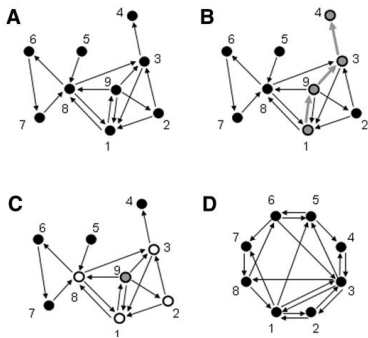


CS 4884: Small World of Brain Networks



T. M. Murali

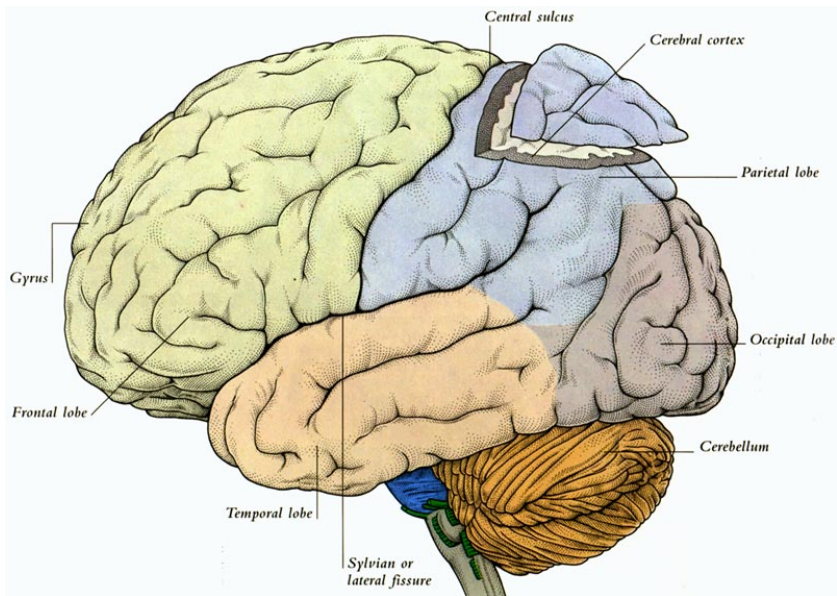
February 10, 2022

Motivation

- The Watts-Strogatz paper set off a storm of research.
- It has 38,000 citations. Even in 2004, it had more than 2,100 citations.
- The *C. elegans* neuronal network is small-world.

Do mammalian brain networks have the small world property?

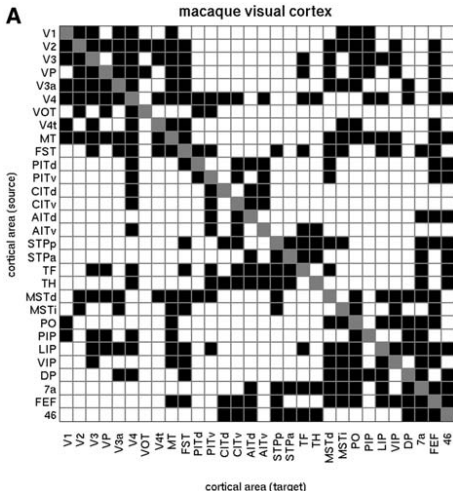
Visual and Cerebral Cortices



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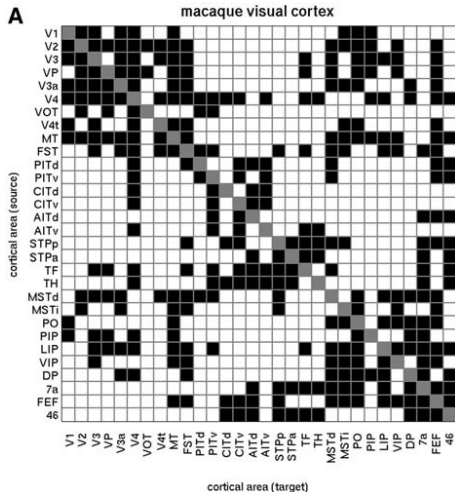


Datasets: Macaque Visual Cortex



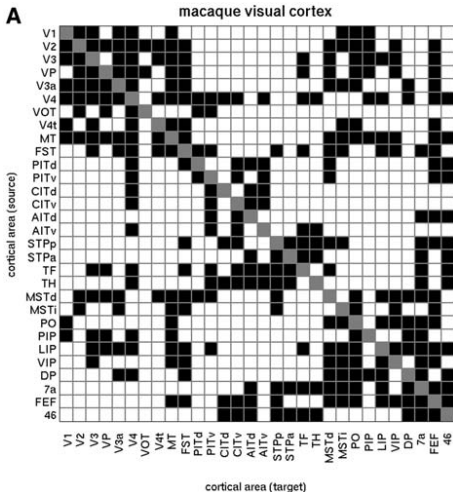
- Each row (*efferent*) and column (*afferent*)

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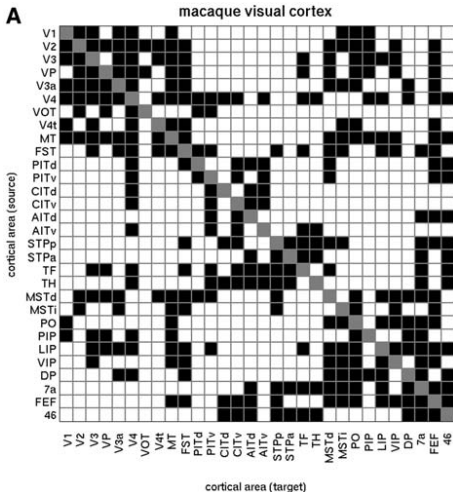
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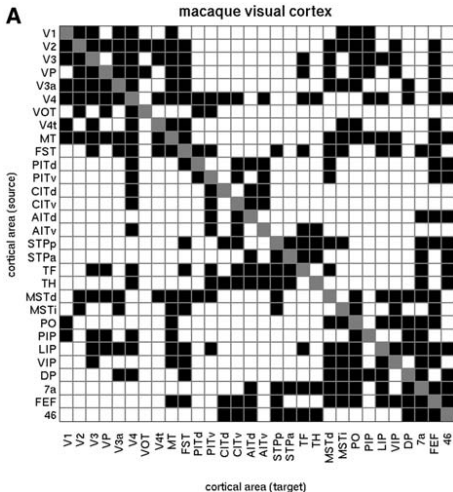
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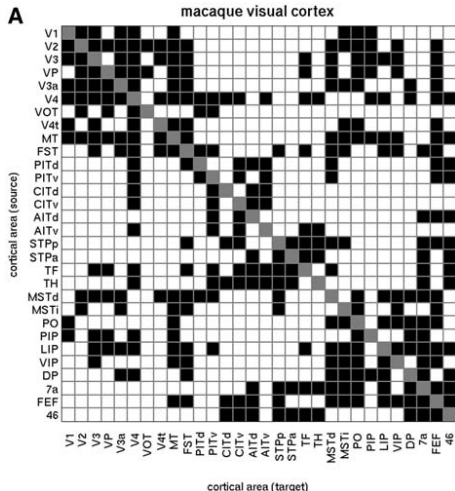
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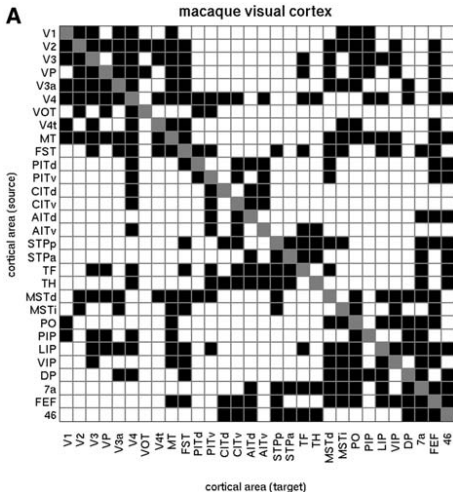
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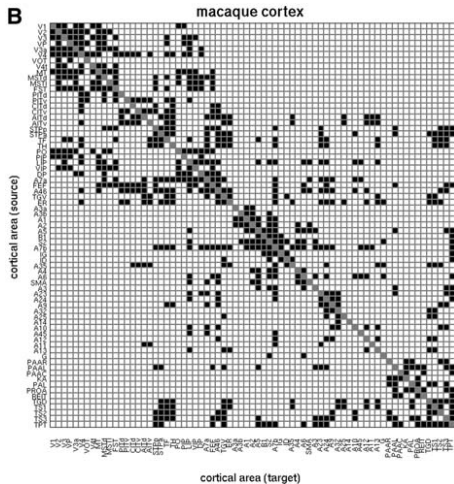
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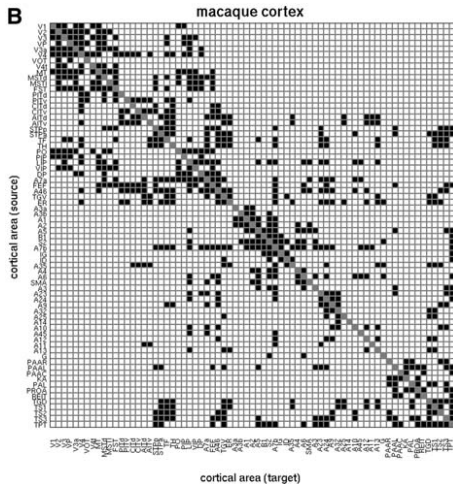
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- $n = 30$, $m = 311$. (The authors use N for the number of nodes and K for the number of edges.)

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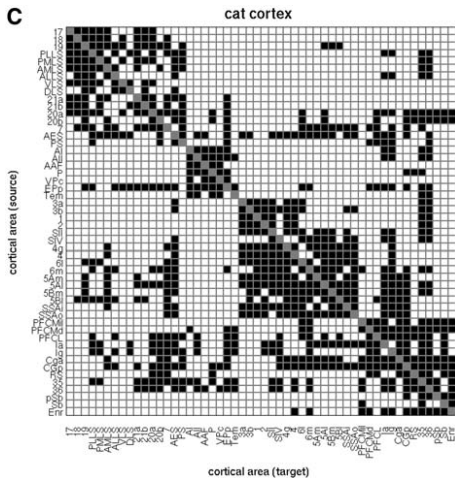
- $n = \dots, m = \dots$

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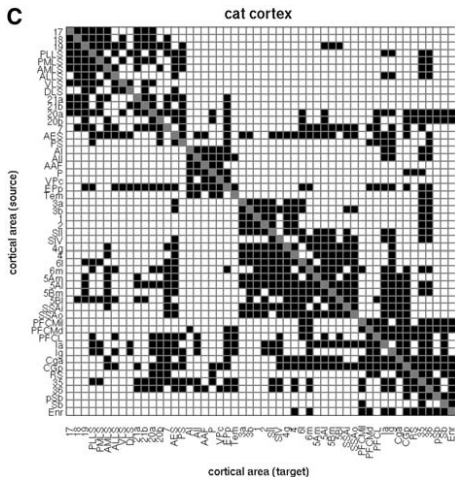
- $n = 71, m = 746$.
- What is the relation between this graph and the one for the macaque visual cortex? Most of the edges in the previous graph are in this one.

Datasets: Cat Cerebral Cortex



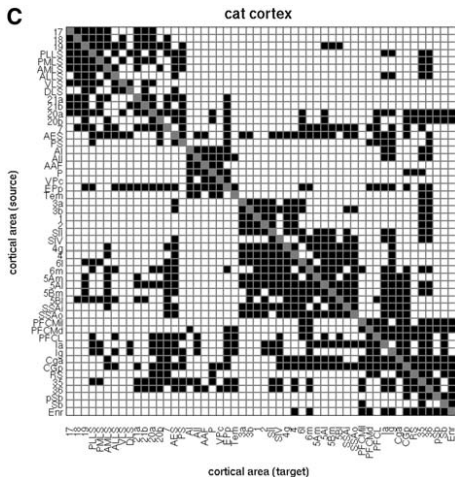
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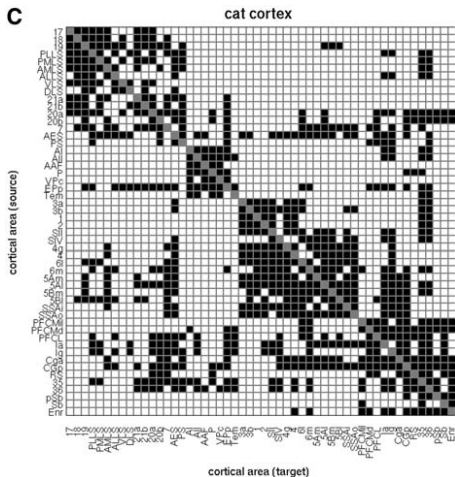
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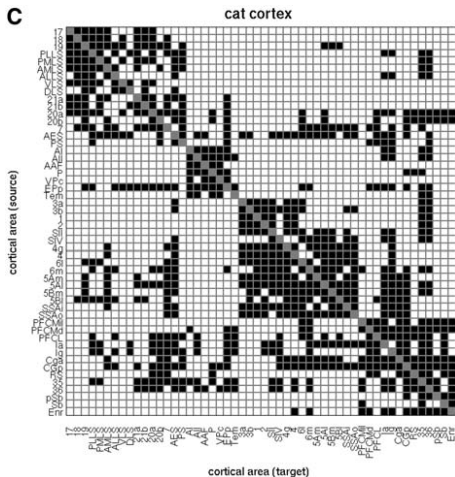
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- We will ignore density-based connectivity data sets.

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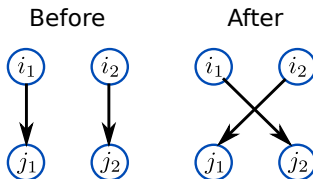
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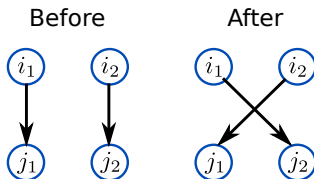
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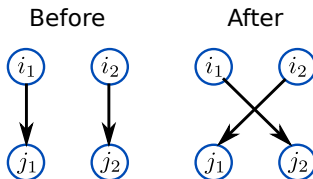
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- Degree-preserving lattice matrix:

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- Degree-preserving lattice matrix: Very poorly specified. Will need to read the code to understand.

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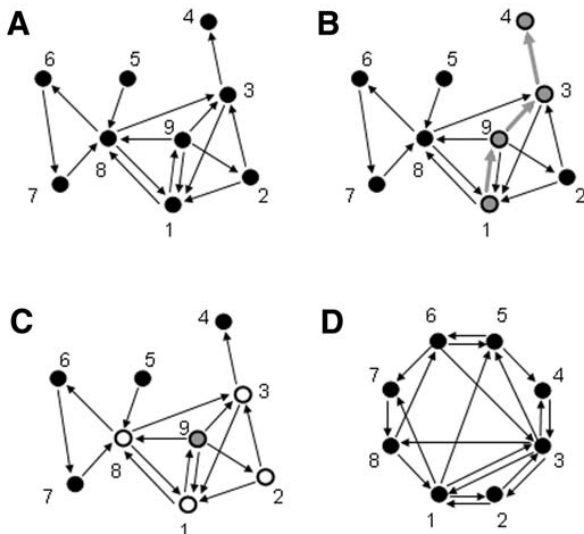
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 - ▶ Clustering coefficient, $c(v)$, called “cluster index” and $\gamma(v)$ in this paper.

Computing $l(G)$ and $c(G)$ for Directed Graphs



- Appropriately generalise definitions for undirected graphs.

Scaling $l(G)$ and $c(G)$

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- How do we interpret these quantities? A small world network will have small $l_{\text{scl}}(G)$ and large $c_{\text{scl}}(G)$.

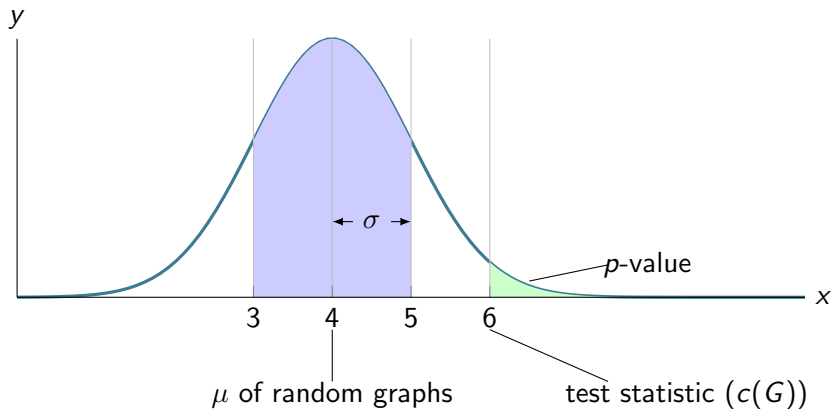
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Table 1. Path Length (λ , λ_{scl}) and Cluster Index (γ , γ_{scl}) for Large-Scale Connection Matrices of Cortico-Cortical Pathways

<i>Topology</i>	λ	γ	λ_{scl}	γ_{scl}
MVC	1.7256	0.5313	0.2188	0.5645
$R_{30,311}$	1.6680 (0.0038)*	0.3616 (0.0048) *		
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$Rio_{30,311}$	1.6880 (0.0033)*	0.4305 (0.0059) *		
$Lio_{30,311}$	1.8190 (0.0391)	0.6214 (0.0243)		
MC	2.3769	0.4614	0.1927	0.6117
$R_{71,746}$	2.0310 (0.0051)*	0.1497 (0.0030) *		
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CC	1.8114	0.5514	0.2498	0.6292
$R_{52,820}$	1.7014 (0.0013)*	0.3103 (0.0026) *		
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Statistical Significance



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L _{52,820}	2.1418 (0.0024)*	0.6933 (0.0000) *	< 0.5	> 0.5
Rio _{52,820}	1.7217 (0.0037)*	0.4023 (0.0030) *		
Lio _{52,820}	1.8570 (0.0283)	0.5893 (0.0172)		

Measures for reference cases represent means and standard deviations (in brackets) for 10 exemplars. Topologies: MVC = macaque visual cortex (Fig. 1A), MC = macaque cortex (Fig. 1B), CC = cat cortex (Fig. 1C), R_{N,K} = random, L_{N,K} = lattice, Rio_{N,K}, Lio_{N,K} = random, lattice matrices with in-degree and out-degree distribution preserved. Statistical significance for all comparisons between cortical matrices and random, lattice, Rio, or Lio matrices marked by "*" are $p < 0.001$, the remaining comparisons are $p < 0.05$.

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- **Caveats:**
 - ▶ p -values are likely to be underestimated. They should be estimated from empirical distributions built from many more random samples.
 - ▶ No indication of correction for testing multiple hypotheses.
 - ▶ No p -value associated with the scaled values of $l(G)$ and $c(G)$.

Computing Small-Worldness

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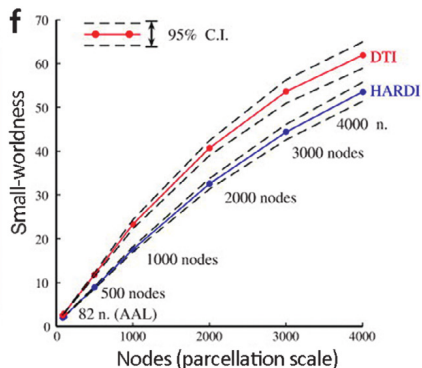
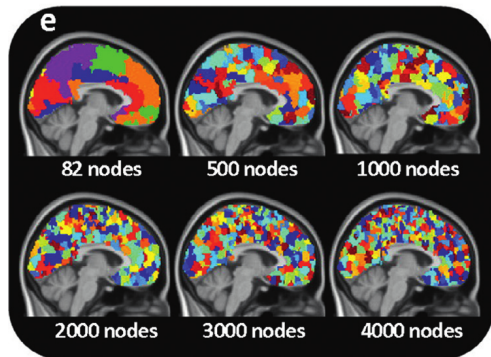
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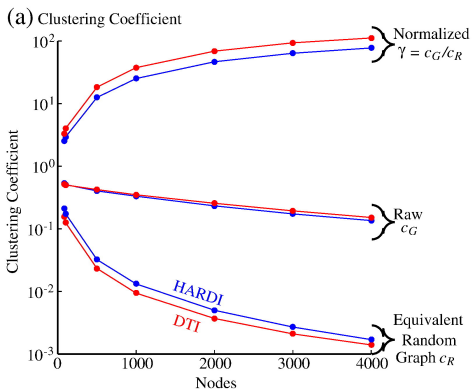
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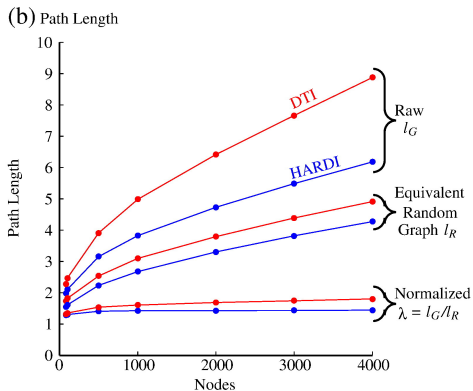
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- New tract-tracing techniques have yielded networks with a density of 0.66, with no small world features ([Markov et al. 2013](#)).
- These global measures obscure more meaningful variations occurring at the level of nodes or subgraphs.