Linear-Time Graph Algorithms

T. M. Murali

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- Claim: For any two nodes s and t in a graph, their connected components are either equal or disjoint. Read proof in page 86 of your textbook.

Computing All Connected Components

1. Pick an arbitrary node s in G.
2. Compute its connected component using BFS (or DFS).
3. Find a node (say v, not already visited) and repeat the BFS from v.
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- Connectivity in directed graphs: Read Chapter 3.5 of your textbook.

Bipartite Graphs

- A graph $G = (V, E)$ is *bipartite* if V can be partitioned into two subsets X and Y such that every edge in E has one endpoint in X and one endpoint in Y.
  - $(X \times X) \cap E = \emptyset$ and $(Y \times Y) \cap E = \emptyset$.
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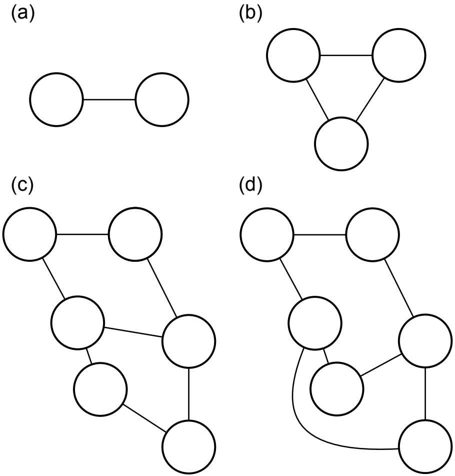
- Examples of bipartite graphs: medical residents and hospitals, COVID-19 vaccines and countries in which they are being adminsitered, jobs and processors they can be scheduled on, professors and courses they can teach.

TestBipartiteness

INSTANCE: An undirected graph $G = (V, E)$

QUESTION: Is G bipartite?

Examples



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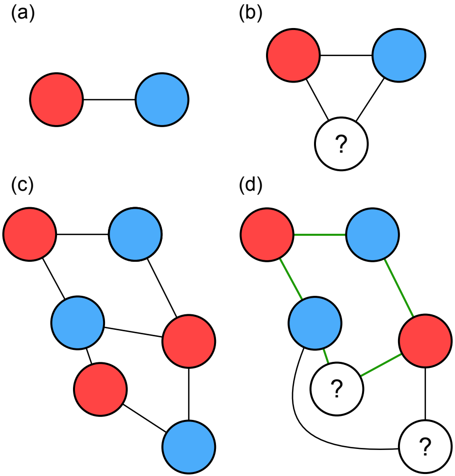

- A triangle is not bipartite.
- Generalisation: No cycle of odd length is bipartite.
- Claim: If a graph is bipartite, then it cannot contain a cycle of odd length.

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  1. Run BFS on G. Maintain an additional array `Colour`.
  2. When we add a node v to a layer i, set `Colour`$[v]$ to red if i is even, otherwise to blue.
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  3. At the end of BFS, scan all the edges to check if there is any edge both of whose endpoints received the same colour.
- Running time of this algorithm is $O(n + m)$, since we do a constant amount of work per node in addition to the time spent by BFS.

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- Let G be a graph and let $L_0, L_1, L_2, \ldots L_k$ be the layers produced by BFS, starting at node s. Then exactly one of the following statements is true:
    1. No edge of G joins two nodes in the same layer:
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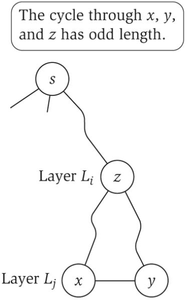

Figure 3.6 If two nodes x and y in the same layer are joined by an edge, then the cycle through x, y, and their lowest common ancestor z has odd length, demonstrating that the graph cannot be bipartite.