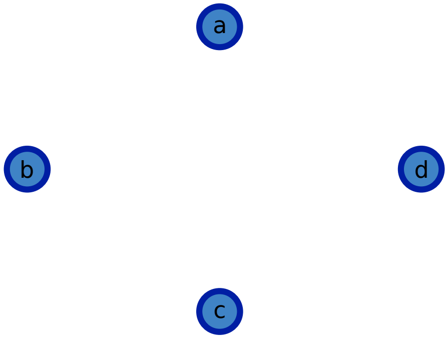

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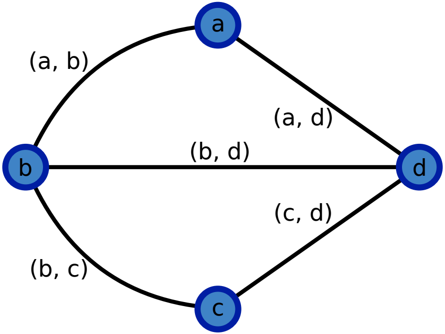

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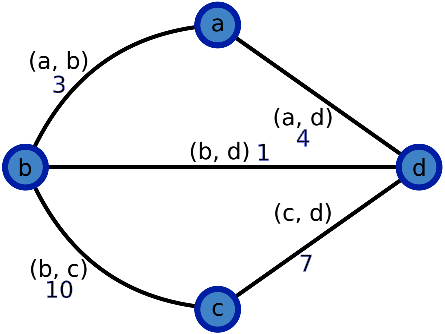

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Mesoscale		
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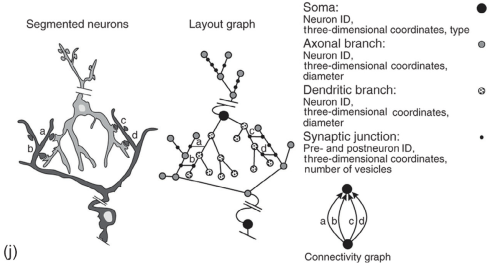

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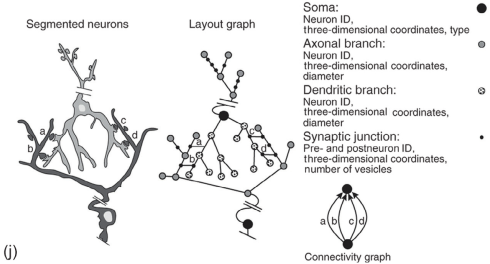

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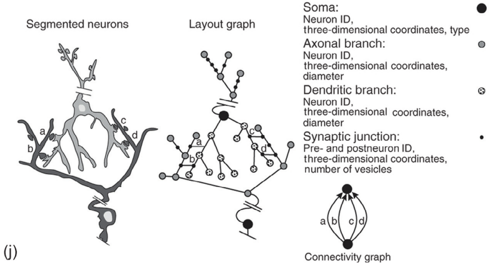

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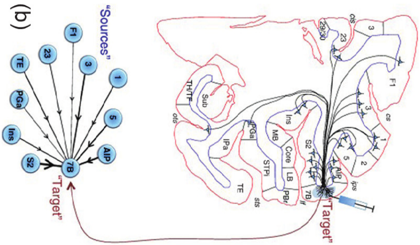

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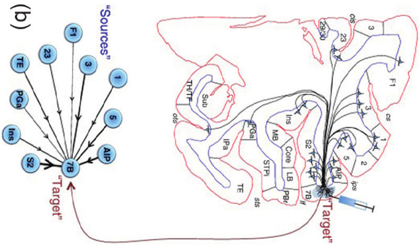

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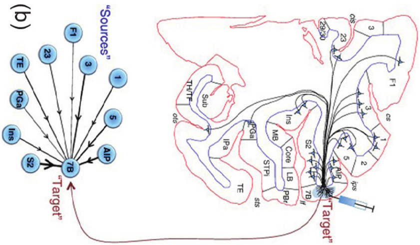

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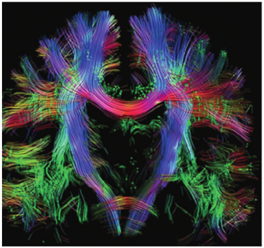

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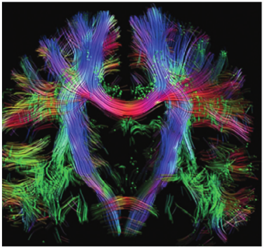

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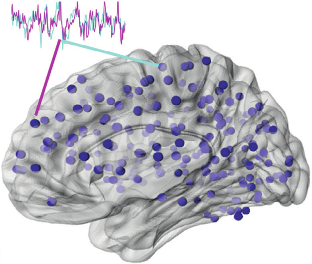

(c)

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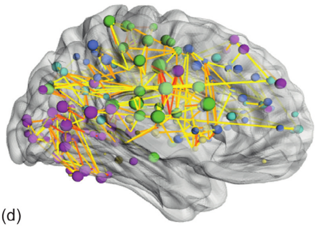

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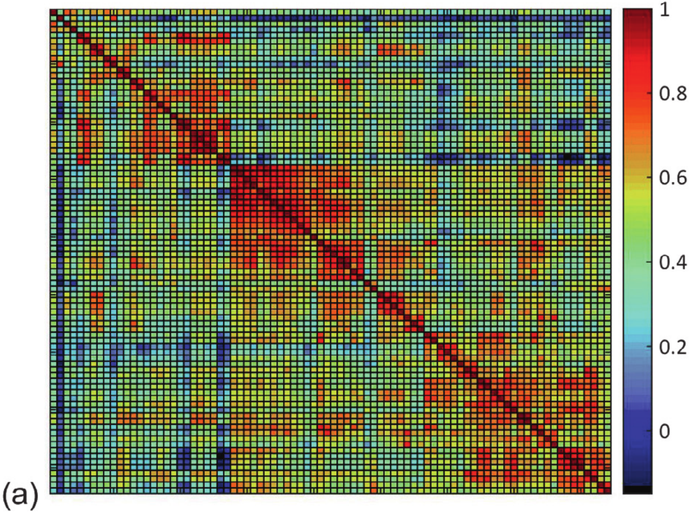
Thresholding and Binarisation



Human functional connectivity matrix from fMRI data.
Every element has a nonzero value.

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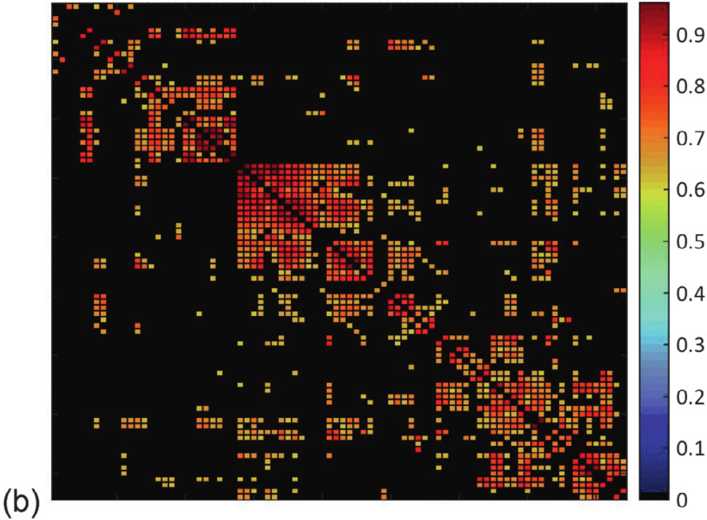

Matrix after thresholding to retain only the 20% strongest weights.

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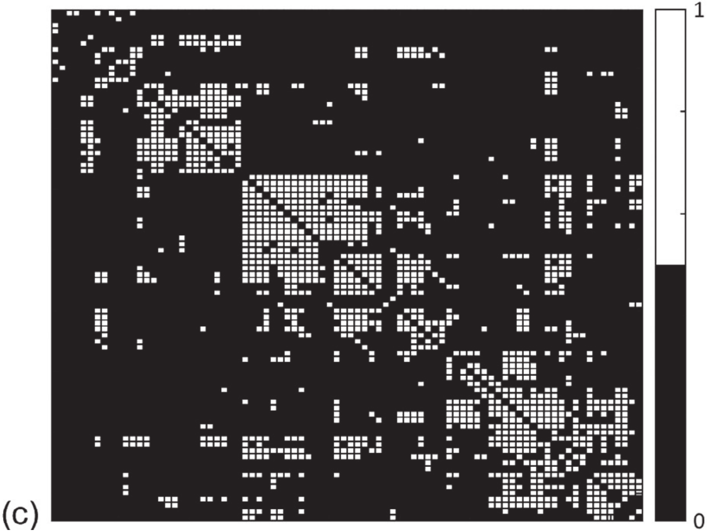

Matrix after thresholding and binarisation.

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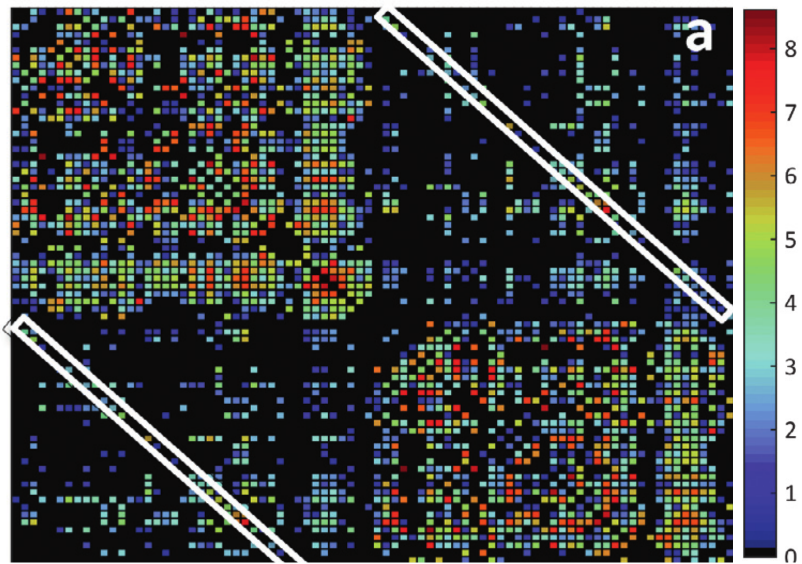
- We can modify these ideas for directed graphs.

Visualising Matrices



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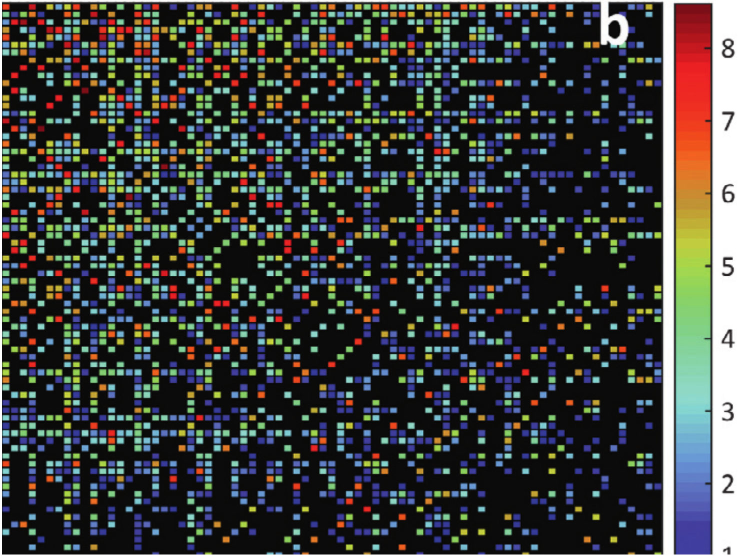

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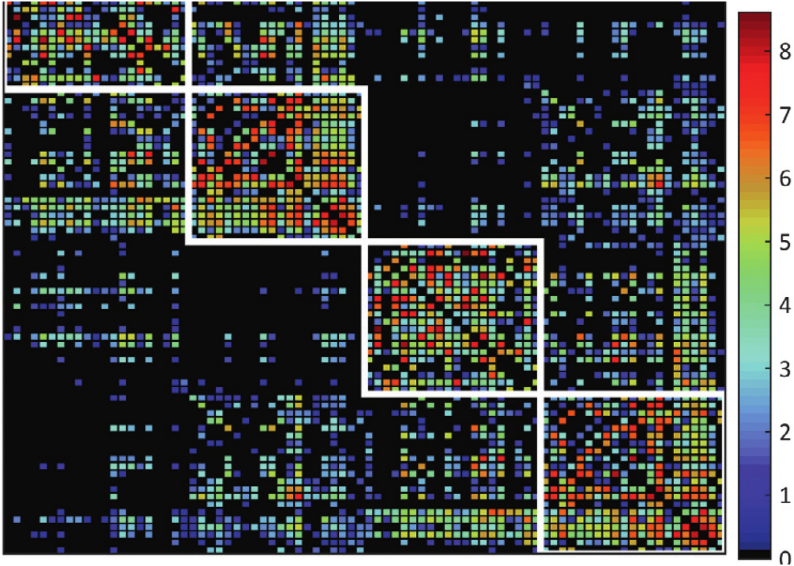
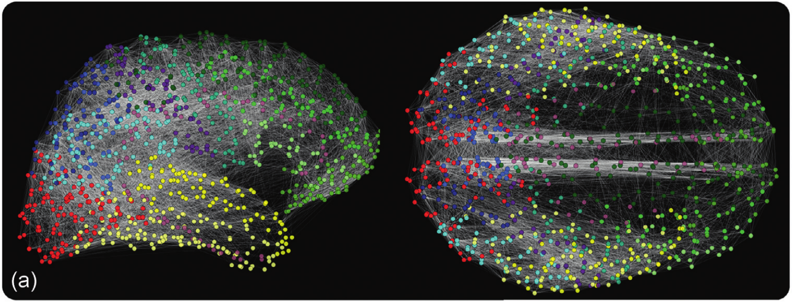
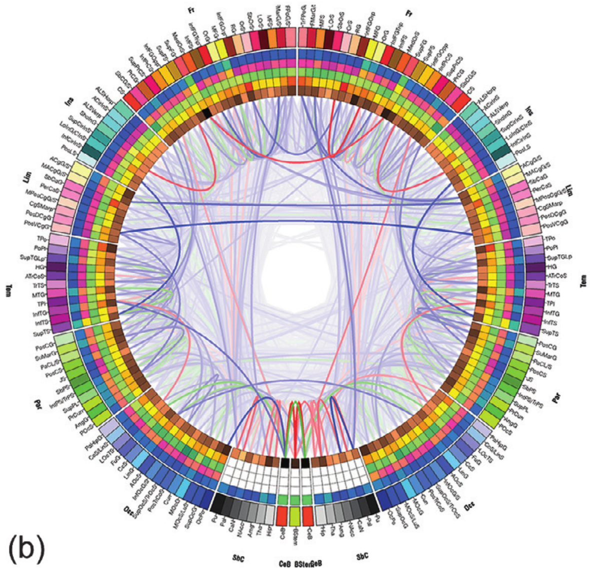


Anatomical Projection

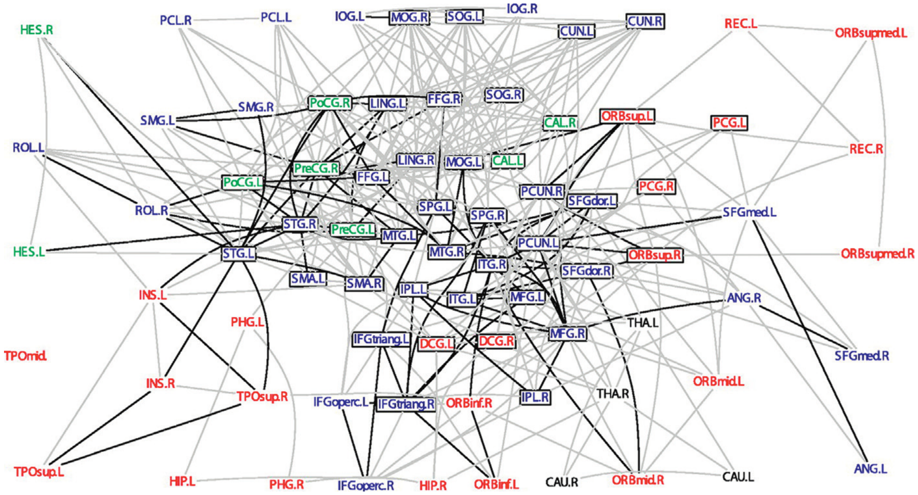


Circular Layout

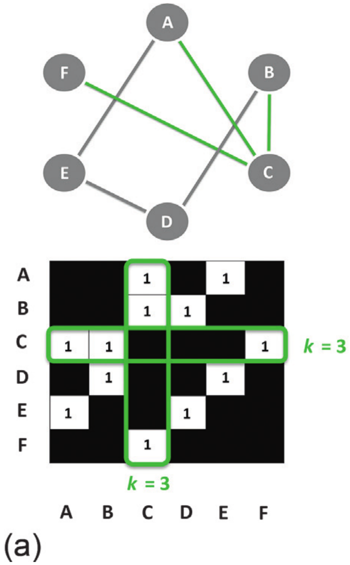


Force-Directed Layout

Spring-Embedded Layout



Node Degree



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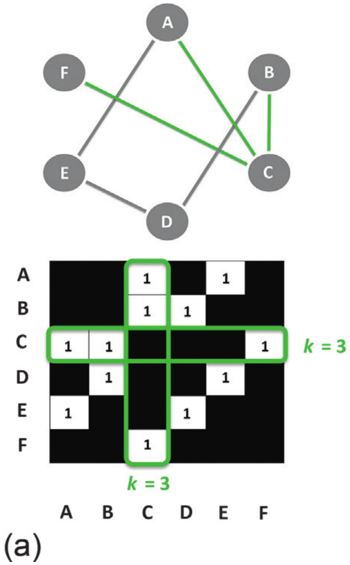

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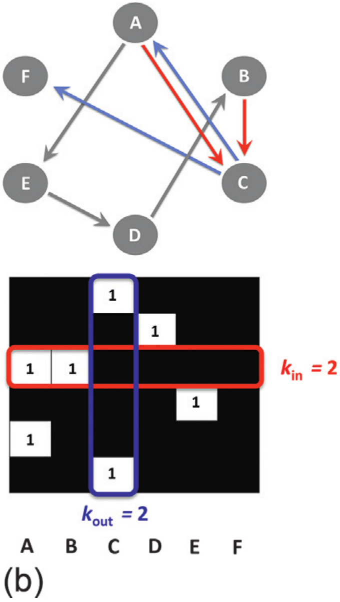

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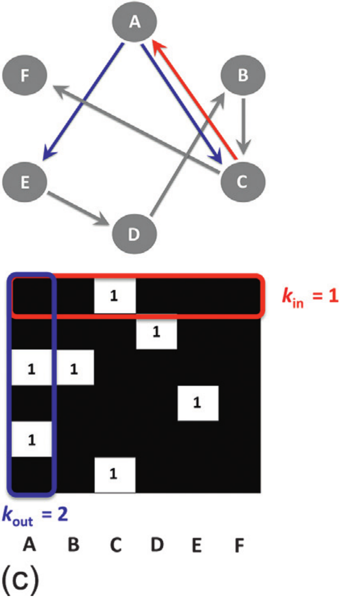

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$$\sum_{k \geq 0} kp(k) = \frac{1}{n} \sum_{k \geq 0} kn(k) = \frac{1}{n} \sum_{v \in V} d(v) = \frac{2m}{n}$$

Degree Distributions of Real-World Networks

- Degree distributions of many real-world networks follow a power law (Barabasi and Albert, 1999).

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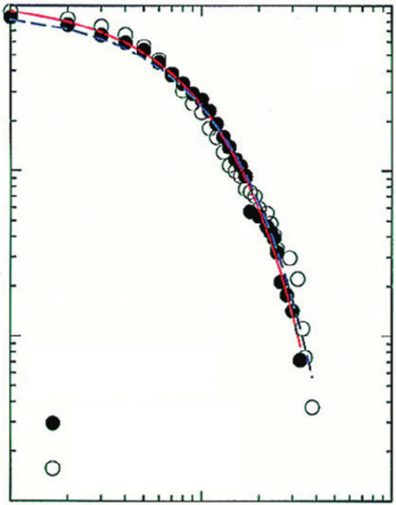
- Broad-scale networks show power law behaviour over limited range of degree, e.g.,

$$p(k) = \Pr\{\text{degree } = k\} \sim k^{-\gamma} e^{-k/k_c}$$

Example of Degree Distributions of Brain Networks



In/out degree distributions of the *C. elegans* neuronal network

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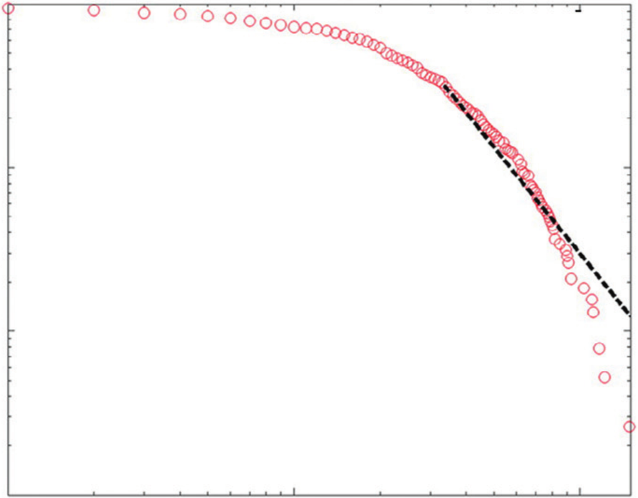

Degree distribution of a 383-region macaque connectome collated from published tract-tracing studies.

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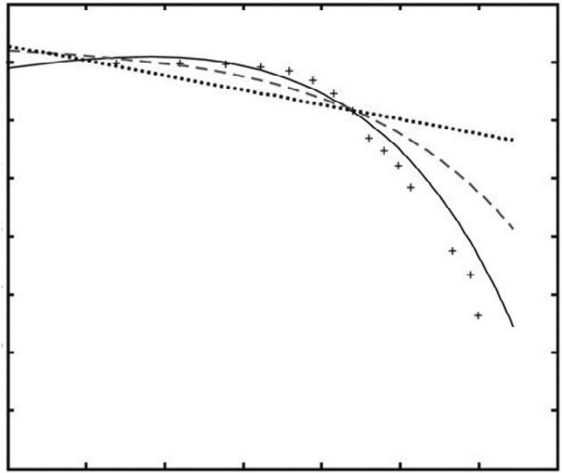

Degree distribution of a 78-region human cortical connectome from diffusion MRI (+: data, solid: exponentially truncated power law, dashed: exponential, dotted: power law).