Review of Priority Queues and Graph Searches

T. M. Murali

January 29, February 5, 7 2024

Results of Poll on Teaching Style

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   - Not used to your handwriting.
3. Polls
   - They help me think (98%)
   - There should be more polls (64%)
4. Other suggestions and comments:
   - Good that you are answering questions on the spot.
   - Like visualisations when you show them.
   - Explanations are a bit confusing ...
   - So far so good, pace is good, like, great, enjoying.

Results of Poll on PQs and Graph Searches

1. Priority queues: Refresher (43%), In detail (38%)
2. Breadth-first search: Refresher (38%), In detail (33%)
3. Depth-first search: Refresher (40%), In detail (31%)

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4. Spend two classes on these three topics
   - Focus on proving their properties.
   - Describe/refresh proof techniques, which will be useful during the rest of the semester.

Motivation: Sort a List of Numbers

Sort

INSTANCE: Nonempty list $x_1, x_2, \ldots, x_n$ of integers.

SOLUTION: A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1},$ for all $1 \leq i < n$.

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  - Insert each number into a data structure D.
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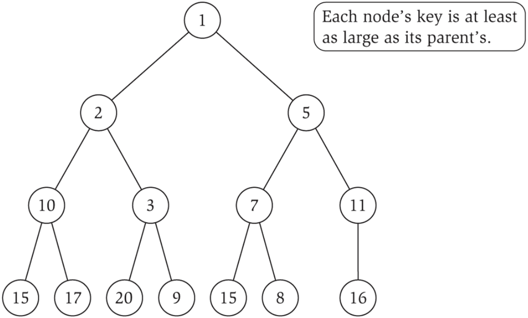
- Possible algorithm:
  - Insert each number into a data structure D.
  - Repeatedly find the smallest number in D, output it, and remove it.
- To get $O(n \log n)$ running time, each "insert" step, "find minimum" step and each "remove" step must take $O(\log n)$ time.

Priority Queue

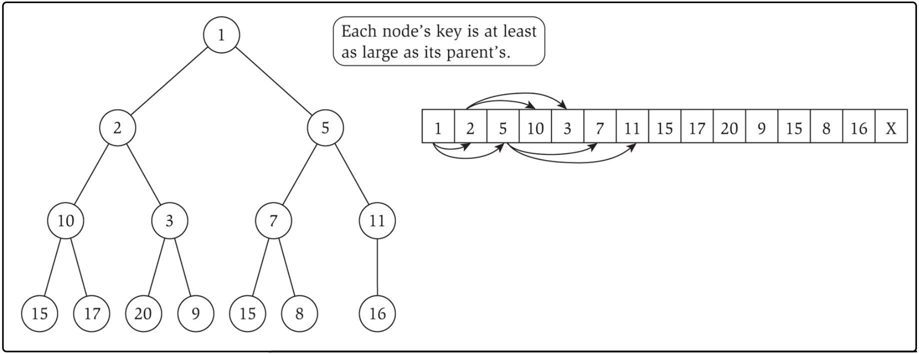
- Store a set S of elements, where each element v has a priority value $\texttt{key}(v)$.
- Smaller key values $\equiv$ higher priorities.
- Operations supported:
  - find the element with smallest key
  - remove the smallest element
  - insert an element
  - delete an element
  - update the key of an element
- Element deletion and key update require knowledge of the position of the element in the priority queue.

Heaps



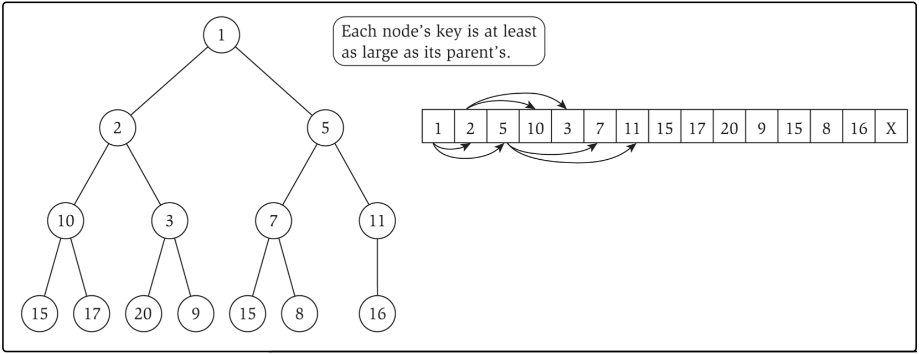
- Combine benefits of both lists and sorted arrays.
- Conceptually, a heap is a balanced binary tree.
- *Heap order*: For every element v at a node i, the element w at i's parent satisfies $\texttt{key}(w) \leq \texttt{key}(v)$.
- We can implement a heap in a pointer-based data structure.

Heaps



- Alternatively, assume maximum number N of elements is known in advance.
- Store nodes of the heap in an array.
  - Node at index i has children at indices $2i$ and $2i+1$ and parent at index $\lfloor i/2 \rfloor$.
  - Index 1 is the root.
  - How do you know that a node at index i is a leaf?

Heaps



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  - Index 1 is the root.
  - How do you know that a node at index i is a leaf? If $2i > n$, where n is the current number of elements in the heap.

Inserting an Element: `Heapify-up`

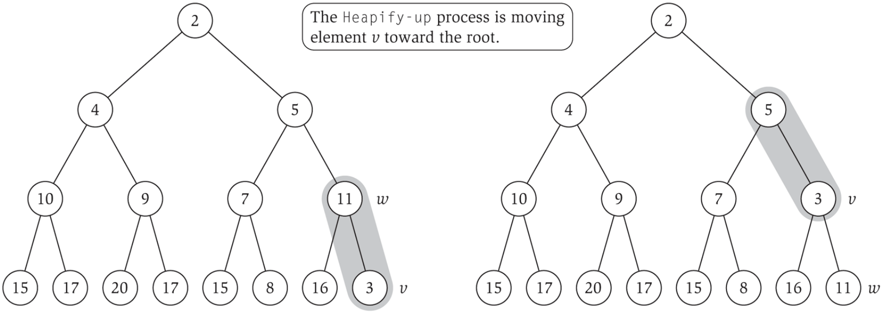

Figure 2.4 The `Heapify-up` process. Key 3 (at position 16) is too small (on the left). After swapping keys 3 and 11, the heap violation moves one step closer to the root of the tree (on the right).

Inserting an Element: `Heapify-up`

1. Insert new element at index $n + 1$.
2. Fix heap order using `Heapify-up`$(H, n + 1)$.

```
Heapify-up(H,i):
  If i > 1 then
    let j = parent(i) = ⌊i/2⌋
    If key[H[i]]<key[H[j]] then
      swap the array entries H[i] and H[j]
      Heapify-up(H,j)
    Endif
  Endif
```

- Proof of correctness: read pages 61–62 of your textbook.

Running time of `Heapify-up`

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  - Each invocation decreases the second argument by a factor of at least 2.

▸ Lecture 4: PQs: factor of 2 decrease

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- Running time of `Heapify-up`(i):
  - Each invocation decreases the second argument by a factor of at least 2.
    Lecture 4: PQs: factor of 2 decrease
  - After k invocations, argument is at most $i/2^k$.
  - Therefore $i/2^k \geq 1$, which implies that $k \leq \log_2 i$.
  - Running time of `Heapify-up`(i) is $O(\log i)$.

Deleting an Element: `Heapify-down`

- Suppose H has $n+1$ elements.

1. Delete element at $H[i]$ by moving element at $H[n+1]$ to $H[i]$.
2. If element at $H[i]$ is too small, fix heap order using `Heapify-up`(H, i).
3. If element at $H[i]$ is too large, fix heap order using `Heapify-down`(H, i).

```
Heapify-down(H,i):
  Let n = length(H)
  If 2i > n then
    Terminate with H unchanged
  Else if 2i < n then
    Let left = 2i, and right = 2i + 1
    Let j be the index that minimizes key[H[left]] and key[H[right]]
  Else if 2i = n then
    Let j = 2i
  Endif
  If key[H[j]] < key[H[i]] then
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- Proof of correctness: read pages 63–64 of your textbook.

Example of `Heapify-down`

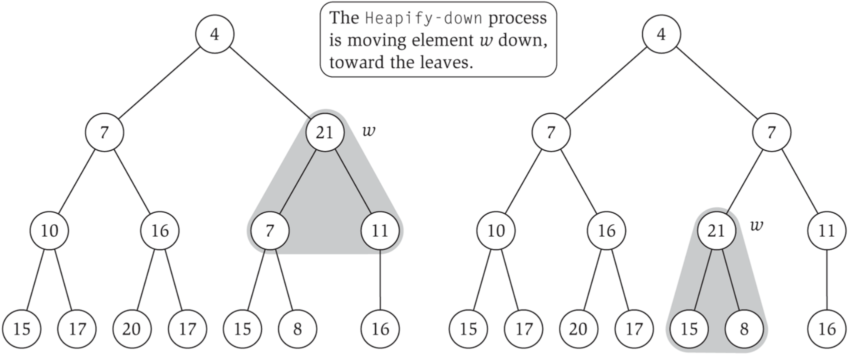

Figure 2.5 The `Heapify-down` process:. Key 21 (at position 3) is too big (on the left). After swapping keys 21 and 7, the heap violation moves one step closer to the bottom of the tree (on the right).

Running time of `Heapify-down`

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- Each invocation of `Heapify-down` increases its second argument by a factor of at least two. ▸ Lecture 4: PQs: factor of 2 increase

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```

- Each invocation of `Heapify-down` increases its second argument by a factor of at least two. ▸ Lecture 4: PQs: factor of 2 increase
- After k invocations argument must be at least $i2^k \leq n$, which implies that $k \leq \log_2 n/i$. Therefore running time is $O(\log_2 n/i)$.

Sorting Numbers with the Priority Queue

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- Final algorithm:
  - Insert each number in a priority queue H.
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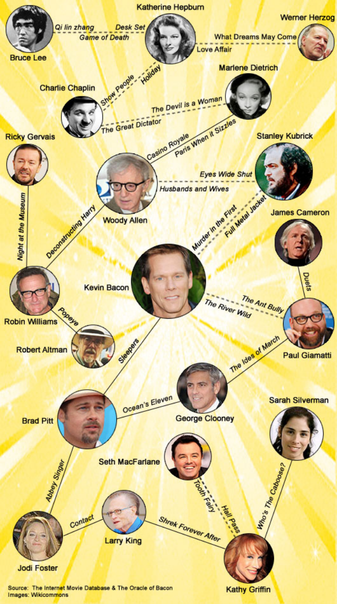
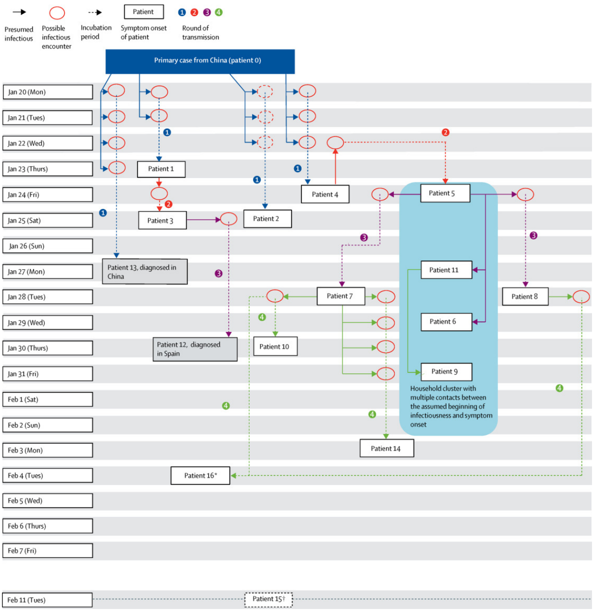
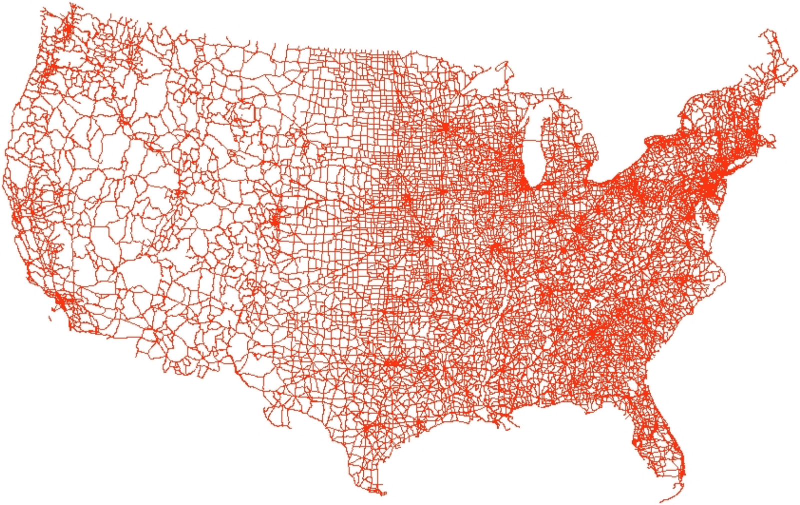
- Final algorithm:
  - Insert each number in a priority queue H.
  - Repeatedly find the smallest number in H, output it, and delete it from H.
- Each insertion and deletion takes $O(\log n)$ time for a total running time of $O(n \log n)$.

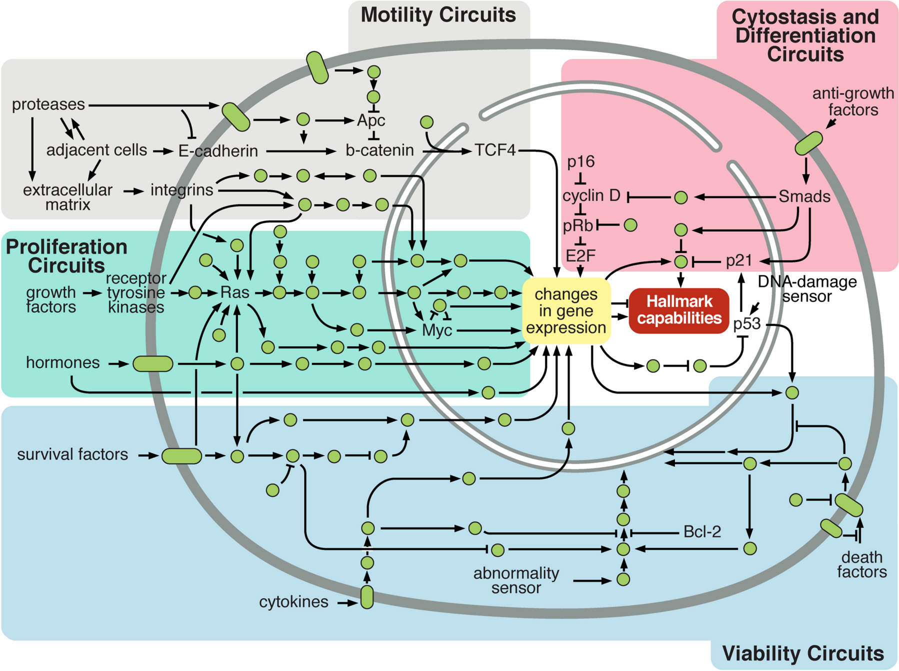


The Oracle of Bacon



(Böhmer *et al.*, *The Lancet*, May 15, 2020)





Motility Circuits
Cytostasis and Differentiation Circuits
Proliferation Circuits
Viability Circuits
proteases
adjacent cells
E-cadherin
extracellular matrix
integrins
Apc
b-catenin
TCF4
p16
cyclin D
pRb
E2F
anti-growth factors
Smads
p21
DNA-damage sensor
p53
growth factors
receptor tyrosine kinases
Ras
Myc
changes in gene expression
Hallmark capabilities
hormones
survival factors
Bcl-2
abnormality sensor
cytokines
death factors

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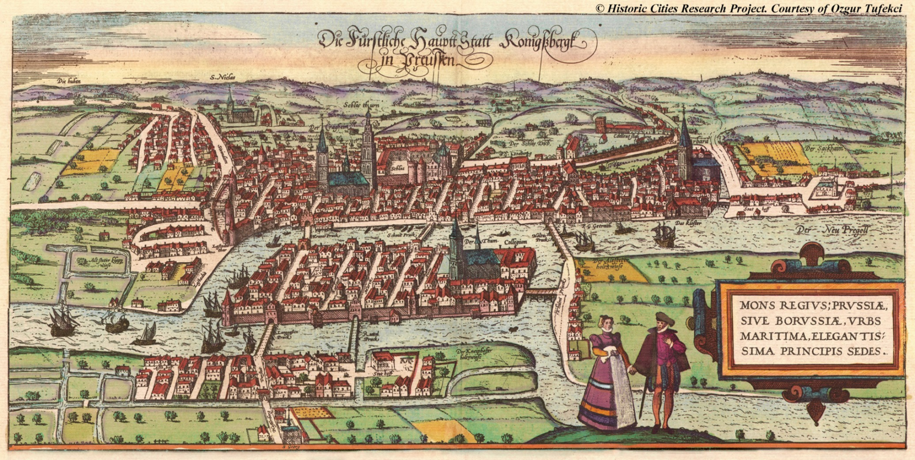
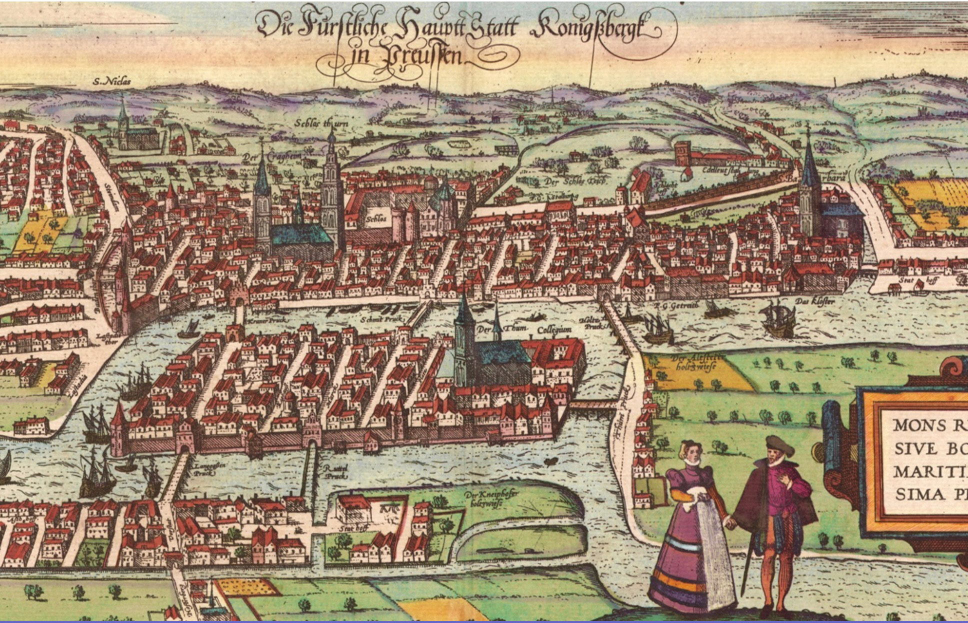
- Problems involving graphs have a rich history dating back to Euler.

Euler and Graphs

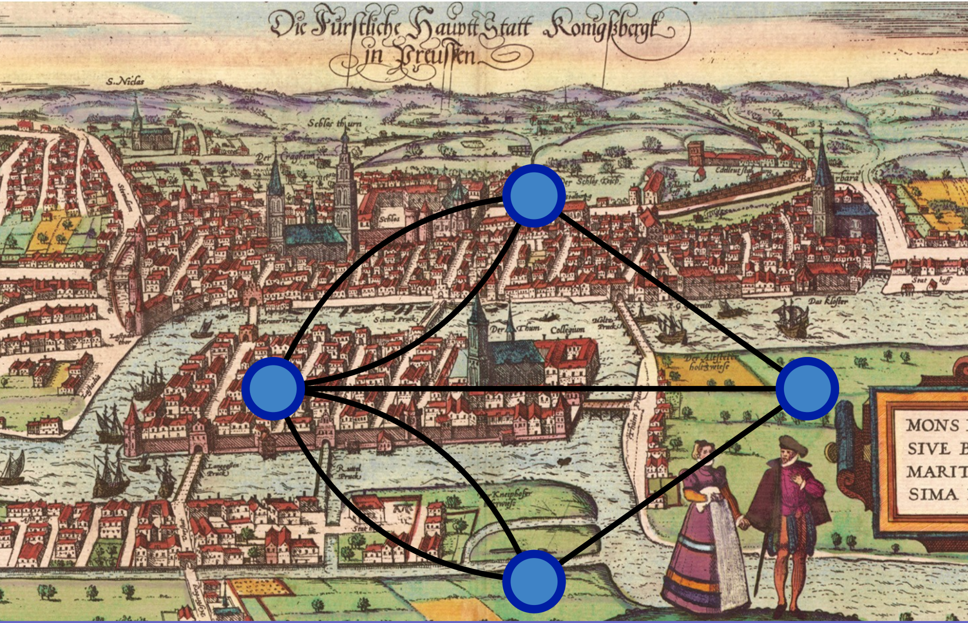


Devise a walk through the city that
crosses each of the seven bridges exactly once.

Euler and Graphs

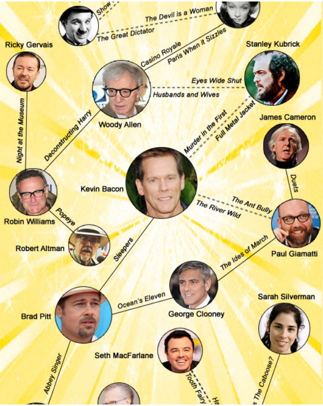


Euler and Graphs



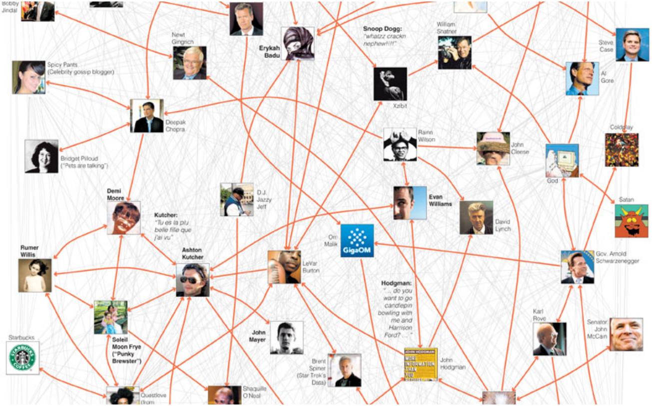
Definition of a Graph

- *Undirected graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$.
  - Elements of E are unordered pairs.
  - Edge (u, v) is *incident* on u, v; u and v are *neighbours* of each other.
  - Exactly one edge between any pair of nodes.
  - G contains no self loops, i.e., no edges of the form (u, u).

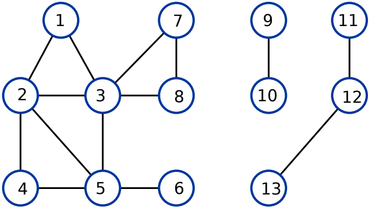


Definition of a Graph

- *Directed graph* $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$.
  - Elements of E are ordered pairs.
  - $e = (u, v)$: u is the *tail* of the edge e, v is its *head*; e is *directed from u to v*.
  - A pair of nodes may be connected by two directed edges: (u, v) and (v, u).
  - G contains no self loops.

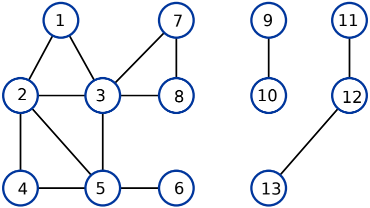


Paths and Connectivity



- A *v_1-v_k path* in an undirected graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k \in V$ such that every consecutive pair of nodes $v_i, v_{i+1}, 1 \leq i < k$ is connected by an edge in E.

Paths and Connectivity



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- A path is *simple* if all its nodes are distinct.

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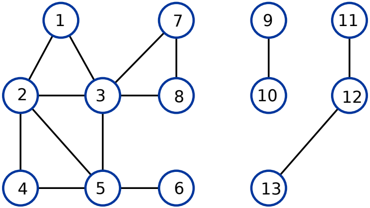

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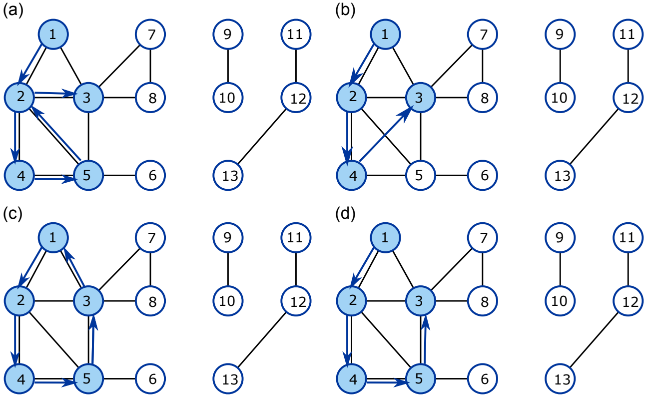

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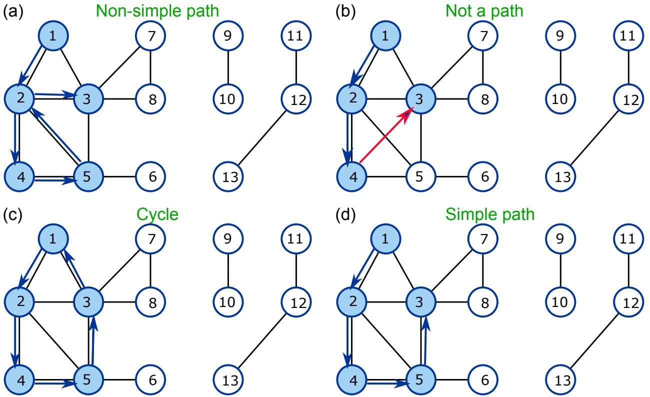

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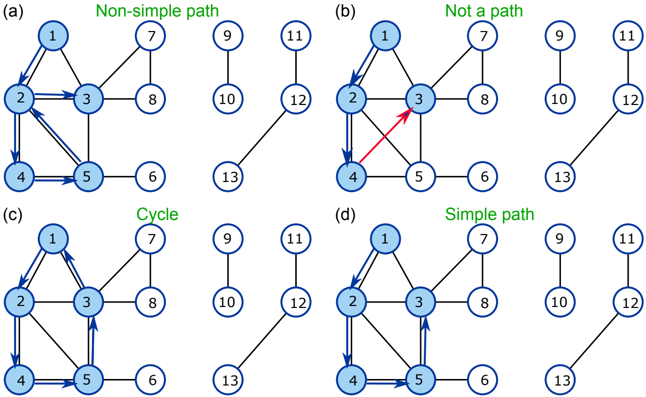

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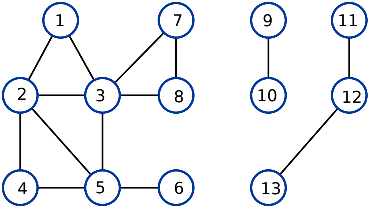
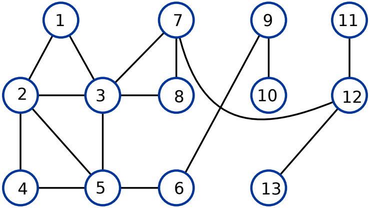
- A path is *simple* if all its nodes are distinct.
- A *cycle* is a path where $k > 2$, the first $k - 1$ nodes are distinct, and $v_1 = v_k$.
- Similar definitions carry over to directed graphs as well.

Connectivity



- An undirected graph G is *connected* if for every pair of nodes $u, v \in V$, there is a path from u to v in G.

Connectivity



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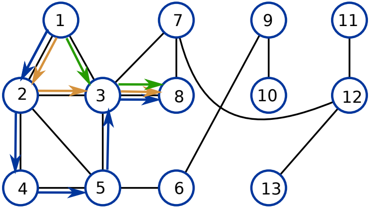
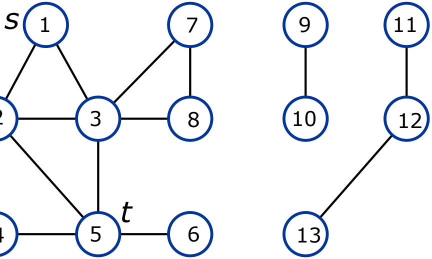


- An undirected graph G is *connected* if for every pair of nodes $u, v \in V$, there is a path from u to v in G.
- *Distance* $d(u, v)$ between two nodes u and v is the minimum number of edges in any u-v path.

s-t Connectivity

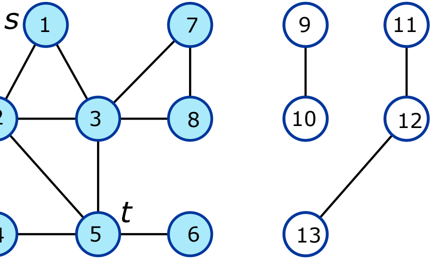


s-t Connectivity

INSTANCE: An undirected graph $G = (V, E)$ and two nodes $s, t \in V$.

QUESTION: Is there an s-t path in G?

s-t Connectivity



s-t Connectivity

INSTANCE: An undirected graph $G = (V, E)$ and two nodes $s, t \in V$.

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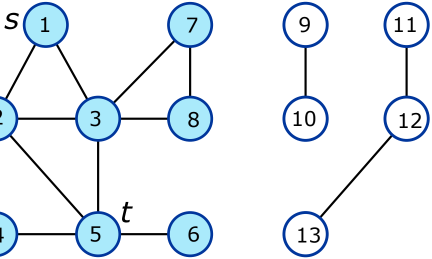

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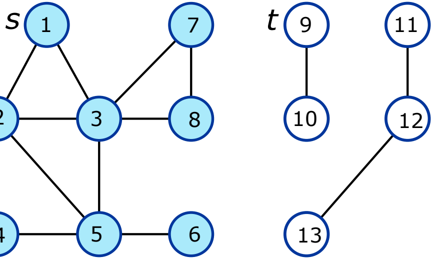

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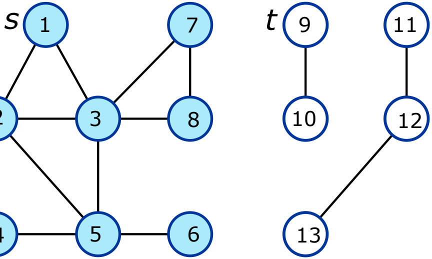

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- Appears to do more work than is strictly necessary.

Computing Connected Components

- Abstract idea for an algorithm, with details to be specified later.
- "Explore" G starting from s and maintain set R of visited nodes.

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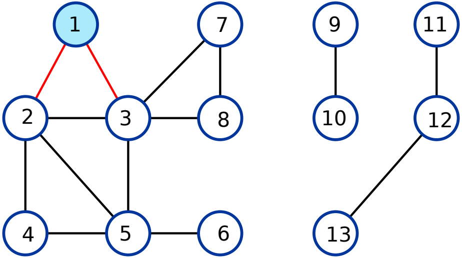

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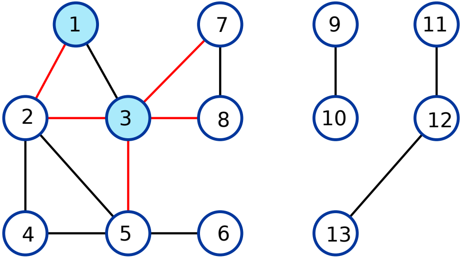

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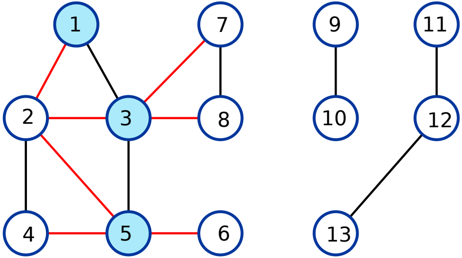

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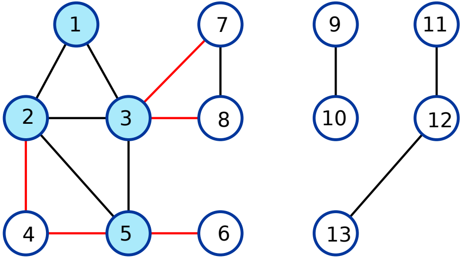

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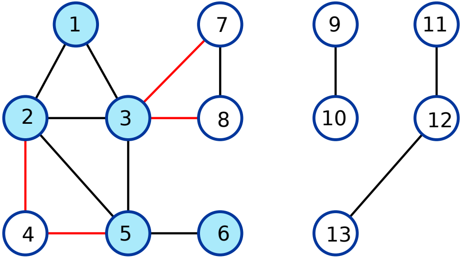

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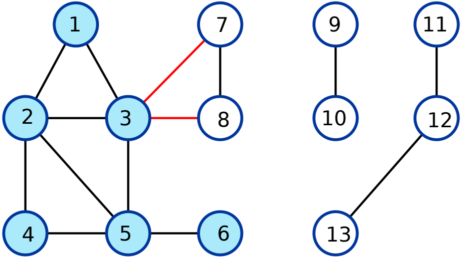

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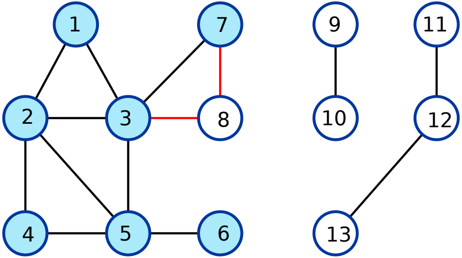

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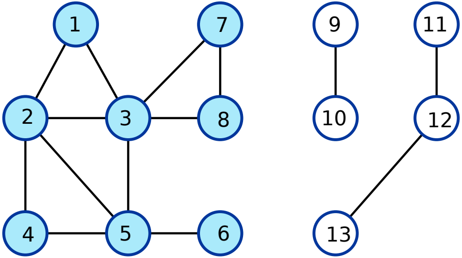

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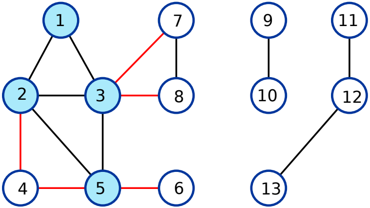

- Why does the algorithm terminate?
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Lectures 4-6: Graph searches: Connected components algorithm

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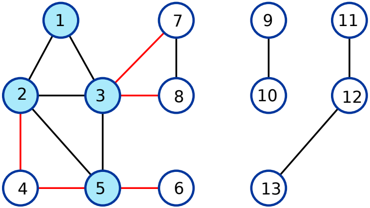

- Why does the algorithm terminate? Each iteration adds a new node to R.
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Lectures 4-6: Graph searches: Connected components algorithm

Correctness of the Algorithm

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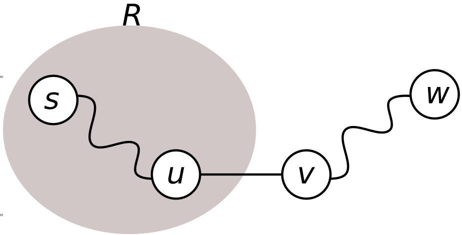

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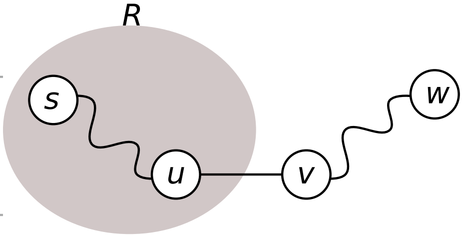

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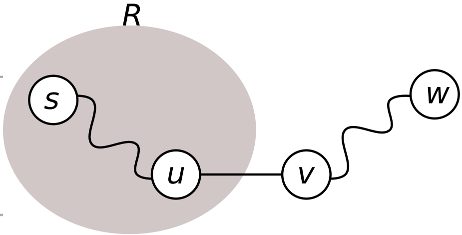

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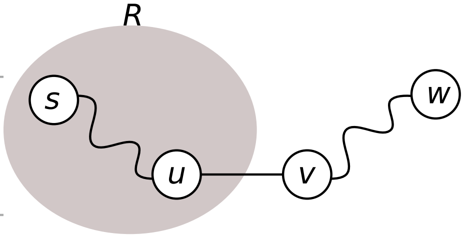

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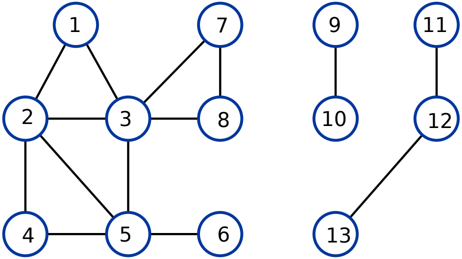
  - Note: wrong to assume that predecessor of w in P is not in R.

Breadth-First Search (BFS)



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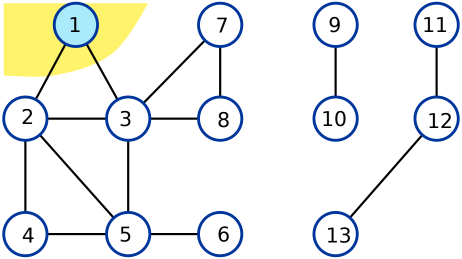

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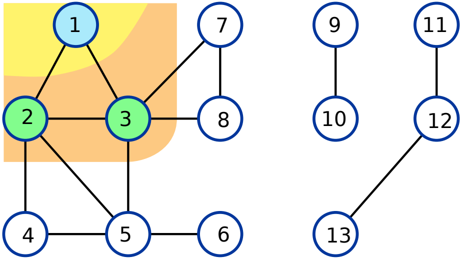

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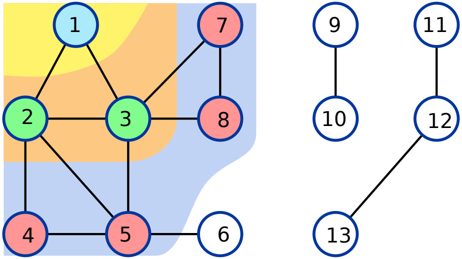

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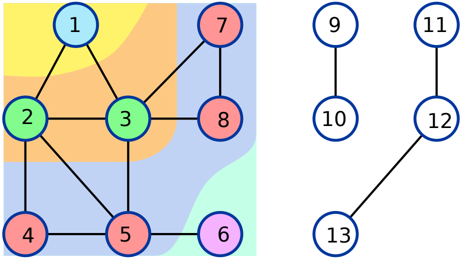

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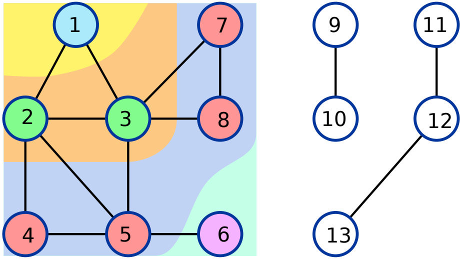
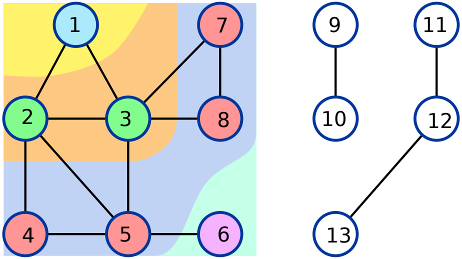
Properties of BFS



- We have not yet described how to compute these layers.
- Claim: For each $j \geq 1$, layer L_j consists of all nodes

Lectures 4-6: Graphs searches: Nodes in layer j

Properties of BFS



- We have not yet described how to compute these layers.
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Lectures 4-6: Graphs searches: Nodes in layer j

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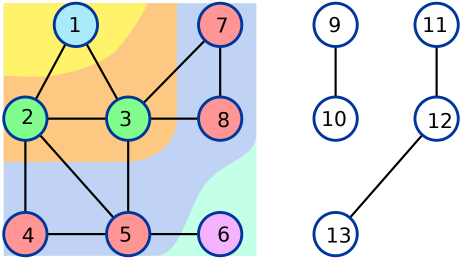

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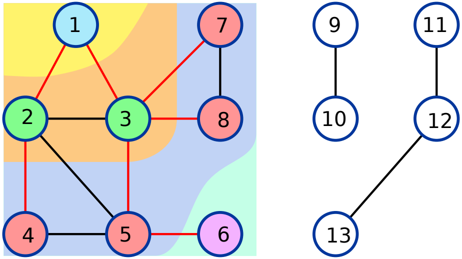

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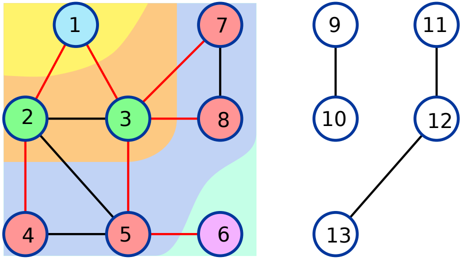

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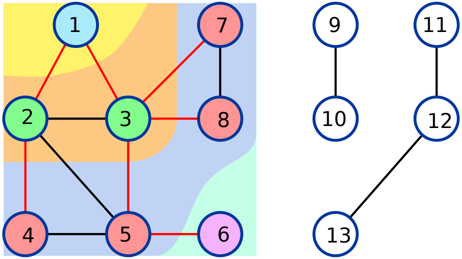

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  - Why is T a tree? It is connected. The number of edges in T is the number of nodes in all the layers minus 1. T is called the *breadth-first search tree*.

Depth-First Search (DFS)

- Explore G as if it were a maze: start from s, traverse first edge out (to node v), traverse first edge out of v, . . . , reach a dead-end, backtrack, ...

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DFS(u):
  Mark u as "Explored" and add u to R
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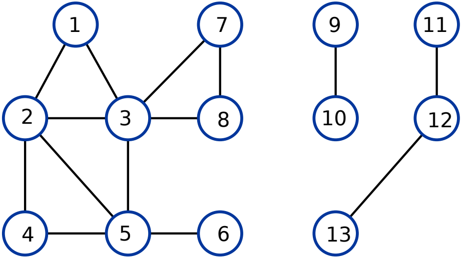
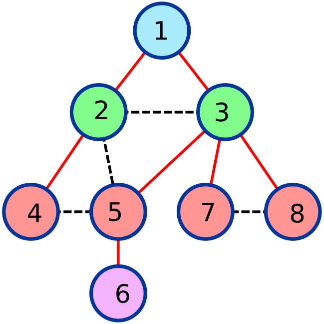
- *Depth-first search tree* is a tree T: when DFS(v) is invoked directly during the call to DFS(v), add edge (u, v) to T.

Example of DFS





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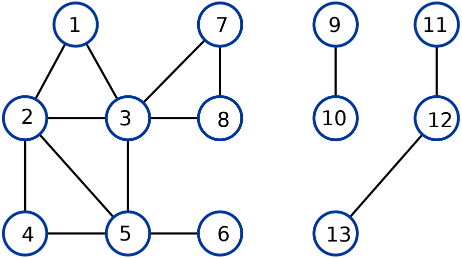

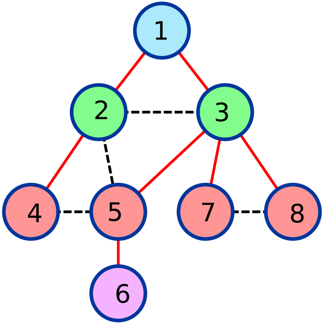

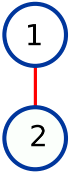

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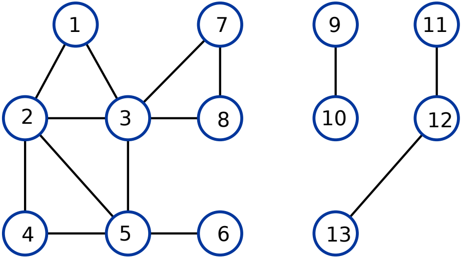

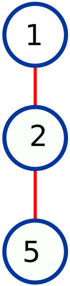

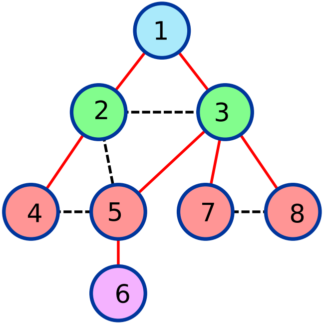

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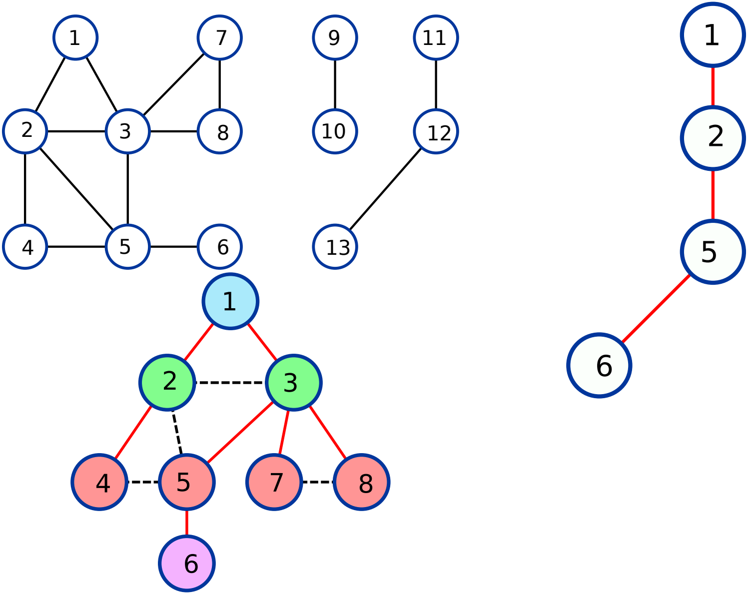

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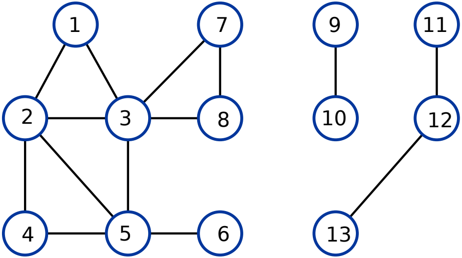

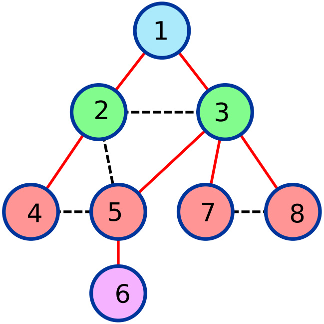

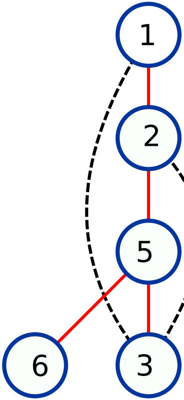

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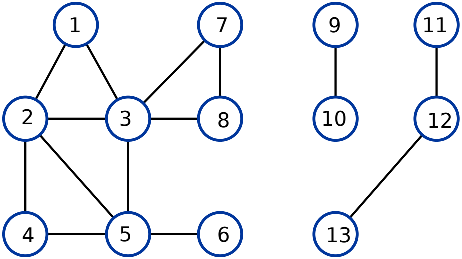

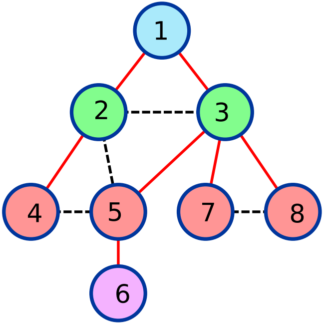

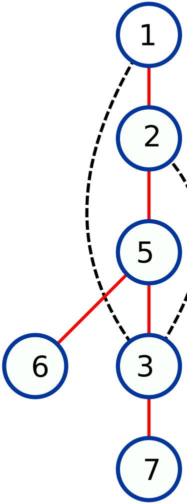

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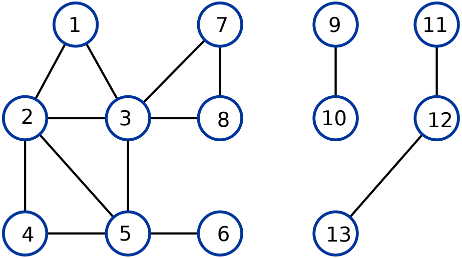

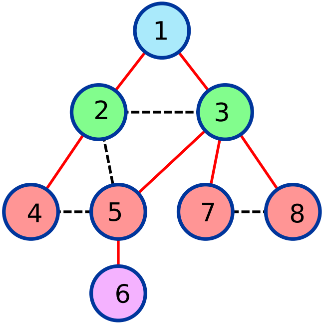

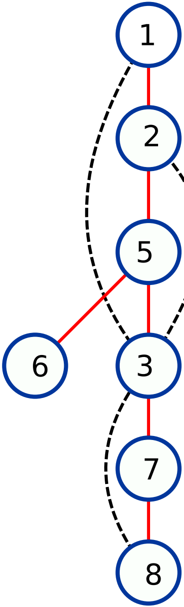

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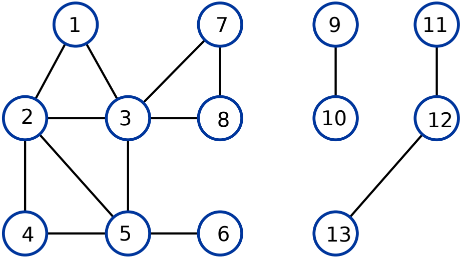
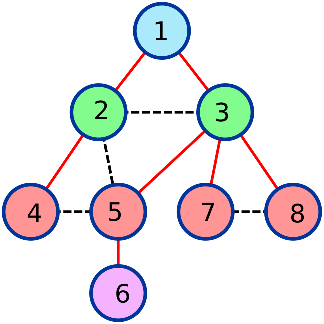
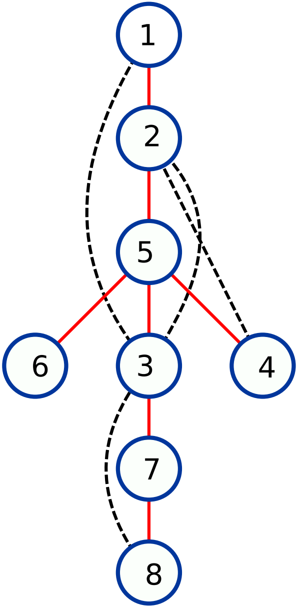
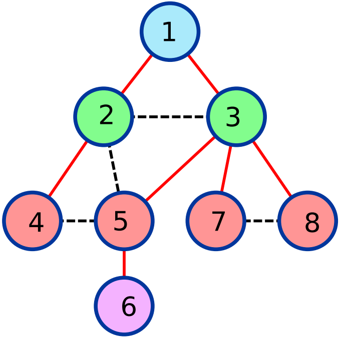
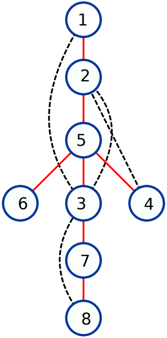






BFS vs. DFS





Lectures 4-6: Graph Searches: BFS vs. DFS

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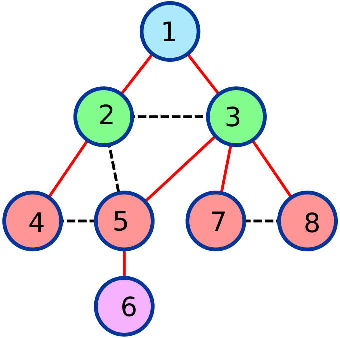

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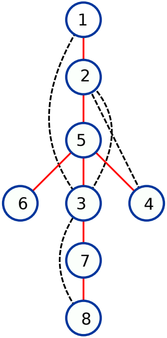

- Both visit the same set of nodes but in a different order.
- Both traverse all the edges in the connected component but in a different order.
- BFS trees have root-to-leaf paths that look as short as possible while paths in DFS trees tend to be long and deep.

Representing Graphs

- Graph $G = (V, E)$ has two input parameters: $|V| = n, |E| = m$.
  - Size of the graph is defined to be $m + n$.
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    Lectures 4-6: Graph Searches: Sum of node degrees
  - Space used and time to iterate over neighbours are optimal for every graph.

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Data Structures for Implementation

- "Implementation" of BFS and DFS: fully specify the algorithms and data structures so that we can obtain provably efficient times.
- Inner loop of both BFS and DFS: process the set of edges incident on a given node and the set of visited nodes.
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- How do we store the set of visited nodes? Order in which we process the nodes is crucial.
  - BFS: store visited nodes in a queue (first-in, first-out).
  - DFS: store visited nodes in a stack (last-in, first-out)

Using a Queue in BFS

- Maintain an array `Discovered` and set `Discovered`$[v]$ = *true* as soon as the algorithm sees v.
- Maintain all the layers in a single queue L.

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BFS(s):
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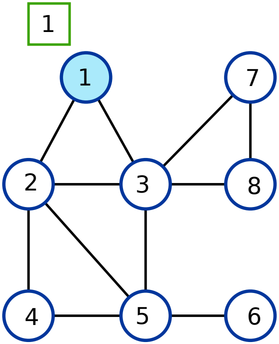

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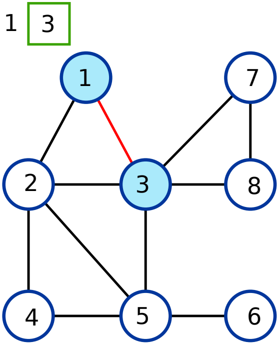

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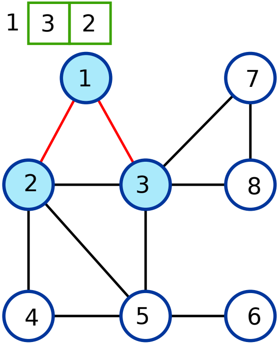

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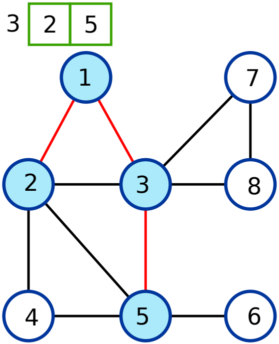

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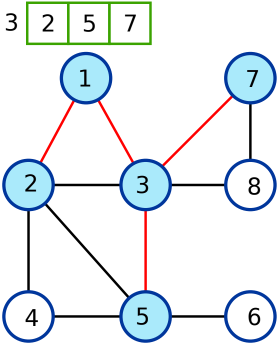

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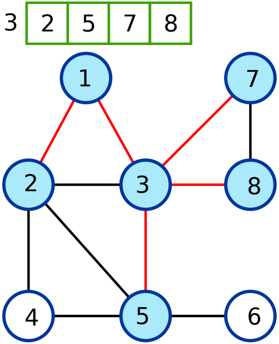

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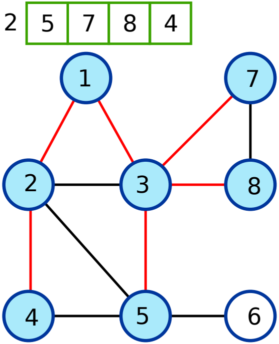

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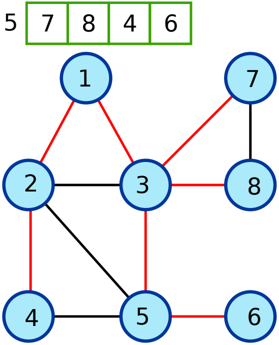

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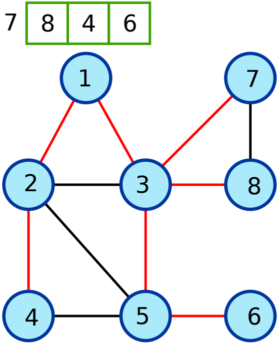

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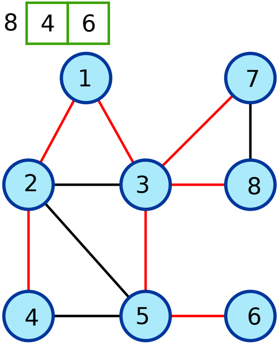

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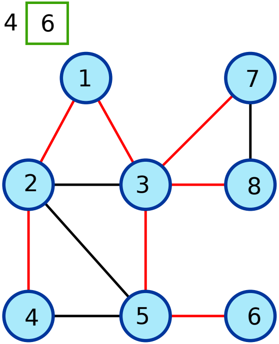

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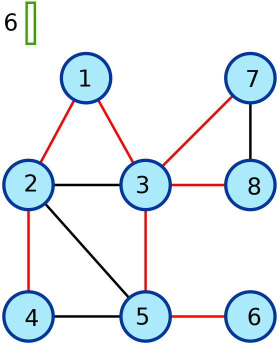

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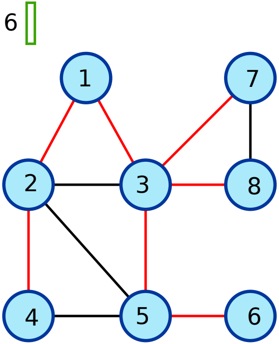
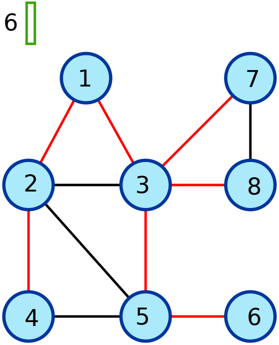


- Can modify this procedure to also keep track of distance to s (layer numbers).

Using a Queue in BFS

- Maintain an array Discovered and set Discovered$[v]$ = *true* as soon as the algorithm sees v.
- Maintain all the layers in a single queue L.

```
BFS(s):
  Set Discovered[s] = true
  Set Discovered[v] = false, for all other nodes v
  Initialize L to consist of the single element s
  While L is not empty
     Pop the node u at the head of L
     For each edge (u, v) incident on u
        If Discovered[v] = false then
           Set Discovered[v] = true
           Add edge (u, v) to the tree T
           Push v to the back of L
        Endif
     Endfor
  Endwhile
```



- Can modify this procedure to also keep track of distance to s (layer numbers). Store the pair (u, l_u), where l_u is the index of the layer containing u.

Analysis of BFS Implementation

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- How many times is a node popped from L?

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- Maintaining layer information: $O(1)$ time per node.
- Total time is $O(n + m)$.

Recursive DFS to Stack-Based DFS

```
DFS(u):
  Mark u as "Explored" and add u to R
  For each edge (u,v) incident to u
    If v is not marked "Explored" then
      Recursively invoke DFS(v)
    Endif
  Endfor
```

- Procedure has "tail recursion": recursive call is the last step.

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```

- Procedure has "tail recursion": recursive call is the last step.
- Can replace the recursion by an iteration: use a stack to explicitly implement the recursion.

Analysing DFS

```
DFS(s):
  Initialize S to be a stack with one element s
  While S is not empty
    Take a node u from S
    If Explored[u] = false then
       Set Explored[u] = true
       For each edge (u,v) incident to u
         Add v to the stack S
       Endfor
    Endif
  Endwhile
```

- How many times is a node's adjacency list scanned?

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- The total amount of time to process edges incident on node u's is $O(n_u)$.
- The total running time of the algorithm is $O(n + m)$.