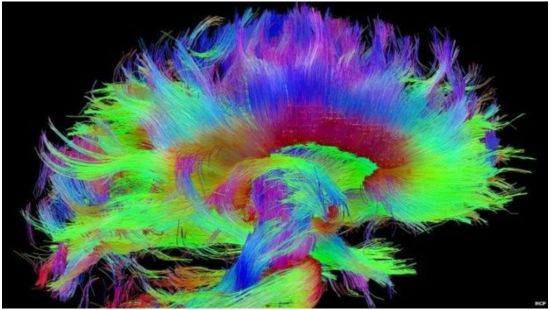
CS 4884: Erdös-Renyi and Small World Networks

T. M. Murali

February 3 and 8, 2022



Collective dynamics of 'small-world' networks

Duncan J. Watts & Steven H. Strogatz

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Specifically, we require $n \gg k \gg \ln(n) \gg 1$, *where* $k \gg \ln(n)$ *guarantees that a random graph will be connected.*

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- Question is under-specified. There are many approaches:
  1. Idea 1: From the set of all graphs of n nodes, pick one uniformly at random.
  2. Idea 2: Specify the number of edges m. From the set of all graphs of n nodes and m edges, pick one uniformly at random.
  3. Idea 3: Specify a probability $0 \leq p \leq 1$. For every pair of nodes, add an edge between the nodes with probability p.

Idea 1 for Creating Random Graphs

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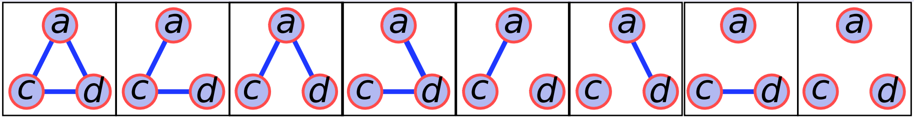

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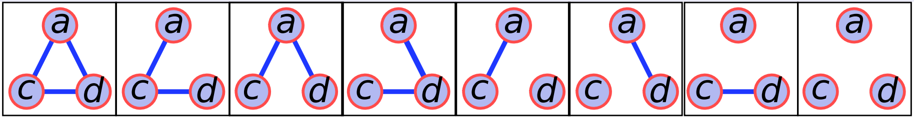

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  - To make a graph, we have two options for each edge: include it or exclude it.
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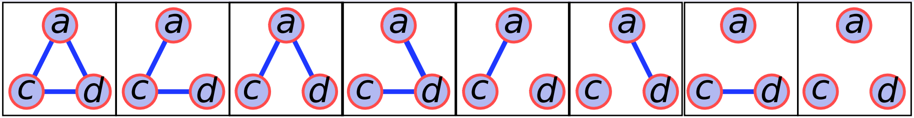

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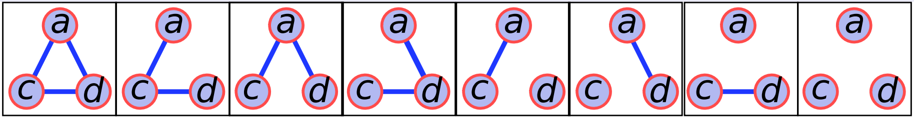

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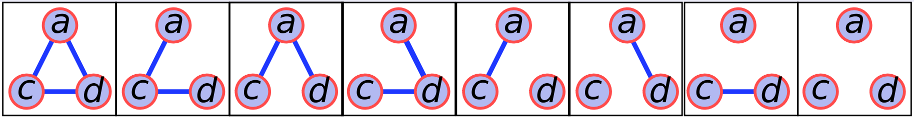

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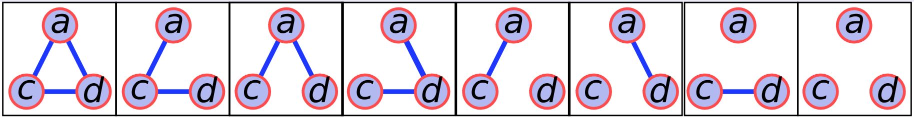

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- For every pair of nodes, add an edge with probability $1/2$. Running time is $O(n^2)$.

Properties of Random Graphs Created by Idea 1

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- What is the expected degree of a node?

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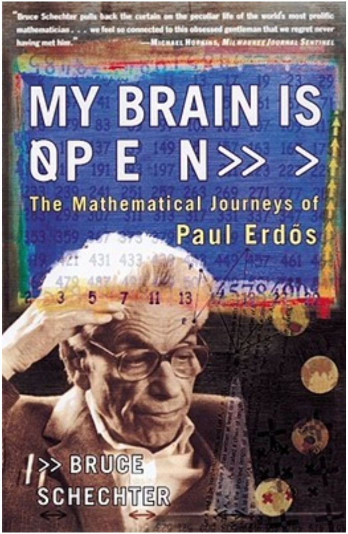
- What is the expected degree of a node? $(n-1)/2$.
- What is the expected number of edges in the graph? $n(n-1)/4$.
- On average, these graphs are very dense.

Erdős-Rényi Graphs



A mathematician is a device for turning coffee into theorems.

Erdős-Rényi Graphs

Idea 3: Specify a probability $0 \leq p \leq 1$.
For every pair of nodes, add an edge between the nodes with probability p.

- Series of papers in the 1960s setting the foundation of random graph theory.
- Framework for generating a random graph.
- $G(n, p)$: an undirected, unweighted graph (family) with n nodes.
- To generate a graph in $G(n, p)$:
  - For each pair (u, v) of $\binom{n}{2}$ node pairs, connect u and v by an edge with probability p.

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  - Generate a random number x between 0 and 1 under the uniform distribution. If $x \leq p$, then "do something", else "do the other thing".

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Degrees and Connectivity in Erdős-Rényi Graphs

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- How many edges does this graph have on average?

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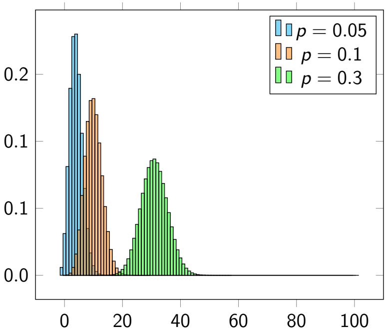

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- The expected degree of a node is $(n-1)p$.
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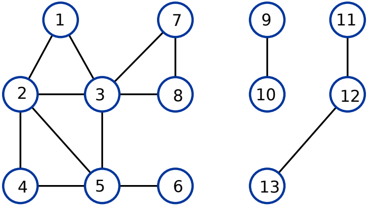
Guarantee that a Random Graph is Connected

Specifically, we require $n \gg k \gg \ln(n) \gg 1$, *where* $k \gg ln(n)$ *guarantees that a random graph will be connected.*



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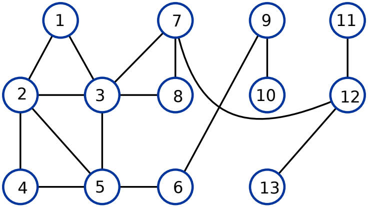

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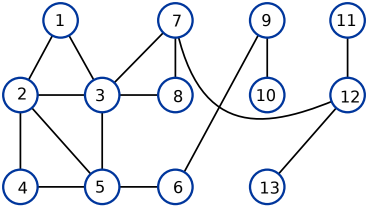

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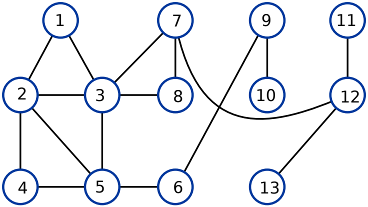

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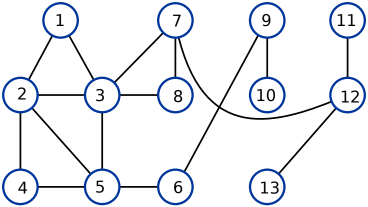

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The evolution of the $G(n, p)$ random graph (Video, 4 min 51 sec)

Phase Transitions

Value of p	Property of $G(n, p)$
$p = 0$	
$p < \frac{(1-\varepsilon)}{n}$ $p > \frac{(1+\varepsilon)}{n}$	
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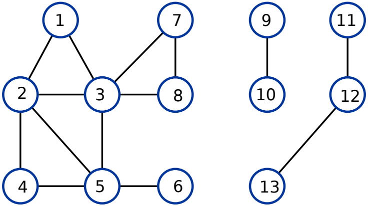
Statements hold with high probability, e.g., if $p > \frac{(1+\varepsilon)\ln n}{n}$, then

$$\Pr\{G(n, p) \text{ is not connected }\} \approx \frac{1}{e^{n^{\varepsilon}}}.$$

Clustering Coefficient



- Measures the extent of clusters/cliques around a node, on average.
- *Clustering coefficient $c(v)$* for a node v is the fraction of pairs of its neighbours that are themselves connected.
- *Clustering coefficient $c(G)$* of a graph G is the average of the clustering coefficients of its nodes.
  - Note that I am using lowercase c (since c is a number), whereas the paper uses uppercase C.
  - Technically, $c(v)$ should have the graph G as an argument, but we will be sloppy and ignore it.

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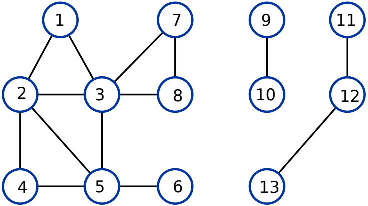

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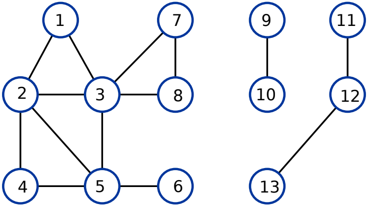

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Clustering Coefficient of an Erdős-Rényi Graph

- Assume $p > \frac{(1+\varepsilon)\ln n}{n}$.
- We know that the average shortest path length in $G(n, p)$ is $\approx \frac{\ln n}{\ln(1+\varepsilon)}$.

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  - What is the probability that two neighbours v and w are connected? p!
  - Hence, the clustering coefficient of $G(n, p)$ is $p < 1$.

Milgram's Experiment

It's a small world! (Video, 1 min 36 sec)

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Criticisms

- Overestimates path lengths.
- Underestimates path lengths.

Milgram's Experiment

It's a small world! (Video, 1 min 36 sec)

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Burning question

How do networks with small average shortest path length arise?

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- Consider two measures for a graph G:
  - $l(G)$, the average shortest path length in G.
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- *Regular ring graph*: n nodes in a ring, each node connected to the next $k/2$ nodes appearing in clockwise order around the ring.

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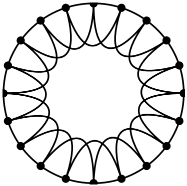

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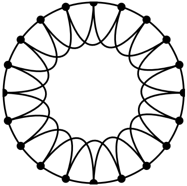

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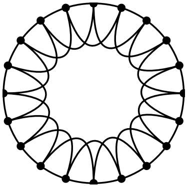
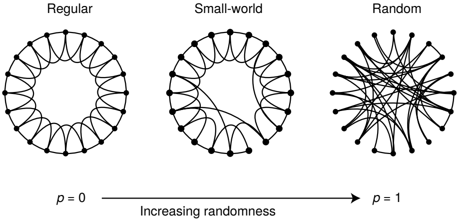


- Real world networks have small average shortest path lengths (like $G(n, p)$) but large clustering coefficients (like ring graph).

Watts-Strogatz Model



- Three parameters: n, number of nodes; k: degree of each node; p: rewiring probability. This p is different from the p in E-R graphs.
- Rewire regular ring graph in $k/2$ rounds. In round j,
  1. For each node i, consider edge $(i, i+j)$.
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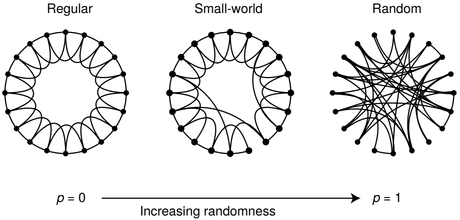

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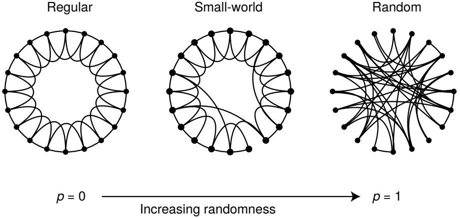
l and c for Watts-Strogatz Graphs



$l(p)$: average shortest path length for ring graph rewired with prob. p.
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$$l(0) =$$

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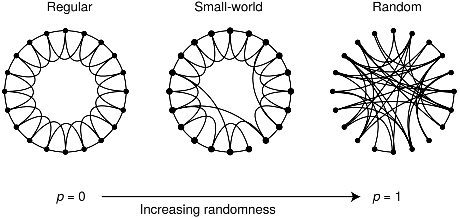

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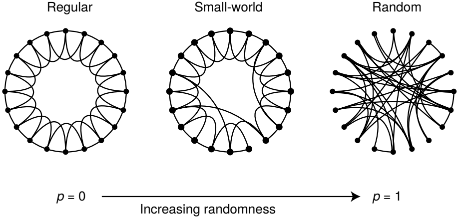

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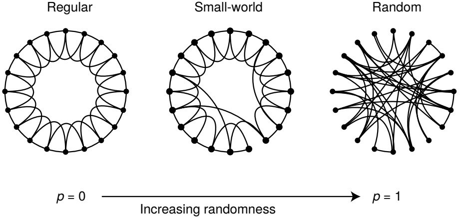

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Ring lattice is large-world and highly clustered.

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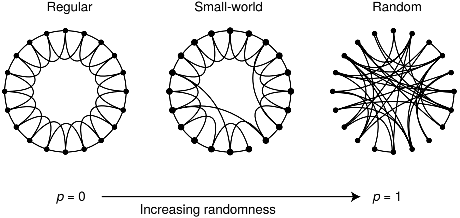

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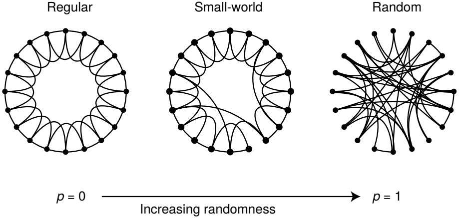

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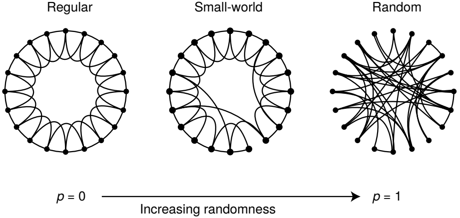

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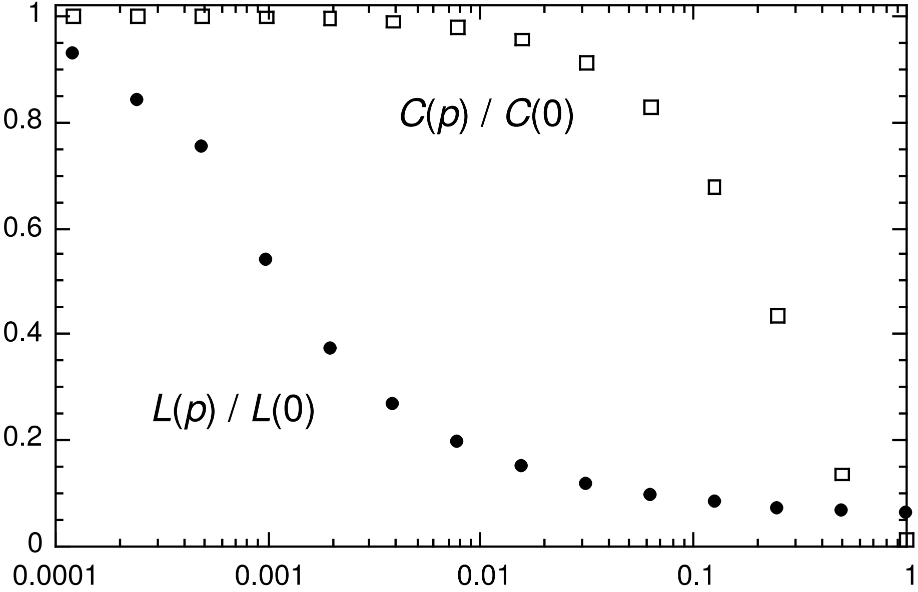
Random ring graph is small-world but poorly clustered.

Are there values of p for which $l(p)$ is small but $c(p)$ is large?

l and c for Watts-Strogatz Graphs



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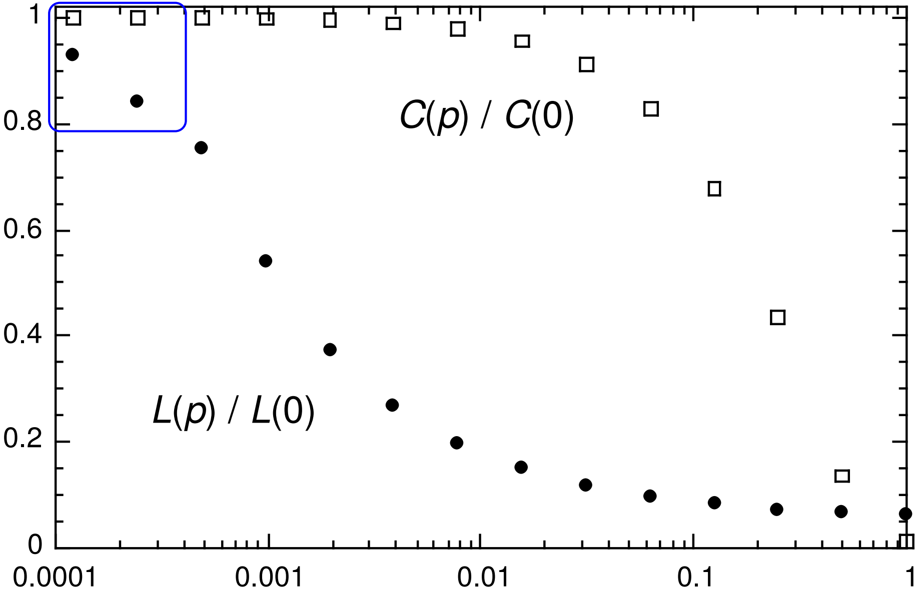

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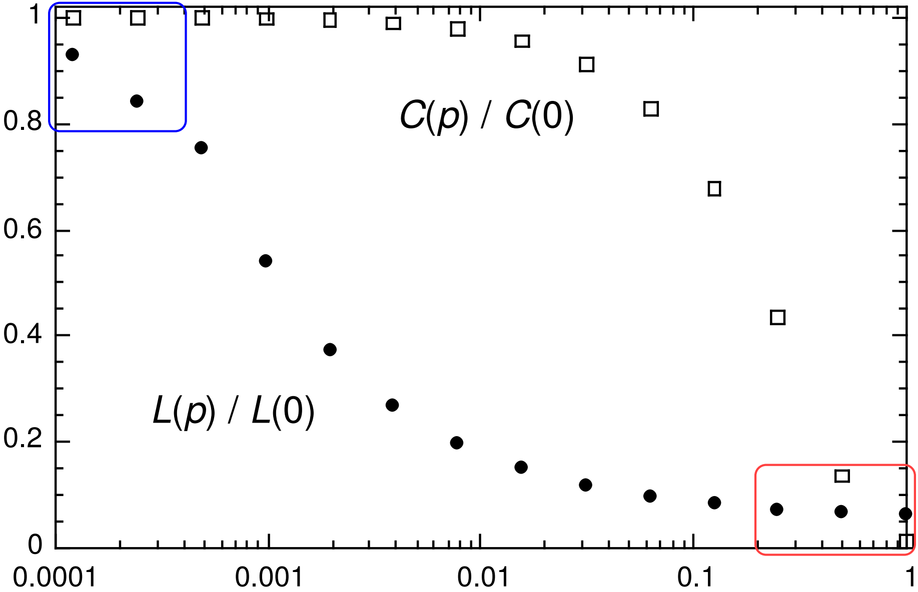

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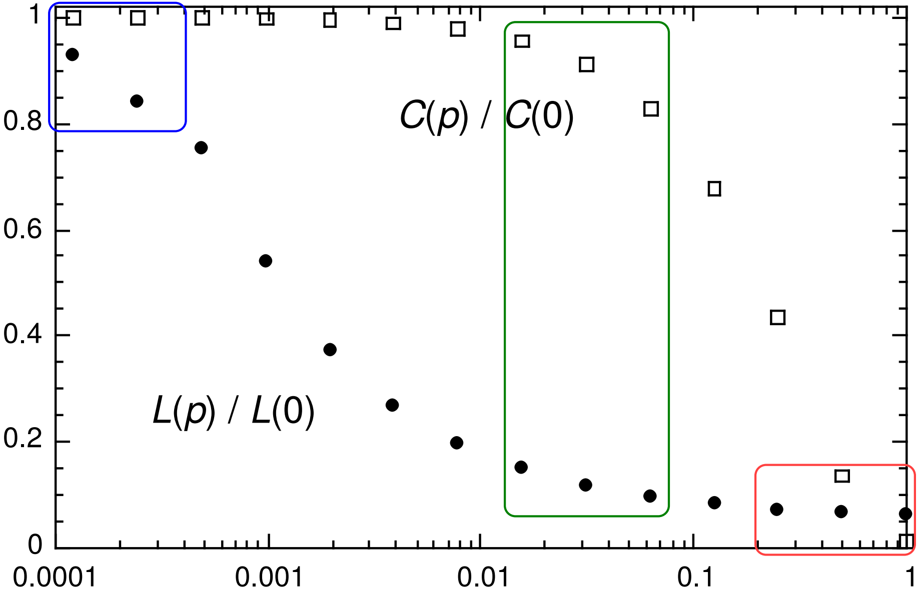

Observations

- $l(p)$ becomes small due to the addition of a small number of "long-range" edges.
- These short cuts connect nodes that would otherwise be very far apart.
- Non-linear effect on $l(p)$: Short cuts also contract the distance between neighbours of the connected nodes, their neighbours, and so on.

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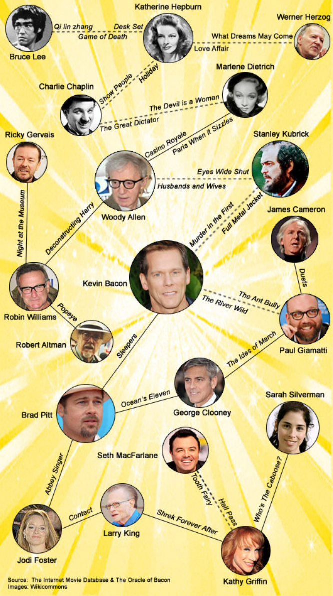
Do real-world networks have small l and large c?

The Science of Six Degrees of Separation (Video, 9 min 22 sec)

Actor Network



Node $\equiv$

Edge $\equiv$

Edge weight $\equiv$

$n =$

$m =$

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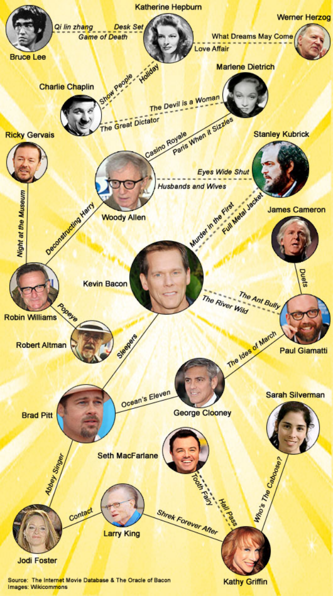
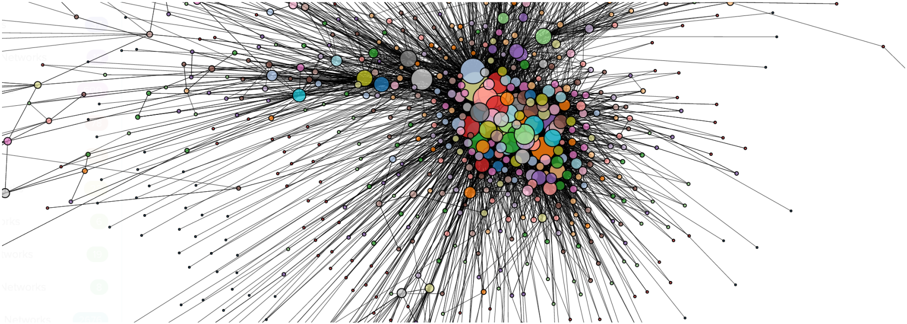


Node $\equiv$ Actor
Edge $\equiv$ Collaboration
Edge weight $\equiv 1$
$n = 225,226$
$m = (225,226 \times 61)/2 = 6,869,393$

Power Network



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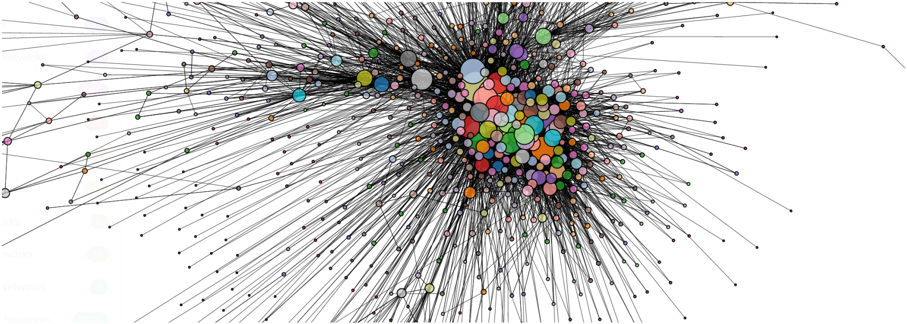


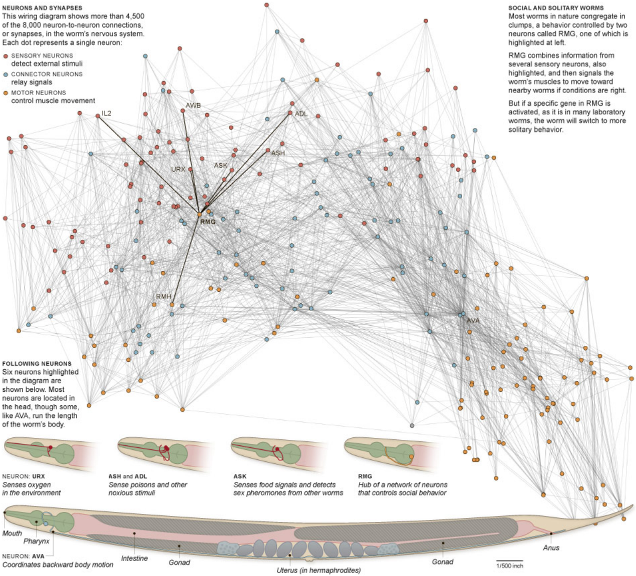
Node ≡ Generators, transformers, and substations

Edge ≡ High-voltage transmission line

Edge weight ≡ 1

$n = 4,941$
$m = (4,941 \times 2.67)/2 = 6,596$

C elegans connectome



Node $\equiv$

Edge $\equiv$

Edge weight $\equiv$

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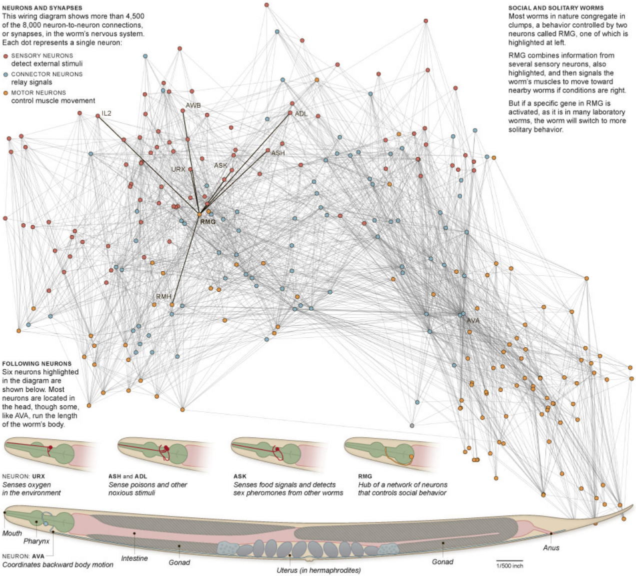

Node $\equiv$ Neuron

Edge $\equiv$ Synpase

Edge weight $\equiv 1$

$n = 282$

$m = (282 \times 14)/2 = 1974$

Real-world Networks are Small World

Table 1 Empirical examples of small-world networks

	L_{actual}	L_{random}	C_{actual}	C_{random}
Film actors	3.65	2.99	0.79	0.00027
Power grid	18.7	12.4	0.080	0.005
C. elegans	2.65	2.25	0.28	0.05

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The pattern in Nature's networks (Video, 3 min 25 sec)