Analysis of Algorithms

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What is Algorithm Analysis?

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- How do we put this notion on a concrete footing?
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Goal

Develop algorithms that provably run quickly and use low amounts of space.

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- *Input size* = number of elements in the input. *Values* in the input do not matter, except for specific algorithms.
- Assume all elementary operations take unit time: assignment, arithmetic on a fixed-size number, comparisons, array lookup, following a pointer, etc.

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- An algorithm has a *polynomial* running time if there exist constants $c > 0$ and $d > 0$ such that on every input of size n, the running time of the algorithm is bounded by cn^d steps.

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Definition

An algorithm is *efficient* if it has a polynomial running time.

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- Bubble sort and insertion sort take roughly n^2 comparisons while quick sort (only on average) and merge sort take roughly $n \log_2 n$ comparisons.
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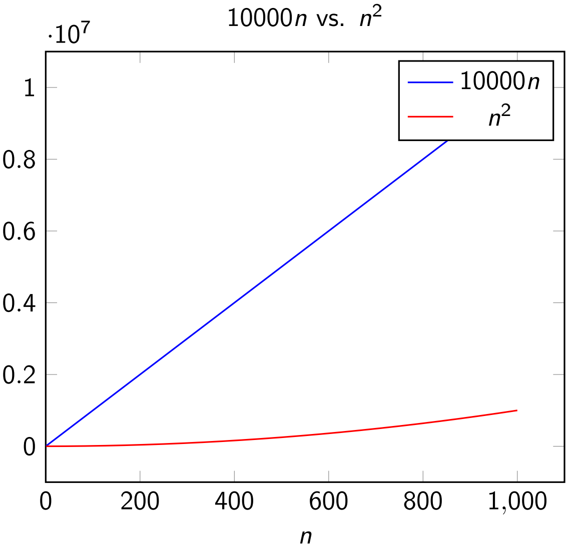
- How can make statements such as the following, in order to compare the running times of different algorithms?
  - $100n \log_2 n \leq n^2$
  - $10000n \leq n^2$
  - $5n^2 - 4n \geq 1000n \log n$

"$10000n \leq n^2$"



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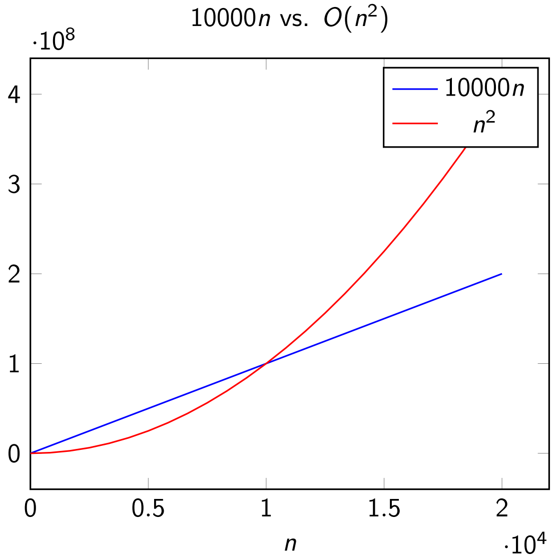

Upper Bound

Definition

Asymptotic upper bound: A function $f(n)$ is $O(g(n))$ if

for all n , $f(n) \leq$ $g(n)$.

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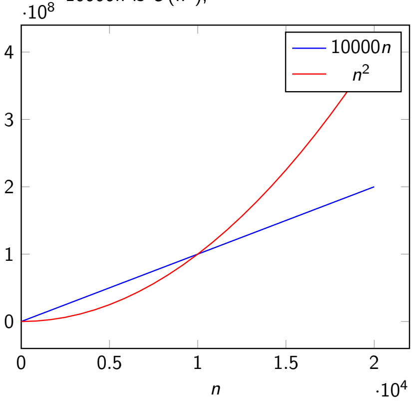

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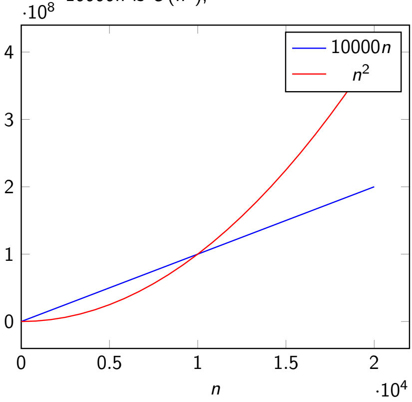

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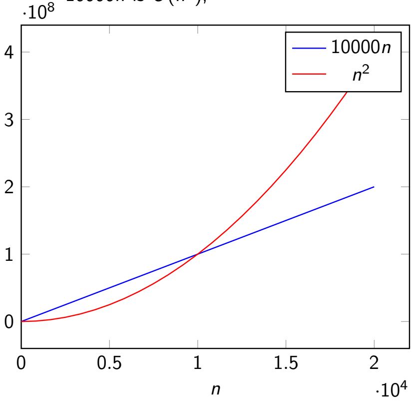

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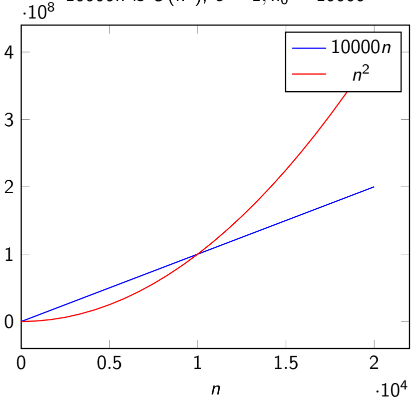
$10000n$ is $O(n^2)$, $c = 1, n_0 = 10000$



$100n \log_2 n$ and n^2

▸ Lecture 2: Analysis: Upper bound

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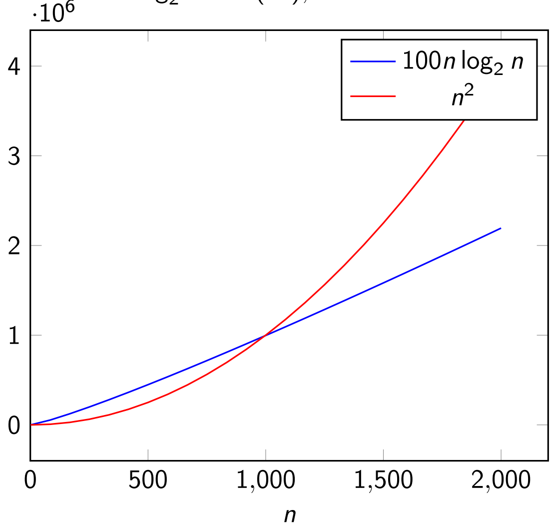

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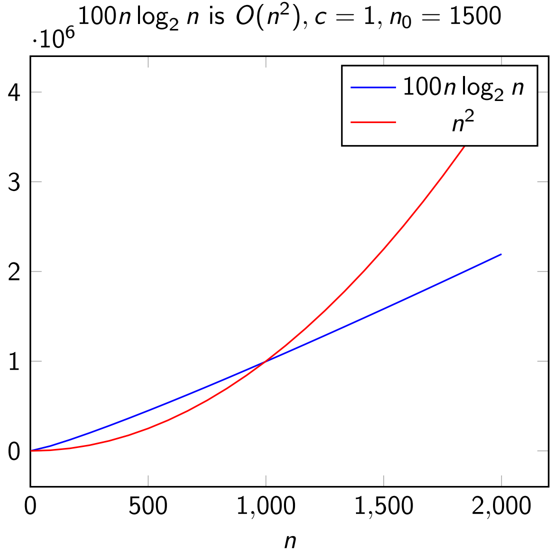

$100n \log_2 n$ and n^2

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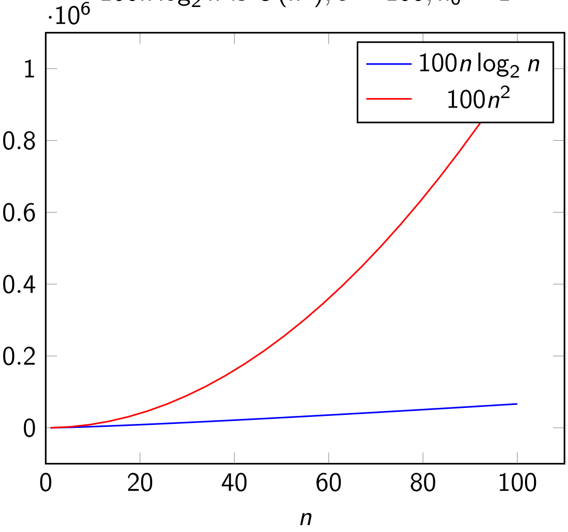
$100n \log_2 n$ is $O(n^2)$, $c = 100$, $n_0 = 1$



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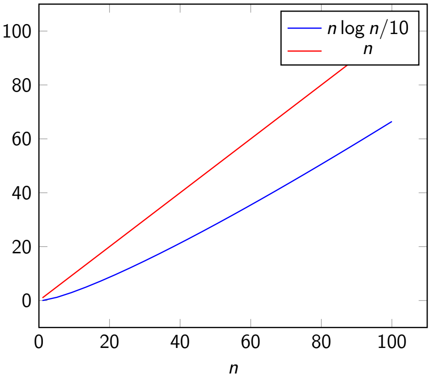
▸ Lecture 2: Analysis: Lower bound

$(n \log_2 n)/10$ and $\Omega(n)$



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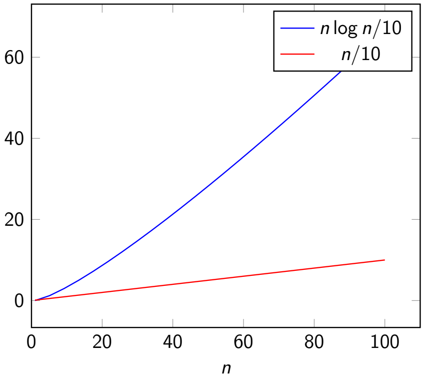
▸ Lecture 2: Analysis: Lower bound

$n \log_2 n/10$ and $\Omega(n)$, $c = 1/10$, $n_0 = 2$



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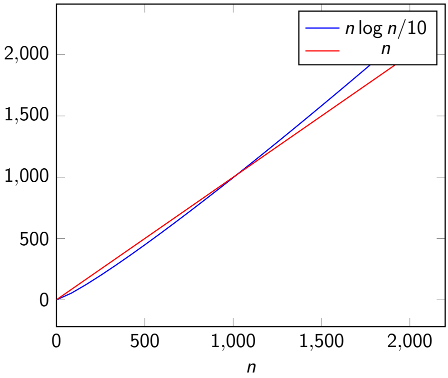

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  - The *stable matching problem* has a lower bound of $\Omega(n^2)$. For *any* algorithm, there is at least one input for which the algorithm will take $\Omega(n^2)$ steps, even if all the preference matrices are already stored in memory (Ng and Hirschberg, *SIAM J. Comput.*, 1990).

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- In all these definitions, c and n_0 are constants independent of n.
- Abuse of notation: say $g(n) = O(f(n))$, $g(n) = \Omega(f(n))$, $g(n) = \Theta(f(n))$.

Properties of Asymptotic Growth Rates

Dropping argument n on this slide for visual clarity.

Transitivity

- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
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- If $f = O(g)$, then $f + g = \Theta(g)$.

Examples

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$pn^2 + qn + r$		
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$\sum_{0 \leq i \leq d} a_i n^i$		
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$\log_a n$		

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$\log_a n$	$O(\log_b n)$	
	Lecture 2: Analysis: logs	

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$O(n^{1.59})$	Polynomial time? Lecture 2: Analysis: Poly time	Yes, since $n^{1.59}$ is $O(n^2)$
$\log_a n$	$O(\log_b n)$ Lecture 2: Analysis: logs	Yes, for any pair of constants $a, b > 1$

- $O(n^d)$ is the definition of *polynomial time*.

Examples

$f(n)$	$g(n)$	**Reason**
$pn^2 + qn + r$	$\Theta(n^2)$	
$pn^2 + qn + r$	$O(n^3)$?	$n^2 \leq n^3$, if $n \geq 1$
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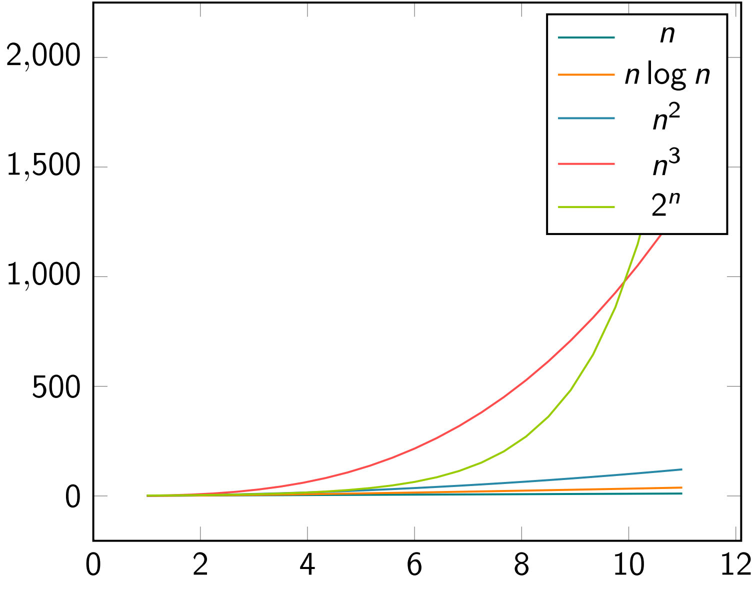
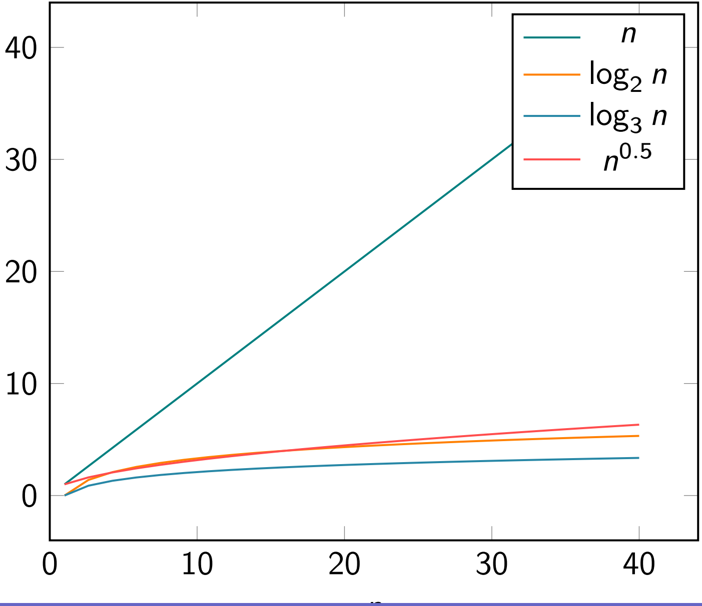
- $O(n^d)$ is the definition of *polynomial time*.
- For every constant $x > 0$, $\log n = O(n^x)$, e.g., $\log n = n^{0.00001}$.
- For every constant $r > 1$ and every constant $d > 0$, $n^d = O(r^n)$, e.g., $n^3 = O(1.1^n)$.

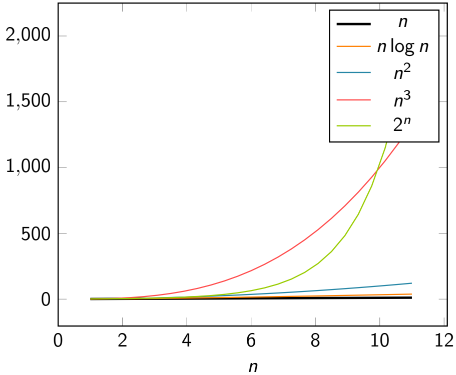
Different functions of n



More functions of n



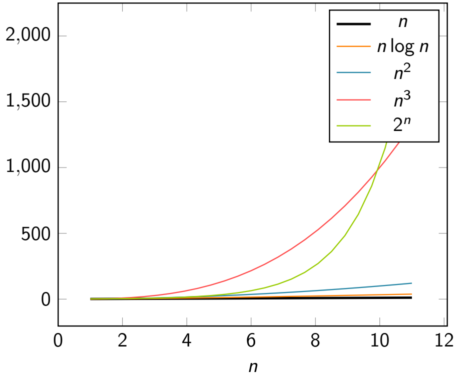
Linear Time



- Running time is at most a constant factor times the size of the input.

Lecture 2: Analysis: Linear time

Linear Time

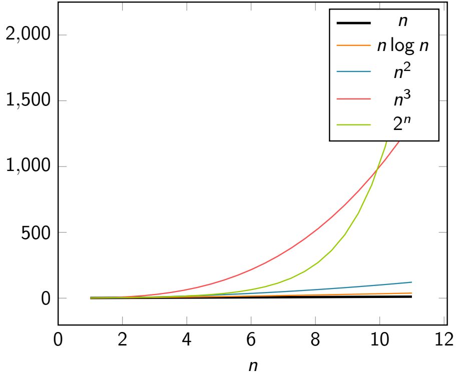


- Running time is at most a constant factor times the size of the input.

  Lecture 2: Analysis: Linear time

- Finding the minimum, merging two sorted lists.

Linear Time



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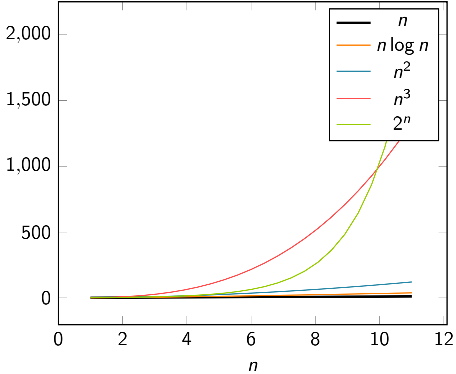
  ▸ Lecture 2: Analysis: Linear time

- Finding the minimum, merging two sorted lists.
- Computing the median (or kth smallest) element in an *unsorted* list.

Linear Time



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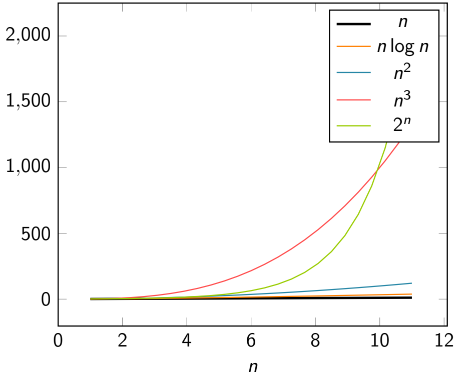
  Lecture 2: Analysis: Linear time
- Finding the minimum, merging two sorted lists.
- Computing the median (or kth smallest) element in an *unsorted* list. "Median-of-medians" algorithm.
- Sub-linear time.

Linear Time



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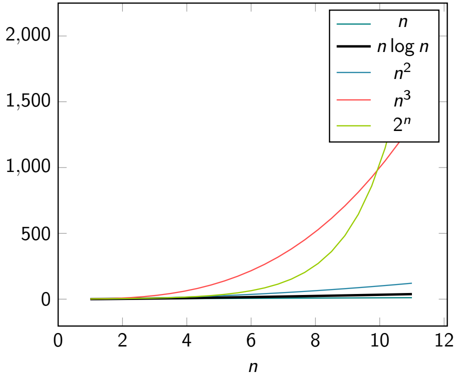
▸ Lecture 2: Analysis: Linear time

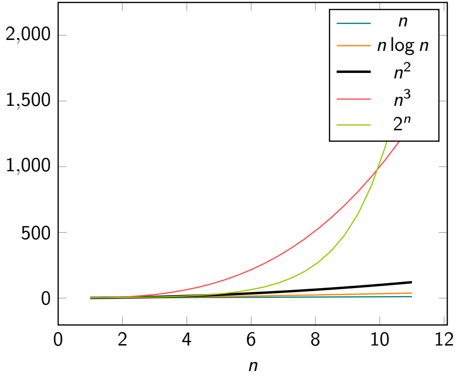
- Finding the minimum, merging two sorted lists.
- Computing the median (or kth smallest) element in an *unsorted* list. "Median-of-medians" algorithm.
- Sub-linear time. Binary search in a sorted array of n numbers takes $O(\log n)$ time.

$O(n \log n)$ Time



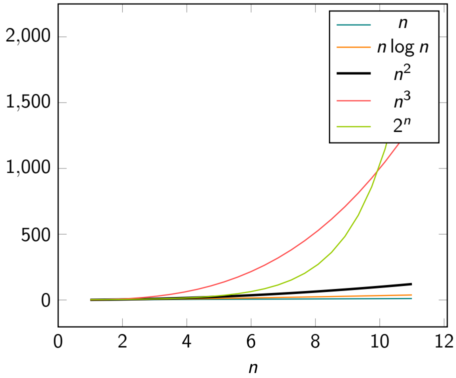
- Any algorithm where the costliest step is sorting.

Quadratic Time



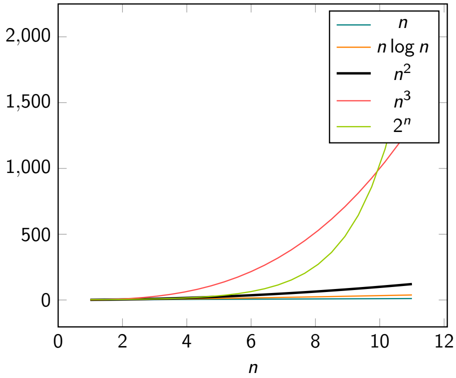
- Enumerate all pairs of elements.

Quadratic Time



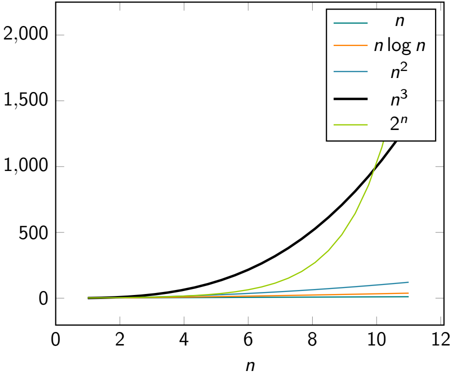
- Enumerate all pairs of elements.
- Given a set of n points in the plane, find the pair that are the closest.

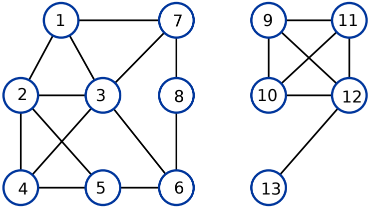
Quadratic Time



- Enumerate all pairs of elements.
- Given a set of n points in the plane, find the pair that are the closest. Surprising fact: will solve this problem in $O(n \log n)$ time later in the semester.

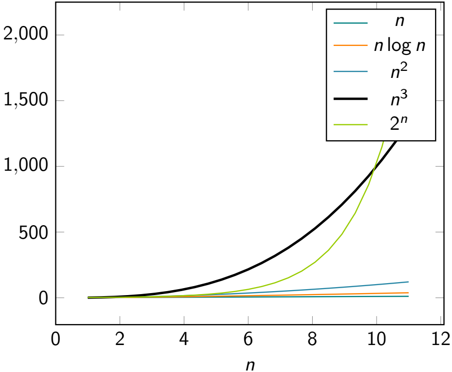
$O(n^k)$ Time

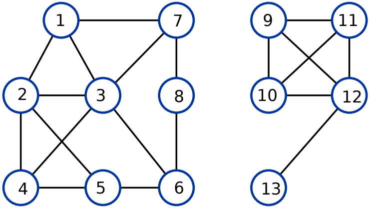




- COVID-19 proximity graph: each node is a person shopping in Kroger, an edge connects two people who came within six feet of each other.
- Some subgraphs can have high potential for virus transmission.

$O(n^k)$ Time





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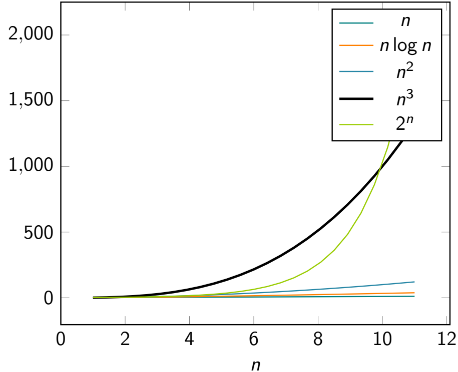
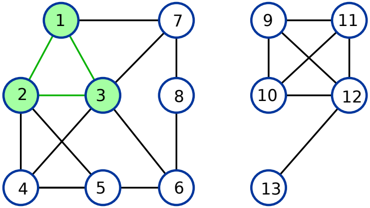
- Some subgraphs can have high potential for virus transmission.
- Does a graph have a *clique* of size k, where k is a constant, i.e. there are k nodes such that every pair is connected by an edge?

$O(n^k)$ Time





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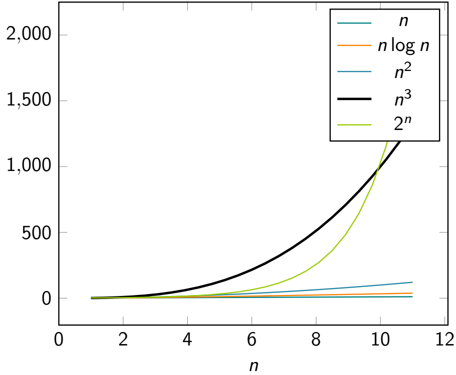

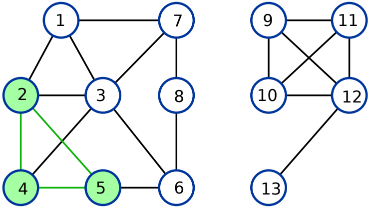

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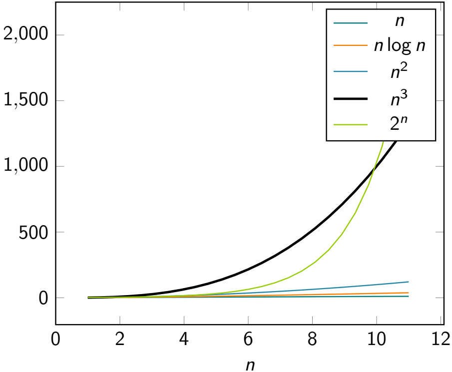

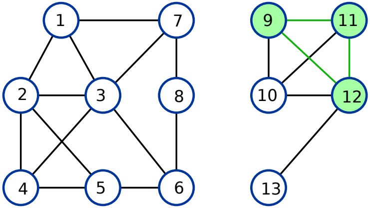

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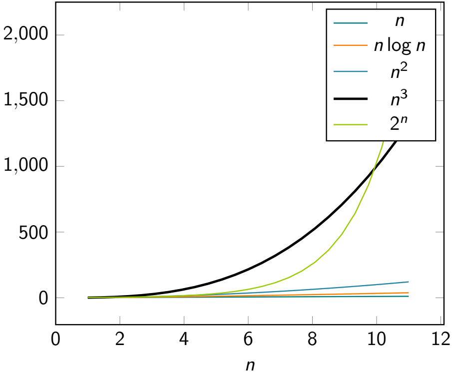

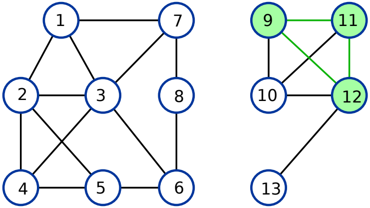

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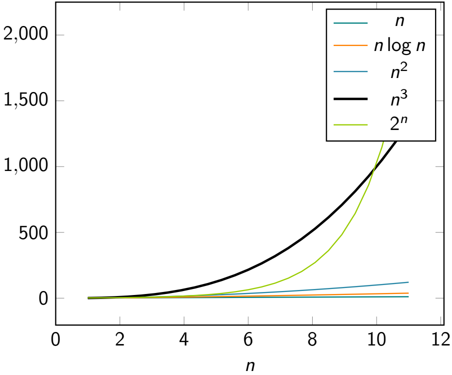
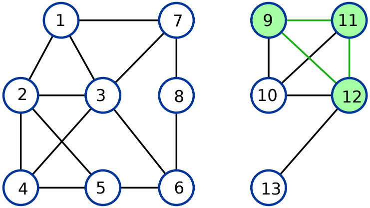
- Algorithm: For each subset S of k nodes, check if S is a clique. If the answer is yes, report it. ▸ Lecture 2: Analysis: Number of subsets

$O(n^k)$ Time





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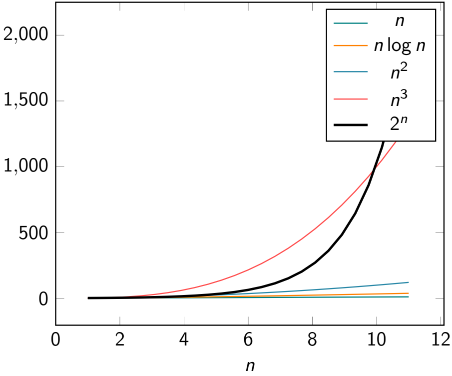
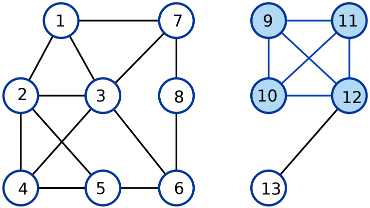
- Algorithm: For each subset S of k nodes, check if S is a clique. If the answer is yes, report it. Lecture 2: Analysis: Number of subsets
- Running time is $O(k^2\binom{n}{k}) = O(n^k)$.

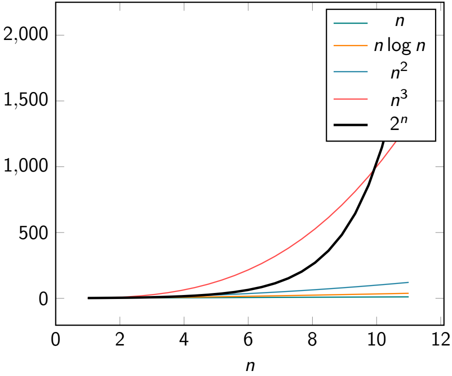
Beyond Polynomial Time

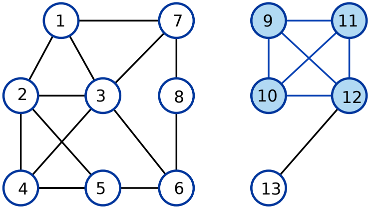




- What is the largest size of a clique in a graph with n nodes?

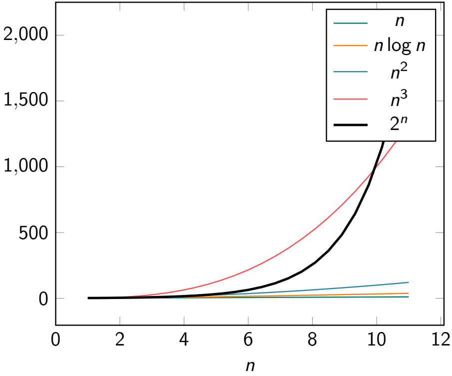
Beyond Polynomial Time

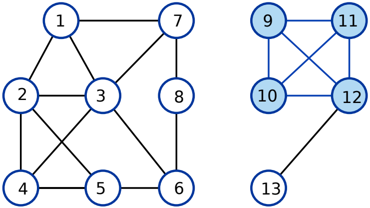




- What is the largest size of a clique in a graph with n nodes?
- Algorithm: For each $1 \leq i \leq n$, check if the graph has a clique of size i. Output largest clique found.

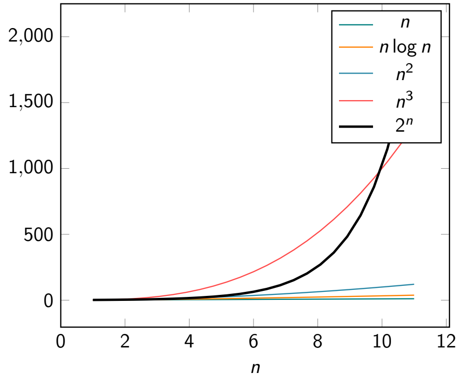
Beyond Polynomial Time

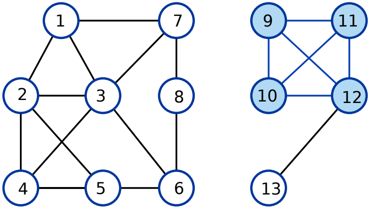




- What is the largest size of a clique in a graph with n nodes?
- Algorithm: For each $1 \leq i \leq n$, check if the graph has a clique of size i. Output largest clique found.
- What is the running time?

Beyond Polynomial Time





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- Algorithm: For each $1 \leq i \leq n$, check if the graph has a clique of size i. Output largest clique found.
- What is the running time? $O(n^2 2^n)$.