Greedy Algorithms

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February 12, 14, 19, 2024

Algorithm Design

- Start discussion of different ways of designing algorithms.
- Greedy algorithms, divide and conquer, dynamic programming.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.

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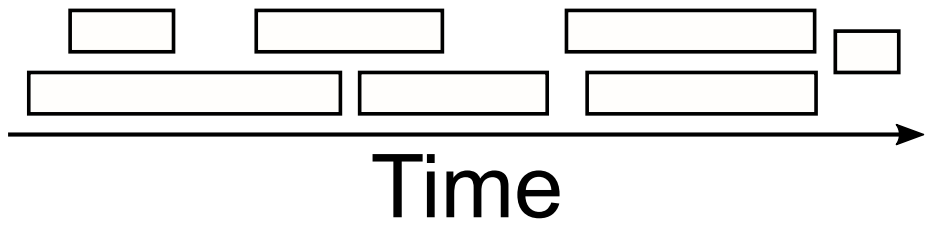
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.
- Greedy algorithms: make the current best choice.

- Input: Start and end time of each ride.
- Constraint: Cannot be in two places at one time.
- Goal: Compute the largest number of rides you can be on in one day.

Interval Scheduling



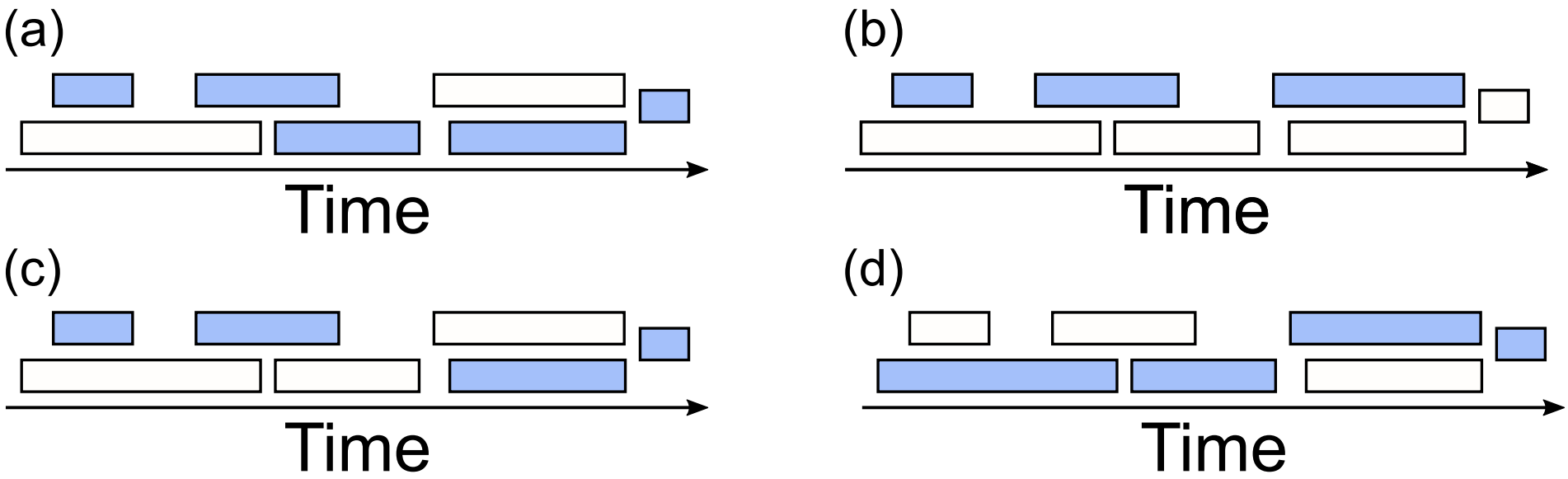
Interval Scheduling

INSTANCE: Set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION: The largest subset of mutually compatible jobs.

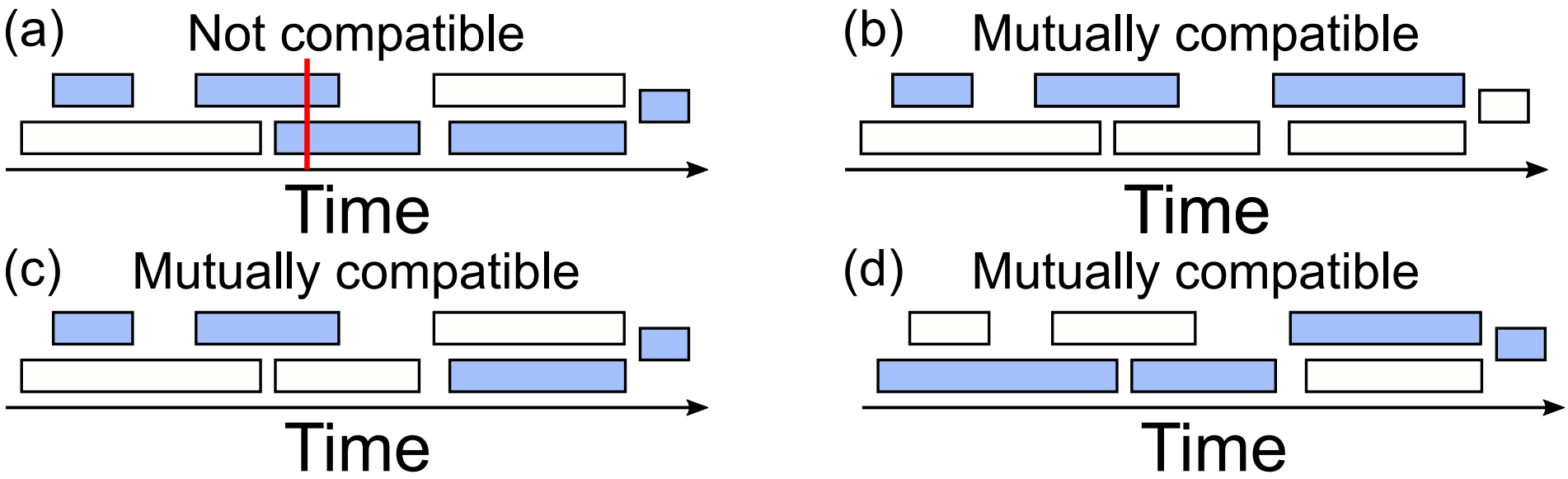
- Two jobs are *compatible* if they do not overlap.
- This problem models the situation where you have a resource, a set of fixed jobs, and you want to schedule as many jobs as possible.
- For any input set of jobs, algorithm must provably compute the largest set of compatible jobs.

Interval Scheduling Example



▸ Lectures 7-9: Greedy scheduling: Interval Scheduling Examples

Interval Scheduling Example



- Solutions (c) and (d) are optimal.
  - Each contains four jobs.
  - No set of mutually compatible jobs can contain more than four jobs.

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  Shortest interval Increasing order of length $f(i) - s(i)$.
  Fewest conflicts Increasing order of the number of conflicting jobs.

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▸ Lectures 7-9: Greedy scheduling: Sub-optimal algorithms

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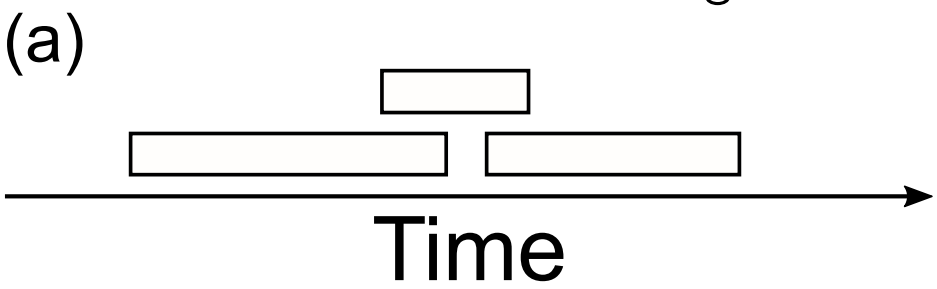

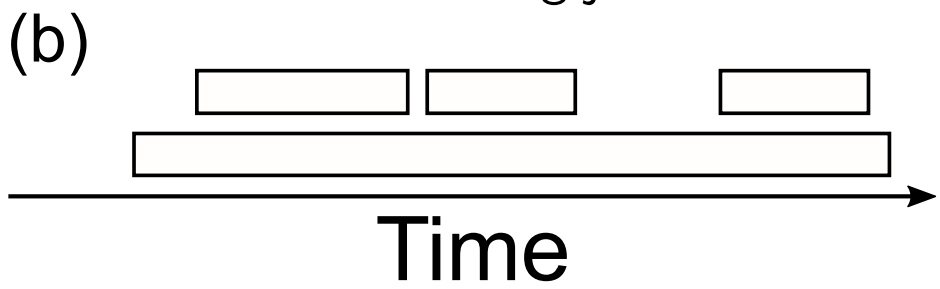

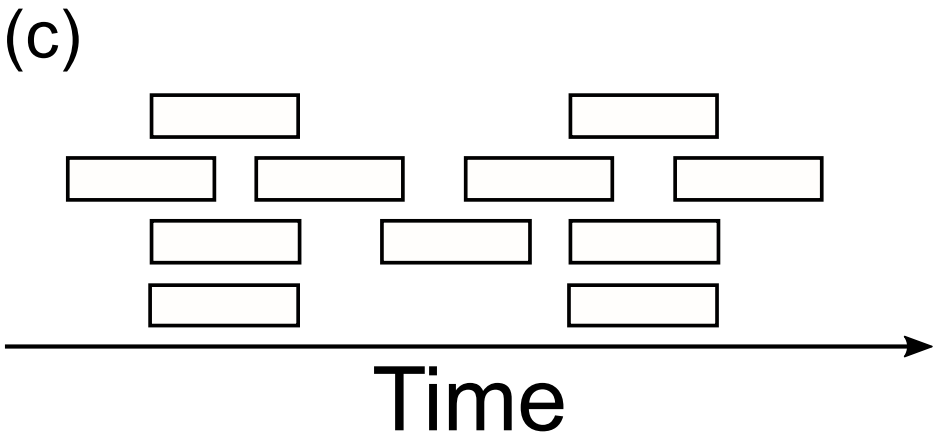

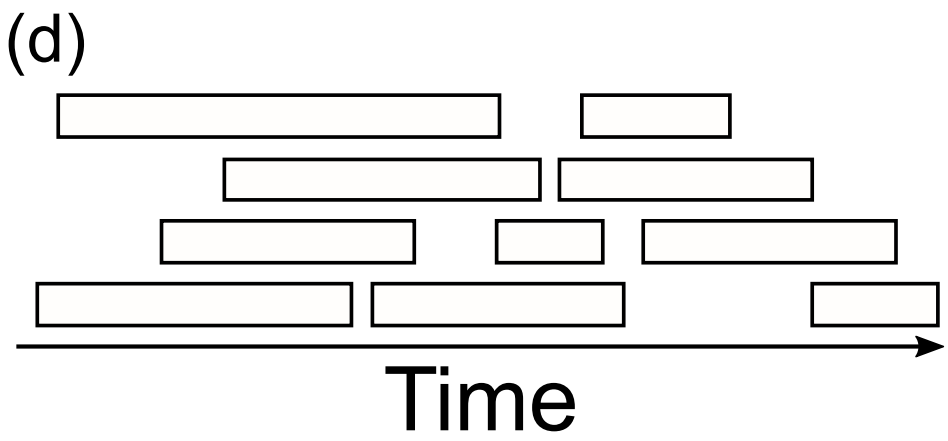

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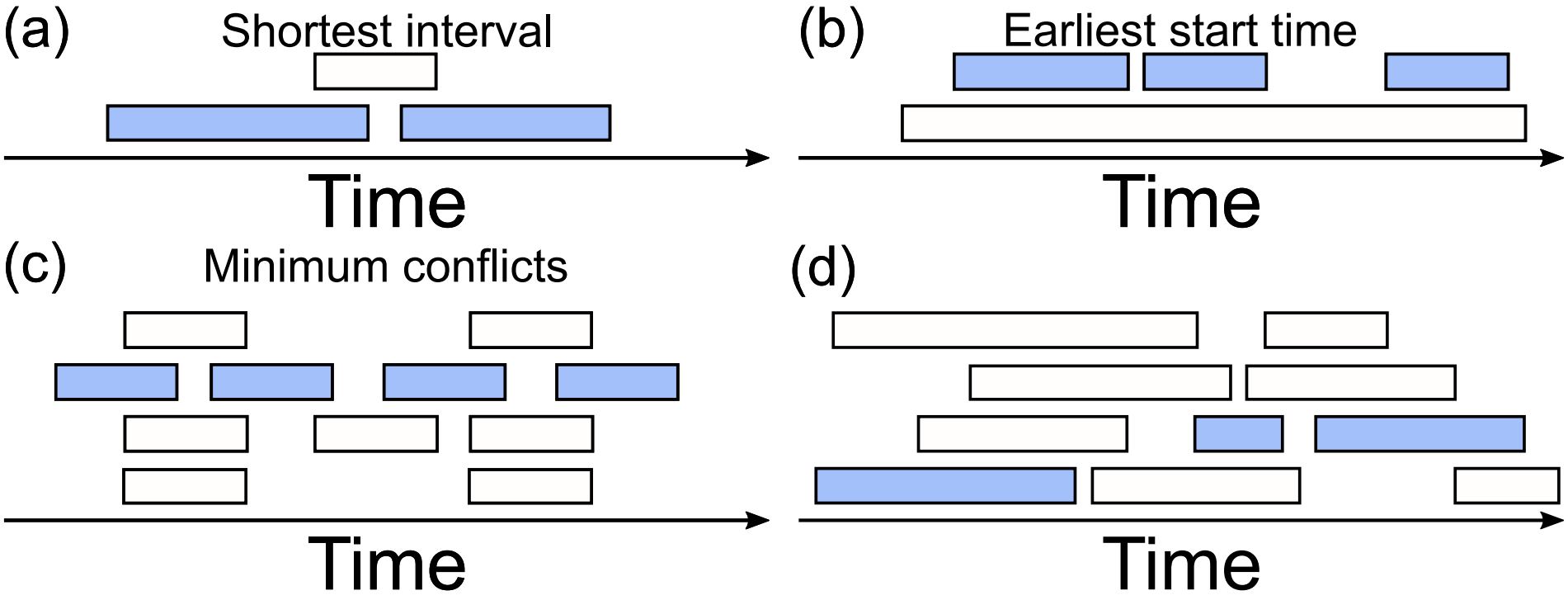

Interval Scheduling Algorithm: Earliest Finish Time

- Schedule jobs in order of earliest finish time (EFT).

Initially let R be the set of all requests, and let A be empty
While R is not yet empty
  Choose a request $i \in R$ that has the smallest finishing time
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- Claim: A is a compatible set of jobs. Proof follows by construction, i.e., the algorithm computes a compatible set of jobs.

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- Proof idea 2: at each step, can we show algorithm has the "better" solution than any other answer?
  - What does "better" mean?
  - How do we measure progress of the algorithm?
- Basic idea of proof:
  - We can sort jobs in any solution in increasing order of their finishing time.
  - Finishing time of job number r selected by $A \leq$ finishing time of job number r selected by any other algorithm.

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- Let $i_1, i_2, \ldots, i_k$ be the set of jobs in A in order.
- Let $j_1, j_2, \ldots, j_m$ be the set of jobs in O in order, $m \geq k$.
- Claim: For all indices $r \leq k$, $f(i_r) \leq f(j_r)$.

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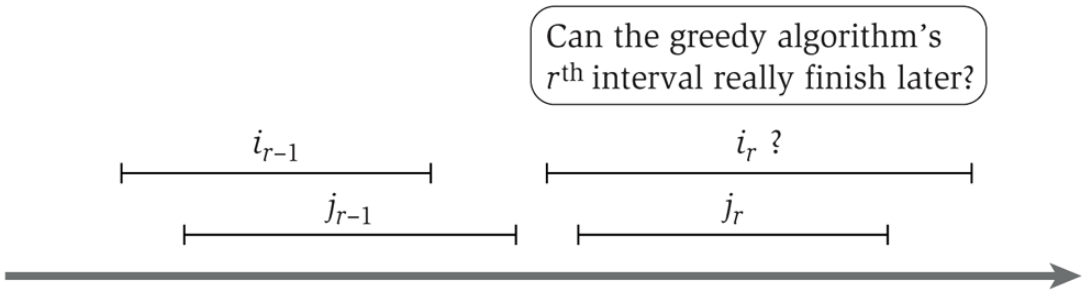

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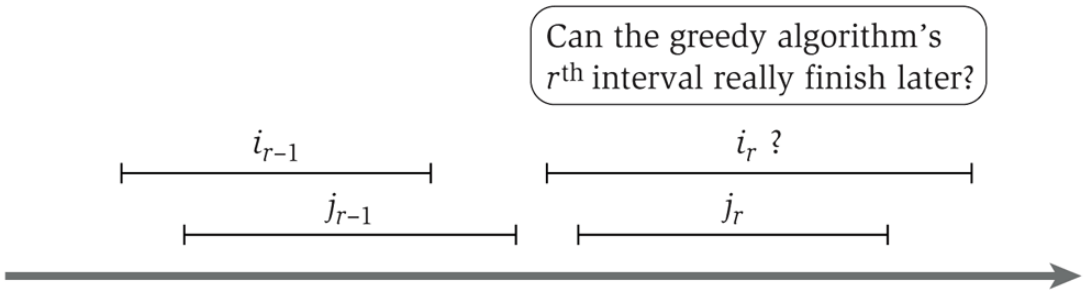

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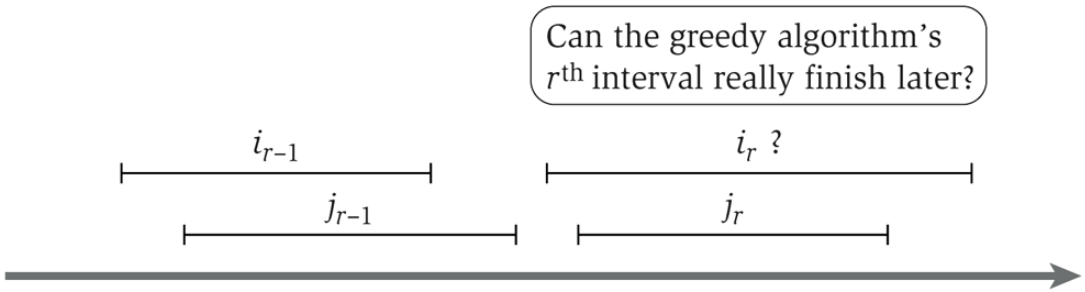
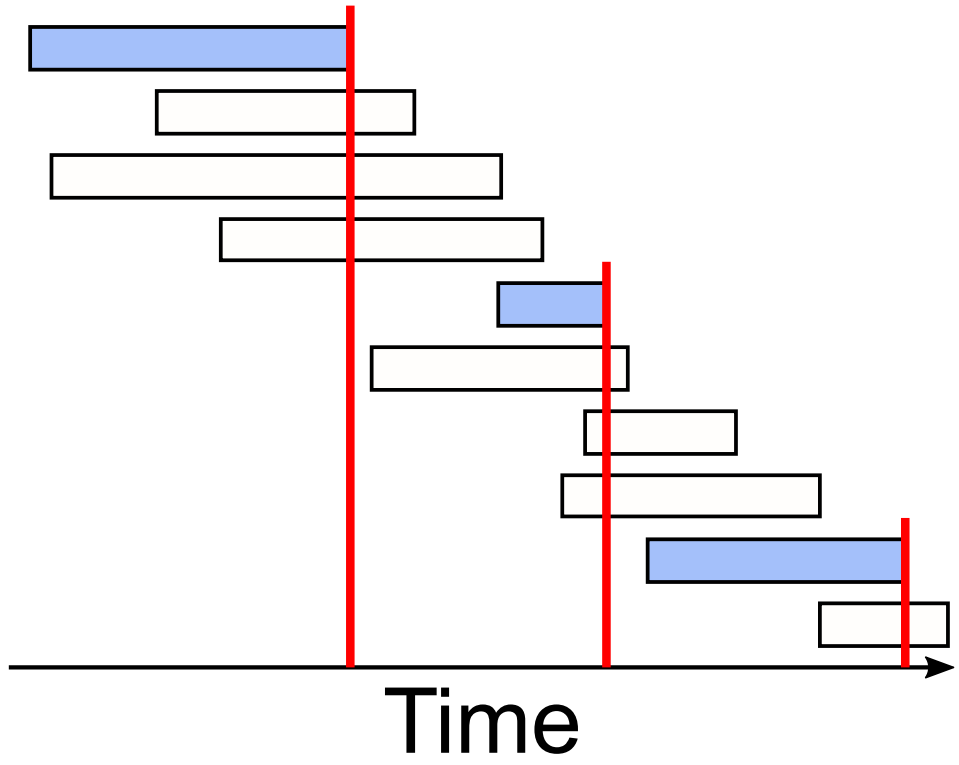


Figure 4.3 The inductive step in the proof that the greedy algorithm stays ahead.

- Claim: $m = k$.
- Claim: The greedy algorithm returns an optimal set A.

Implementing the EFT Algorithm

1. Reorder jobs so that they are in increasing order of finish time.
2. Store starting time of jobs in an array S.
3. $k = 1$.
4. While $k \leq |S|$,
   1. Output job k.
   2. Let finish time of job k be f.
   3. Iterate over S from index k onwards to find the first index i such that $S[i] \geq f$.
   4. $k = i$



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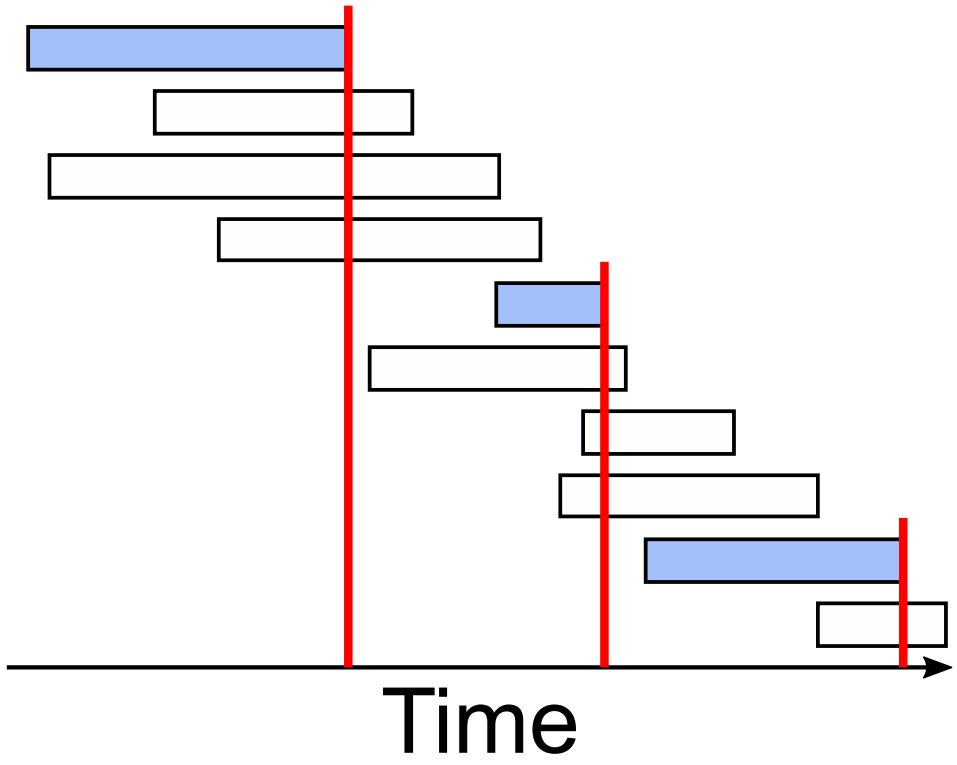

- Must be careful to iterate over S such that we never scan same index more than once.
- Running time is $O(n \log n)$, dominated by sorting.

12616	CS-4104	Data and Algorithm Analysis	L	3	75	CA Shaffer	T R	2:00PM	3:15PM	SURGE 107	14T
18154	CS-4104	Data and Algorithm Analysis	L	3	70	TM Murali	M W	2:30PM	3:45PM	SURGE 104C	14M
12617	CS-4114	Formal Languages	L	3	75	L Zhang	T R	9:30AM	10:45AM	MCB 129	09T
18155	CS-4204	Computer Graphics	L	3	36	D Gracanin	T R	11:00AM	12:15PM	MCB 224	11T
19593	CS-4264	Principles Computer Security	L	3	50	KE Giles	M W	2:30PM	3:45PM	GOODW 135	14M
12618	CS-4284	Systems & Networking Capstone	L	3	40	GV Back	M W	2:30PM	3:45PM	MCB 238	14M
18156	CS-4304	Compiler Design	L	3	50	C Jung	T R	8:00AM	9:15AM	GOODW 125	08T
12620	CS-4604	Int Data Base Mgt Sys	L	3	55	RJ Quintin	M W	4:00PM	5:15PM	SURGE 109	16M
12621	CS-4624	Multimedia/Hypertext	L	3	70	EA Fox	T R	3:30PM	4:45PM	SURGE 109	15T
12622	CS-4644	Creative Computing Studio	L	3	25	SR Harrison	W	2:30PM	5:15PM	MAC 253A	14W
Comments for CRN 12622:		Prerequisite: C or better in CS 3724 OR CS 3744									
12623	CS-4654	Intermed Data Analytics & ML	L	3	50	RB Gramacy	M W	4:00PM	5:15PM	SEITZ 313	16M
12624	CS-4704	Software Engineering Capstone	L	3	15	KR Edmison	M W	4:00PM	5:15PM	NCB 170	16M
Comments for CRN 12624:		Prerequisite: C or better in CS 3704 OR CS 3714									
12625	CS-4784	Human-Computer Interact Capstn	L	3	30	AL Kavanaugh	F	1:00PM	3:45PM	MAC 253A	13F
Comments for CRN 12625:		Prerequisite: CS 3724 required; CS 3714 or 3744 recommended									
19924	CS-4784	Human-Computer Interact Capstn	L	3	0	DS McCrickard	F	12:30PM	3:15PM	MCB 655	12F

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- Input: Start and end time of each class.
- Constraint: Cannot schedule two overlapping classes to the same room.
- Output: Assign each class to a room and use smallest number of rooms possible.

Interval Partitioning

Interval Partitioning

INSTANCE: Set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION: A partition of the jobs into k sets, where each set of jobs is mutually compatible, and k is minimised.

- This problem models the situation where you a set of fixed jobs, and you want to schedule all jobs using as few resources as possible.

Depth of Intervals

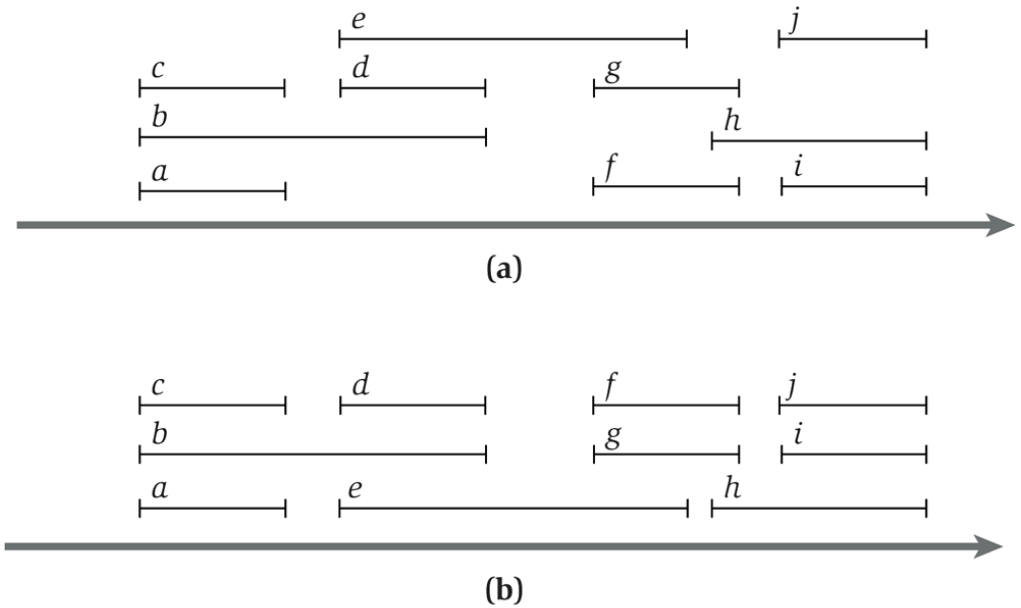

Figure 4.4 (a) An instance of the Interval Partitioning Problem with ten intervals (a through j). (b) A solution in which all intervals are scheduled using three resources: each row represents a set of intervals that can all be scheduled on a single resource.

- The *depth* of a set of intervals is the maximum number of intervals that contain any time point.

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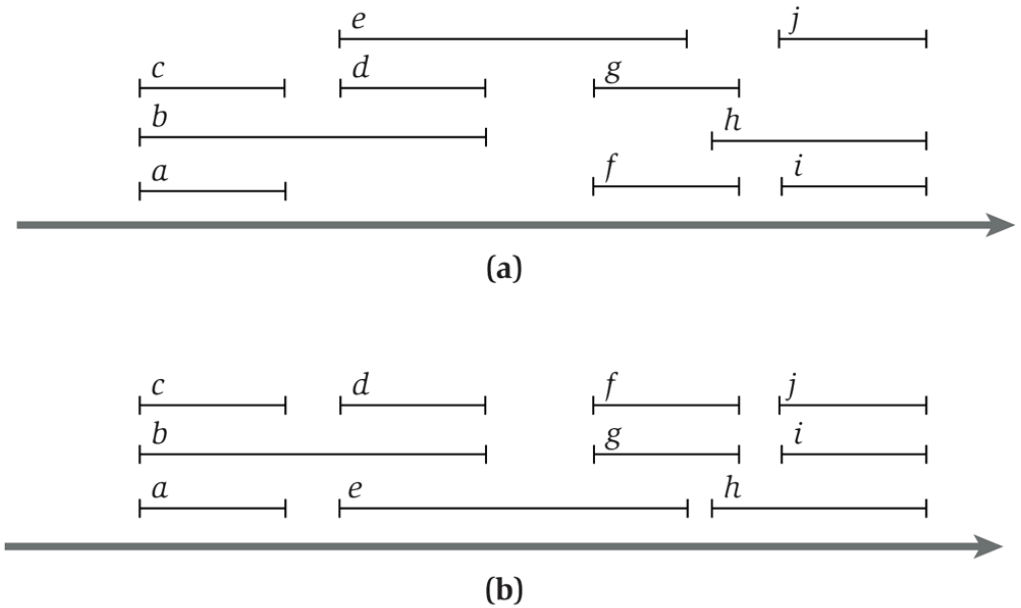

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- Claim: In any instance of INTERVAL PARTITIONING, $k \geq$ depth.

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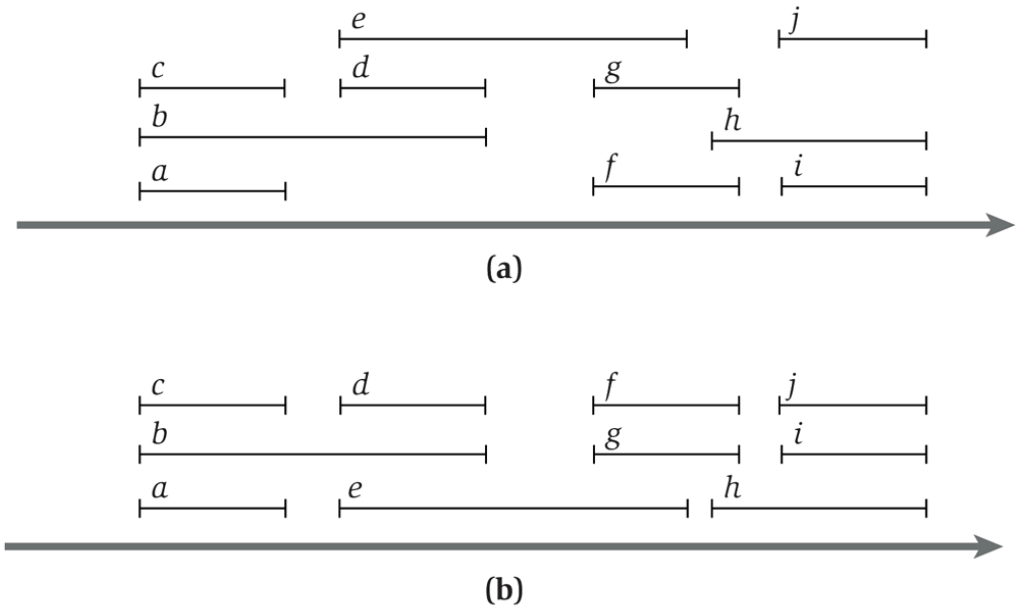

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- The *depth* of a set of intervals is the maximum number of intervals that contain any time point.
- Claim: In any instance of INTERVAL PARTITIONING, $k \geq$ depth.
- Is it possible to compute the depth efficiently? Is $k =$ depth?

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1. Sort the start times and finish times of the jobs into a single list L.
2. $d \leftarrow 0$.
3. For i ranging from 1 to $2n$
   1. If L_i is a start time, increment d by 1.
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- Algorithm runs in $O(n \log n)$ time.

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Sort the intervals by their start times, breaking ties arbitrarily
Let I_1, I_2, ..., I_n denote the intervals in this order
For j = 1, 2, 3, ..., n
  For each interval I_i that precedes I_j in sorted order and overlaps it
     Exclude the label of I_i from consideration for I_j
  Endfor
  If there is any label from {1, 2, ..., d} that has not been excluded then
    Assign a nonexcluded label to I_j
  Else
    Leave I_j unlabeled
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- Claim: Every interval gets a label and no pair of overlapping intervals get the same label.

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- Claim: Every interval gets a label and no pair of overlapping intervals get the same label.
- Claim: The greedy algorithm is optimal.
- The running time of the algorithm is $O(n \log n)$. Can modify algorithm for computing depth to maintain set of available labels and to assign them efficiently.

Scheduling to Minimise Lateness

- Study different model: job i has a length $t(i)$ and a deadline $d(i)$.
- We want to schedule all n jobs on one resource.
- Our goal is to assign a starting time $s(i)$ to each job such that each job is delayed as little as possible.

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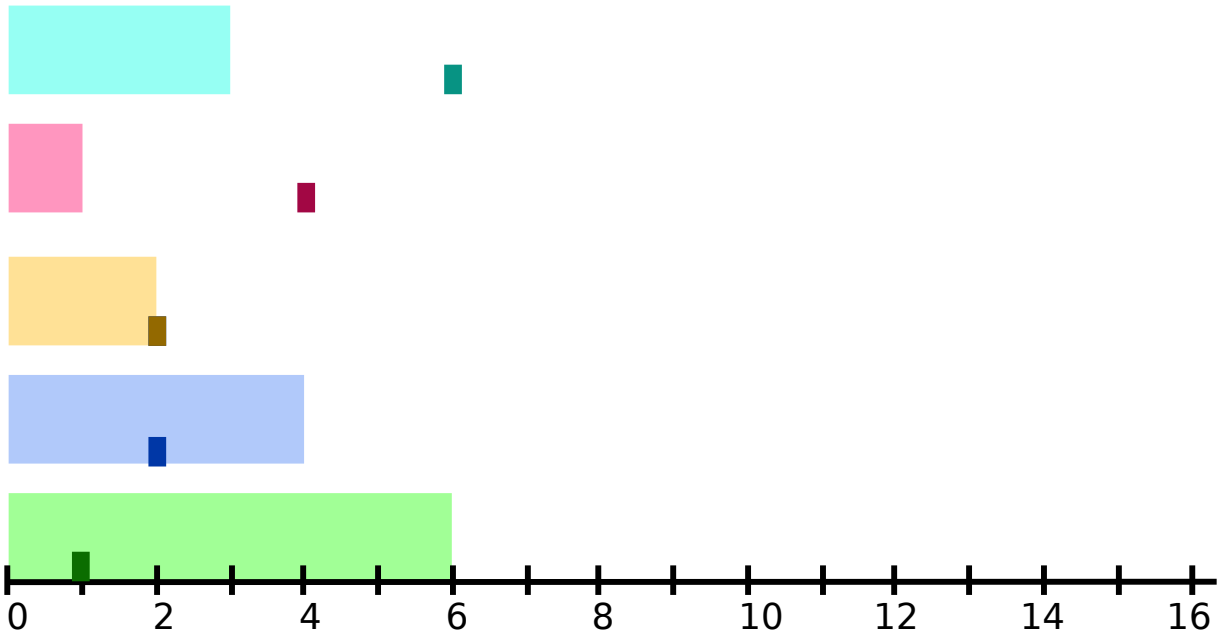
- A job i is *delayed* if $f(i) > d(i)$; the *lateness of the job* is

$$\max(0, f(i) - d(i)).$$

- The *lateness of a schedule* is

$$\max_{1 \leq i \leq n} \left(\max \left(0, f(i) - d(i)\right)\right).$$

Examples of Lateness



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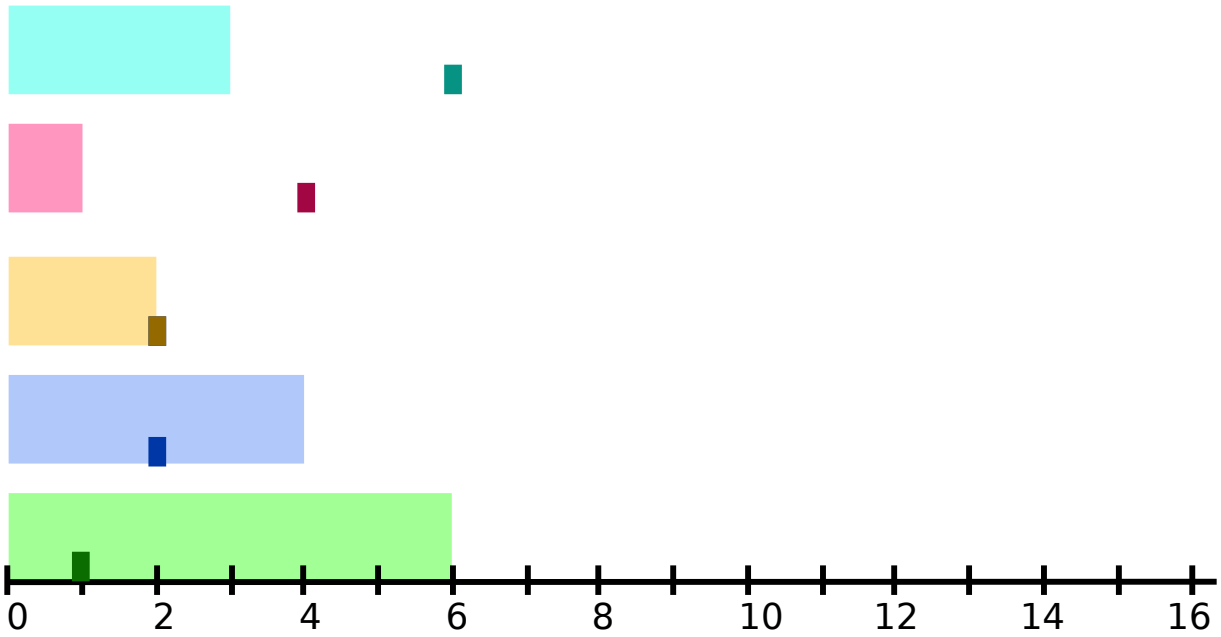

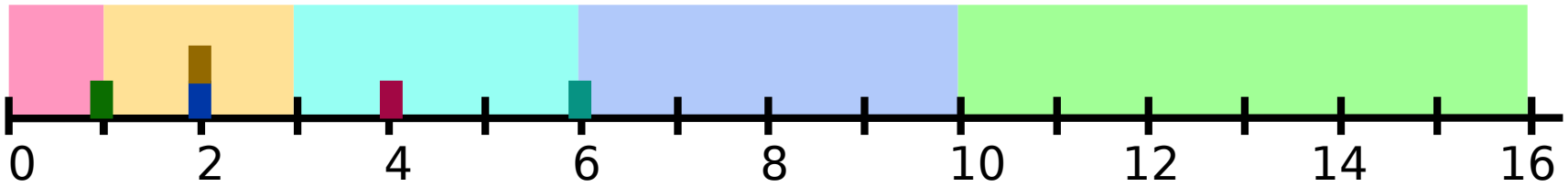

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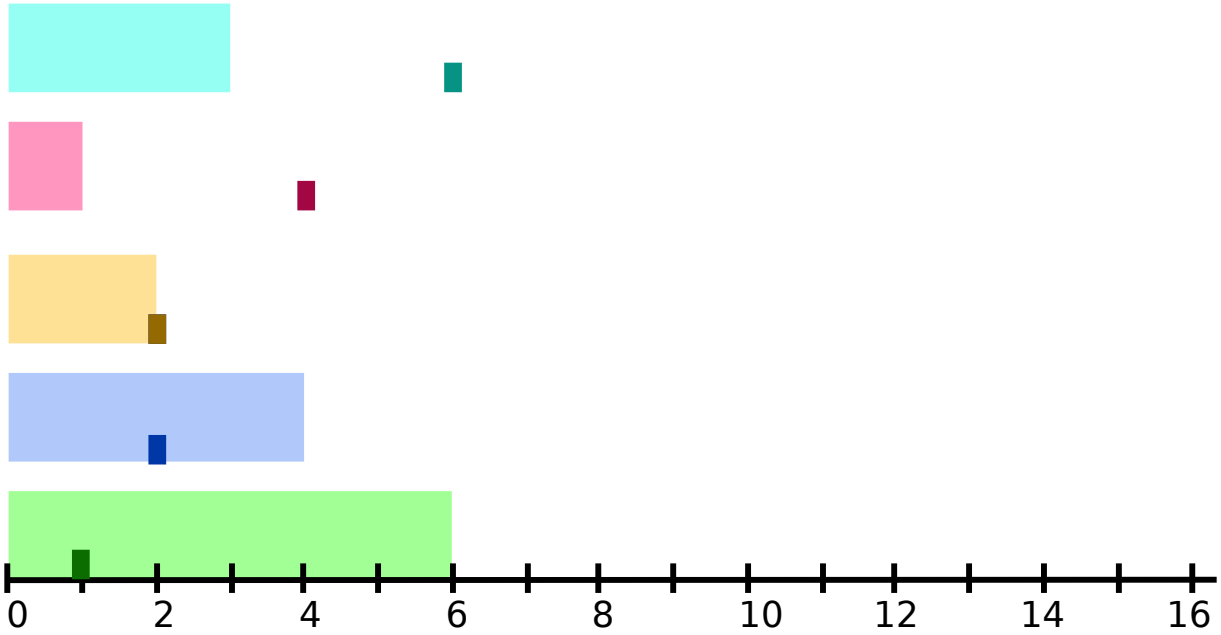

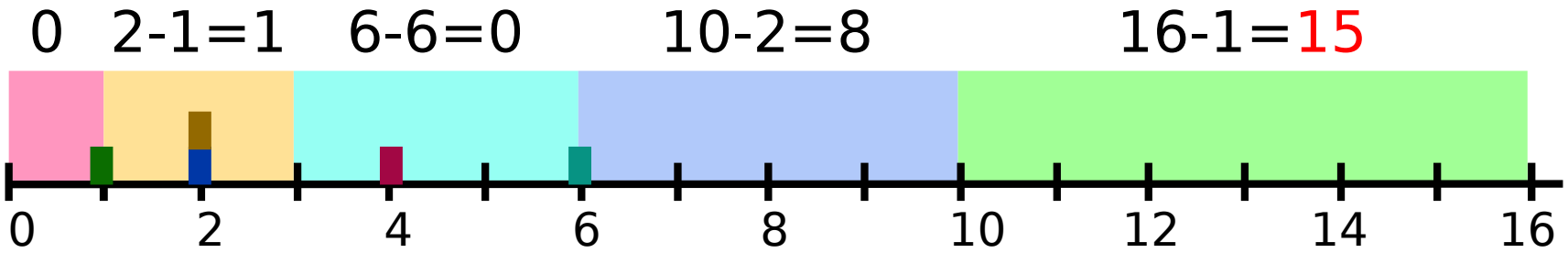

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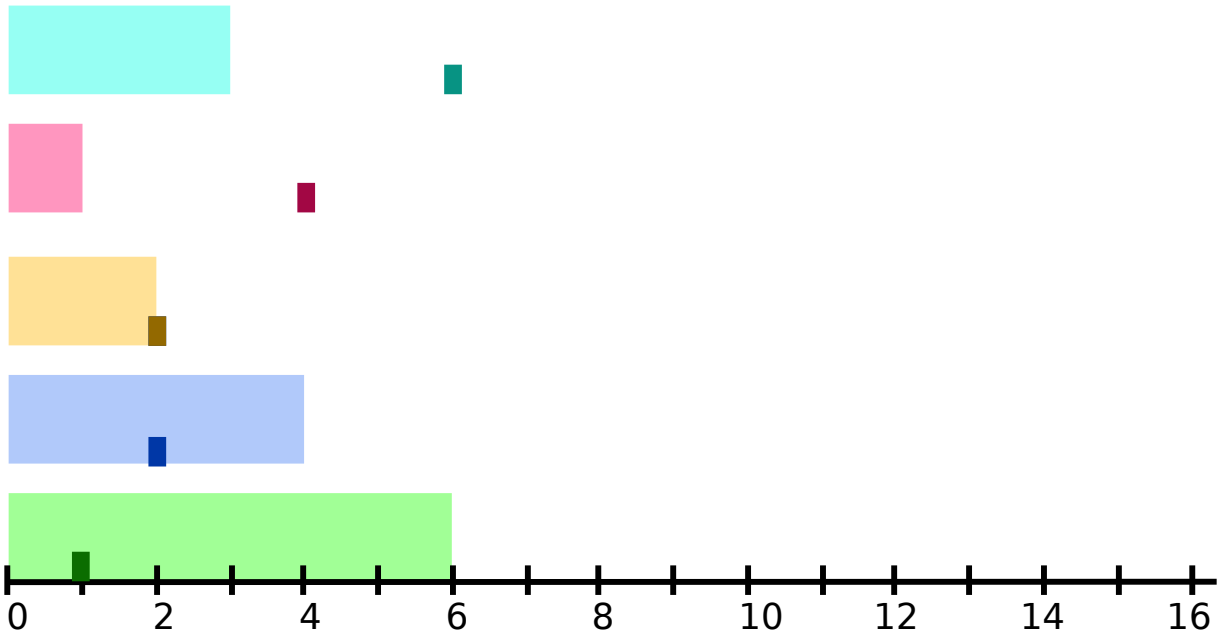
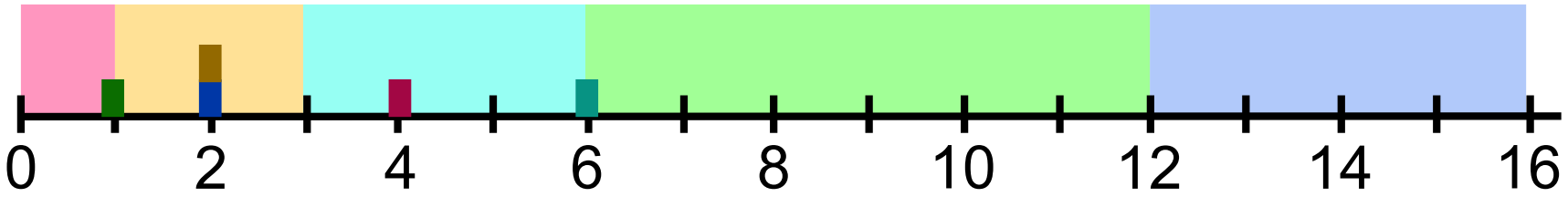
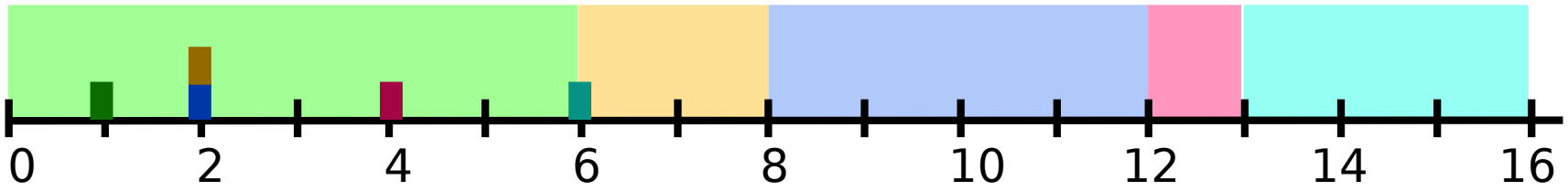


▸ Lectures 7-9: Greedy scheduling: Lateness example 2





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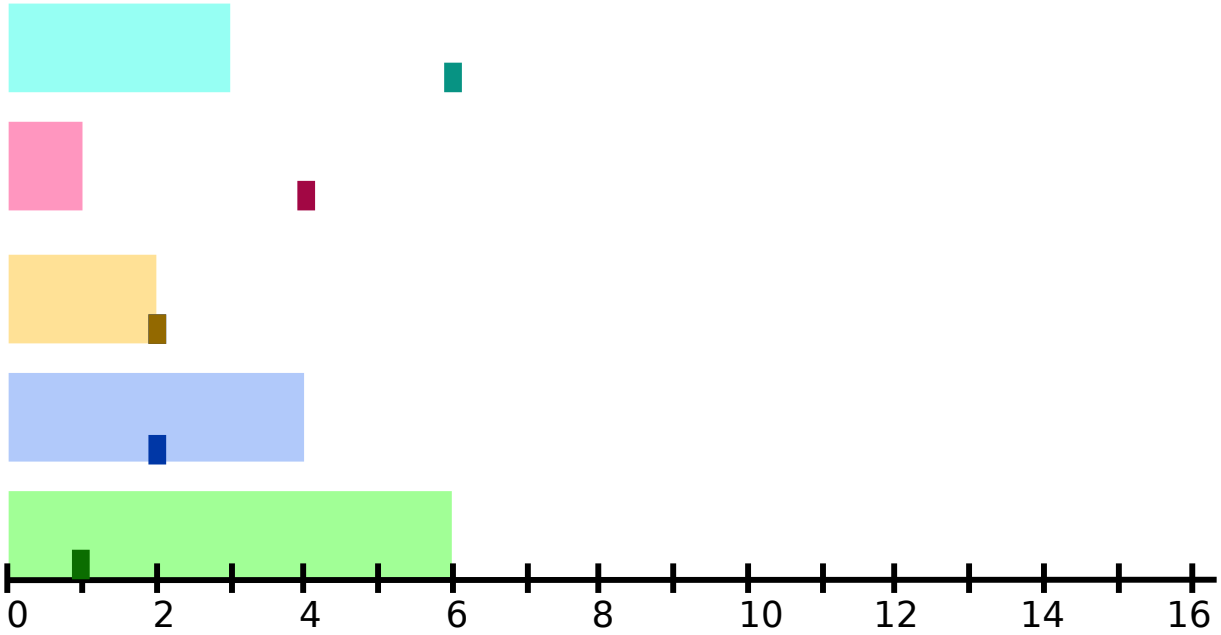
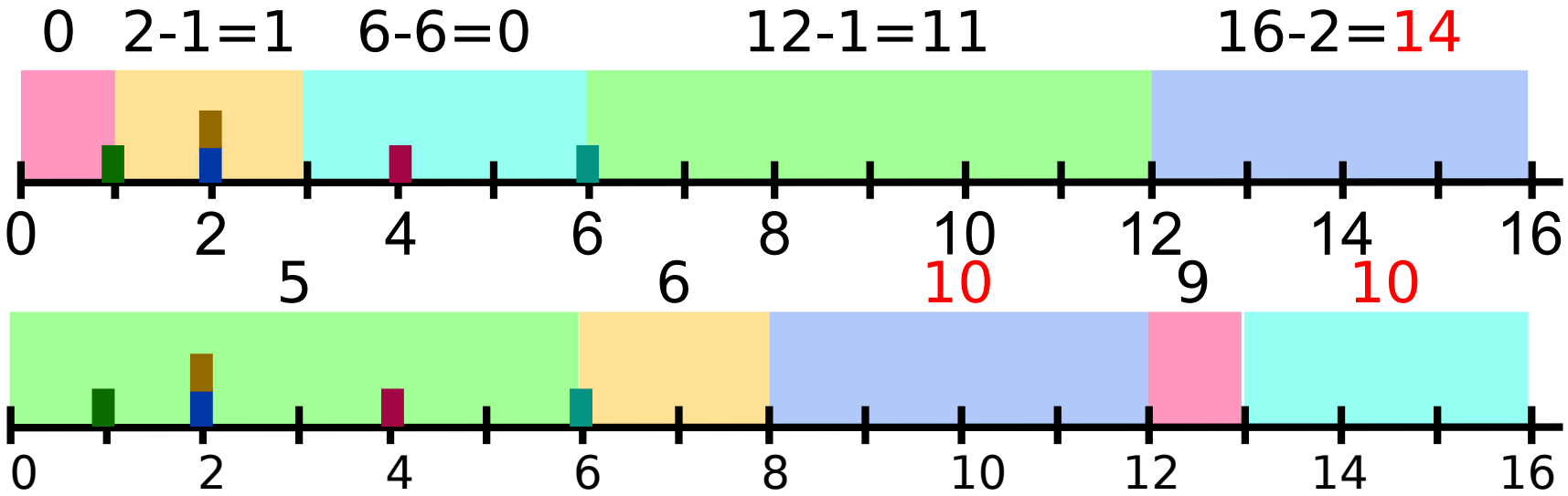


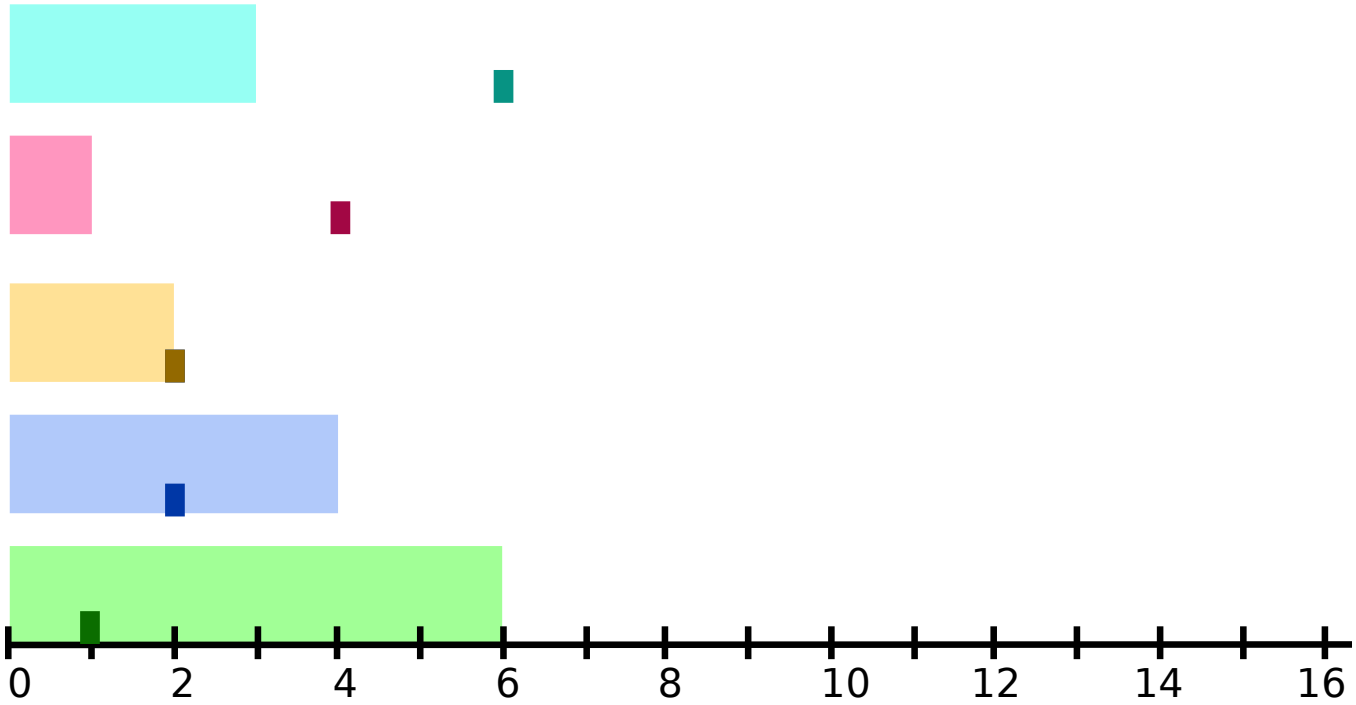


Scheduling to Minimise Lateness

Minimise Lateness

INSTANCE: Set $\{(t(i), d(i)), 1 \leq i \leq n\}$ of lengths and deadlines of n jobs.

SOLUTION: Set $\{s(i), 1 \leq i \leq n\}$ of start times such that $\max_{1 \leq i \leq n} \left(\max \left(0, s(i) + t(i) - d(i)\right)\right)$ is as small as possible.



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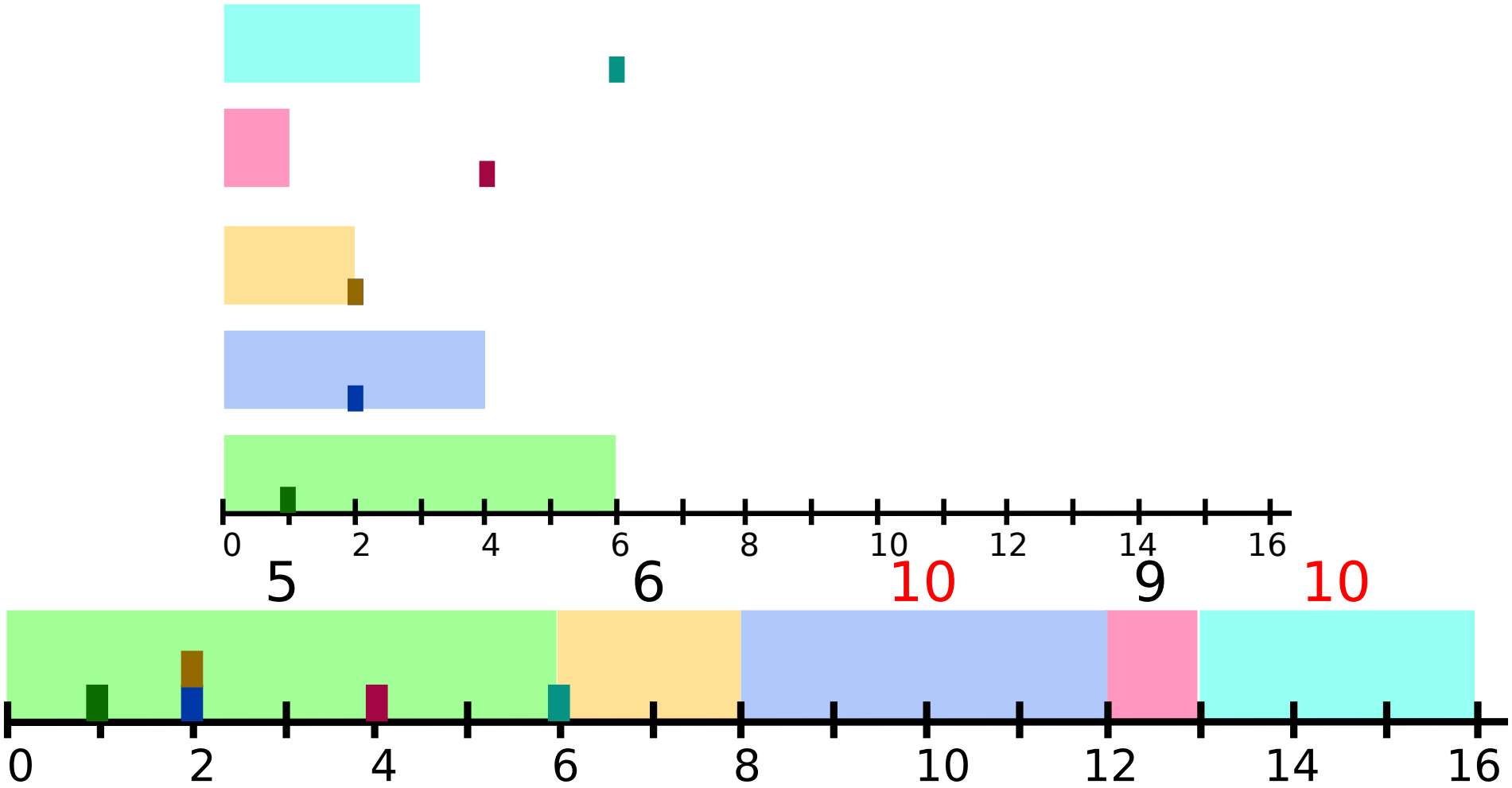

Template for Greedy Algorithm

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  Shortest job may have a very late deadline.

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  Earliest deadline Increasing order of deadline $d(i)$. Correct? Does it make sense to tackle jobs with earliest deadlines first?

Minimising Lateness: Earliest Deadline First

Order the jobs in order of their deadlines

Assume for simplicity of notation that $d_1 \leq \ldots \leq d_n$

Initially, $f = s$

Consider the jobs $i = 1, \ldots, n$ in this order

  Assign job i to the time interval from $s(i) = f$ to $f(i) = f + t_i$

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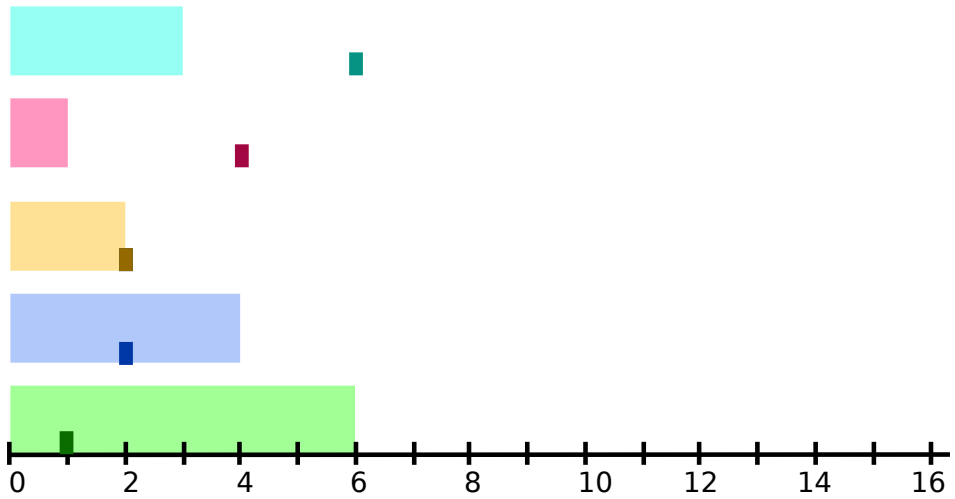
End

Return the set of scheduled intervals $[s(i), f(i)]$ for $i = 1, \ldots, n$



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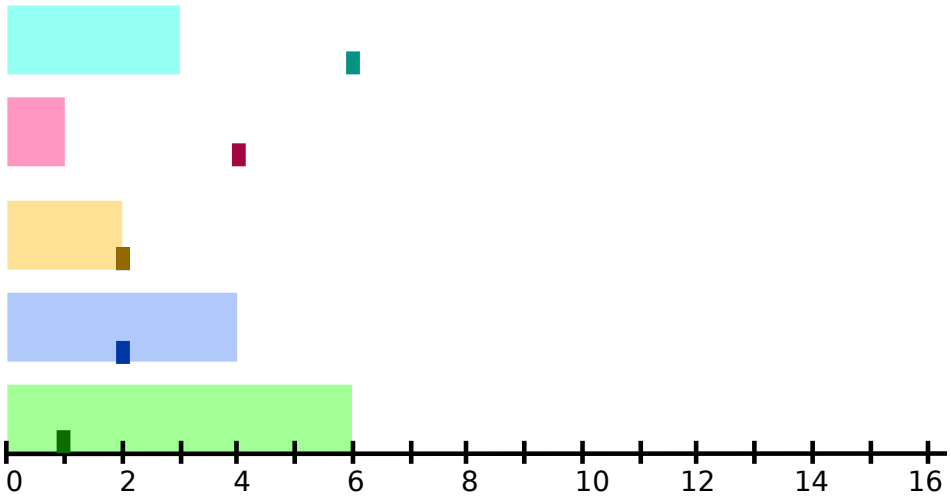

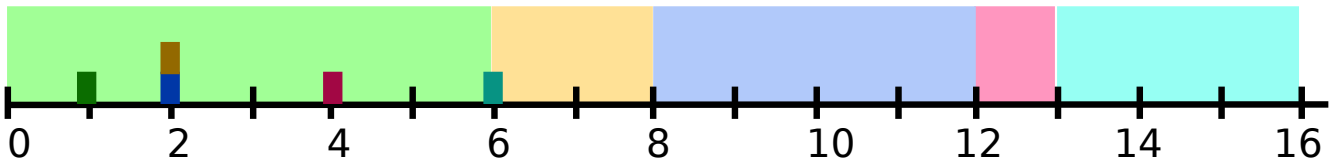

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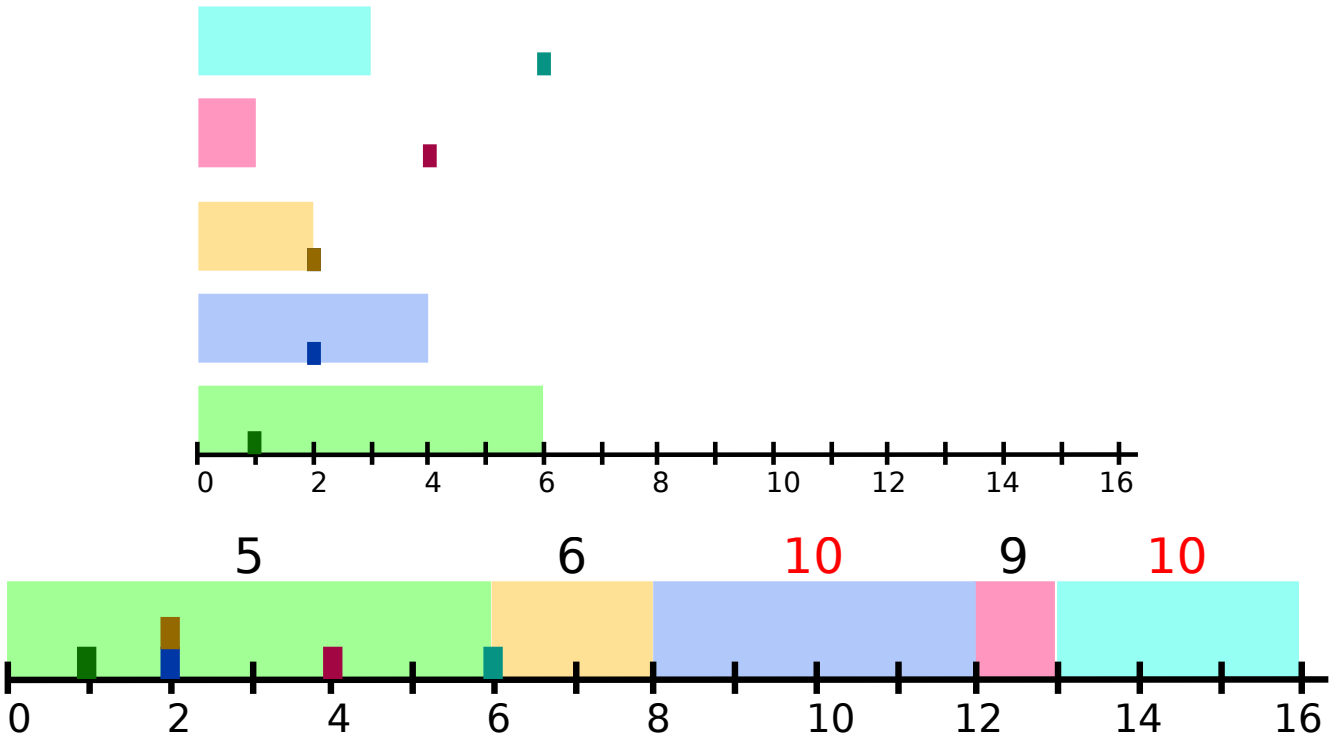
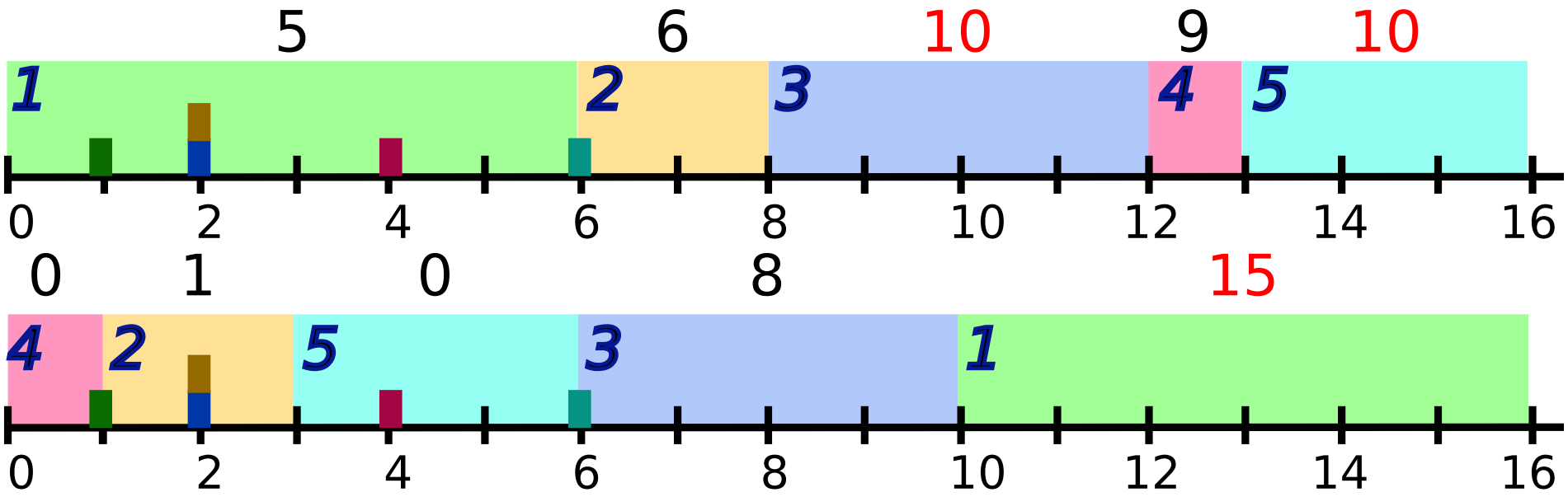


Inversions



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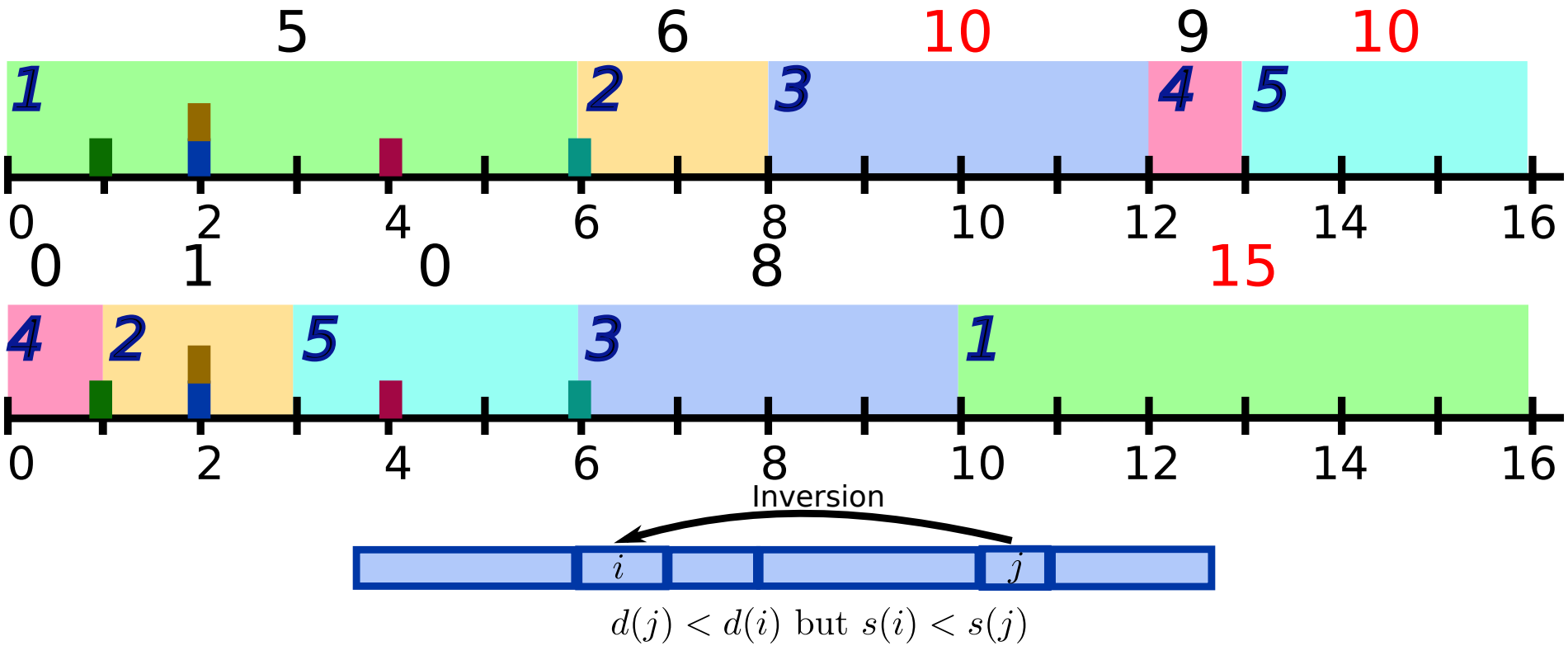
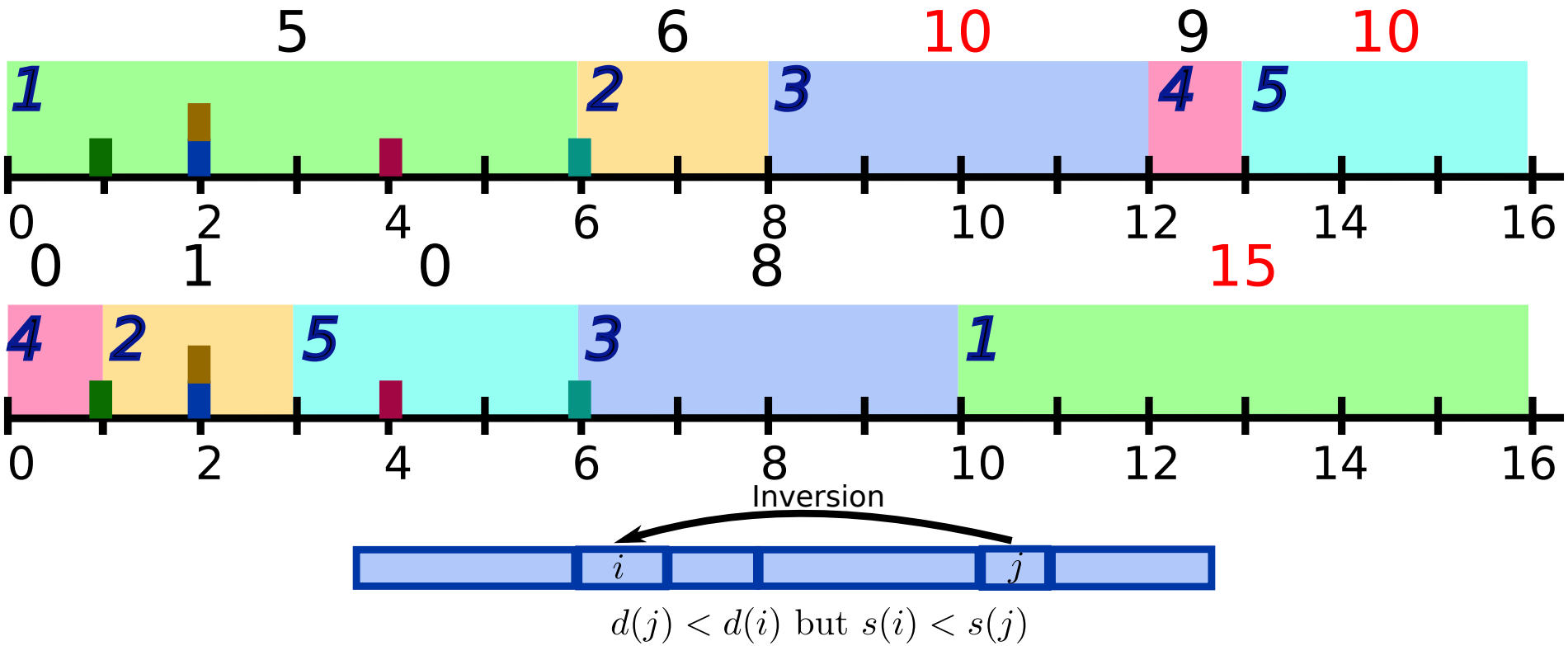


$d(j) < d(i)$ but $s(i) < s(j)$

- A schedule has an *inversion* if a job i with deadline $d(i)$ is scheduled before a job j with an earlier deadline $d(j)$, i.e., $d(j) < d(i)$ and $s(i) < s(j)$.
  - If i and j have the same deadlines, they cannot cause an inversion.
  - Examples: ▸ Lectures 7-9: Greedy scheduling: Inversions in Example

Inversions



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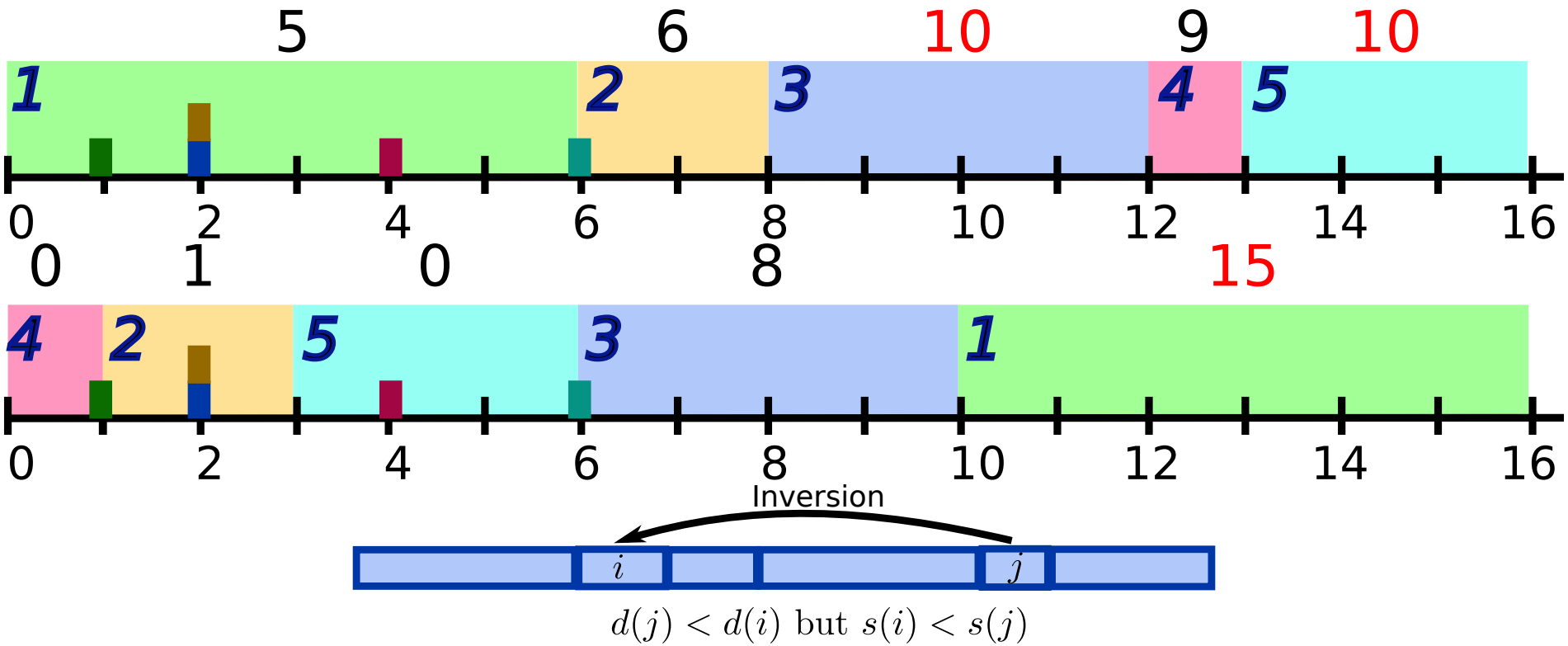

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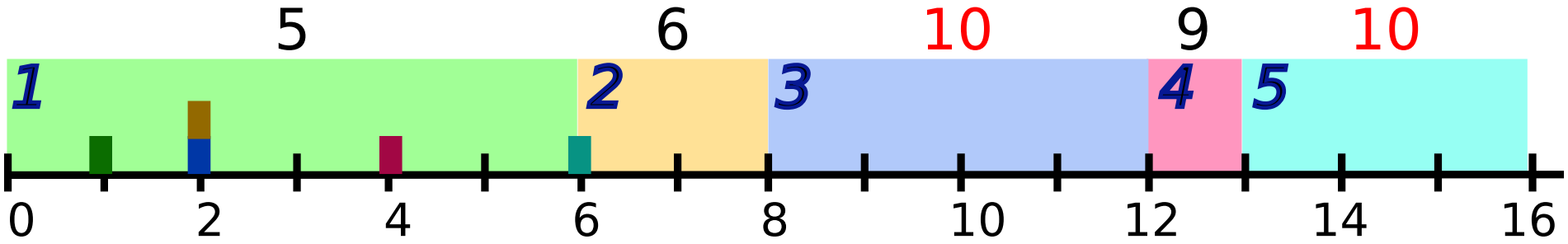
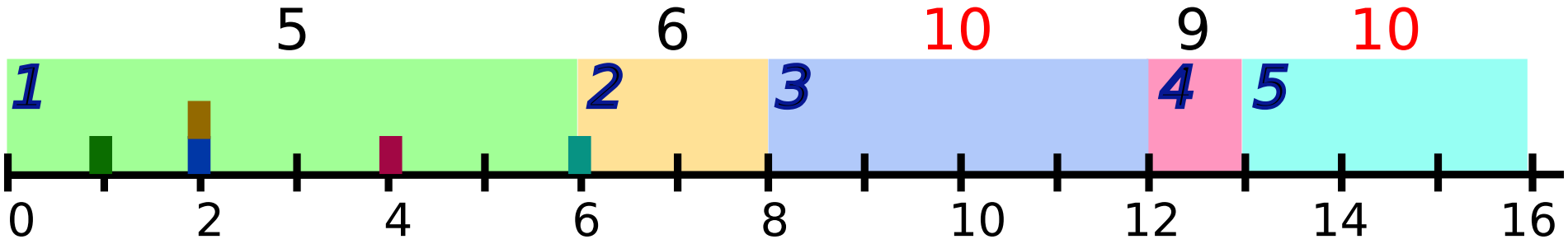
  - Examples: 2 and 1, 3 and 1, 4 and 1, 5 and 1, 4 and 2, 4 and 3, 5 and 3.
- Claim: If a schedule has an inversion, then there is a pair of *consecutive* jobs with an inversion, i.e., there are jobs i and j such that j is scheduled immediately after i and $d(j) < d(i)$.

Properties of Schedules



- Claim 1: The algorithm produces a schedule with no inversions and no idle time.

Properties of Schedules



- Claim 1: The algorithm produces a schedule with no inversions and no idle time.
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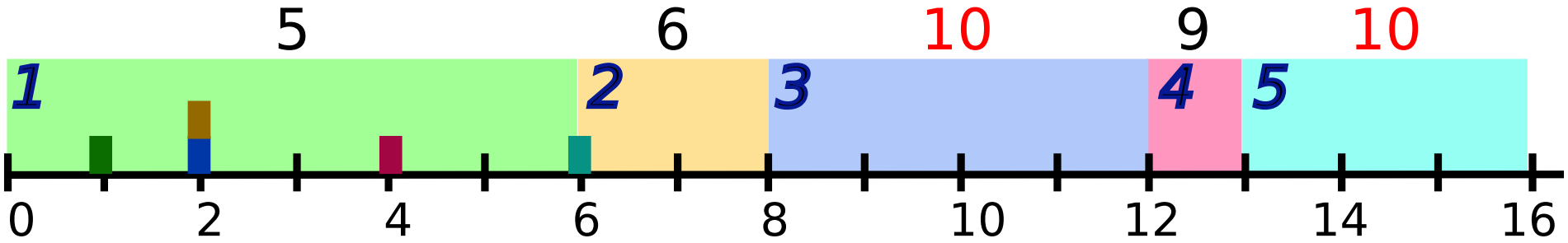

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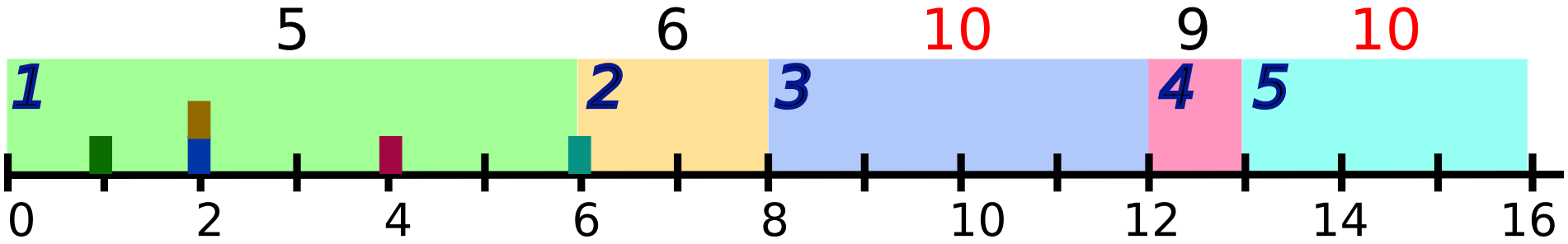
    - Lectures 7-9: Greedy scheduling: Number of schedules with no inversion

Properties of Schedules



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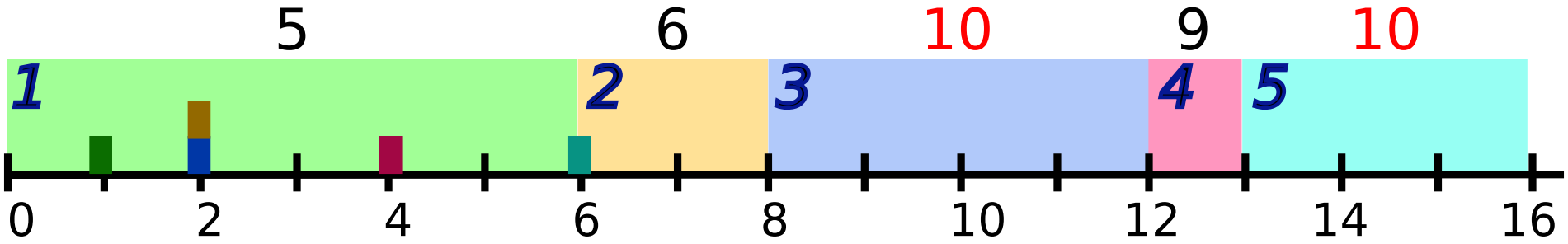

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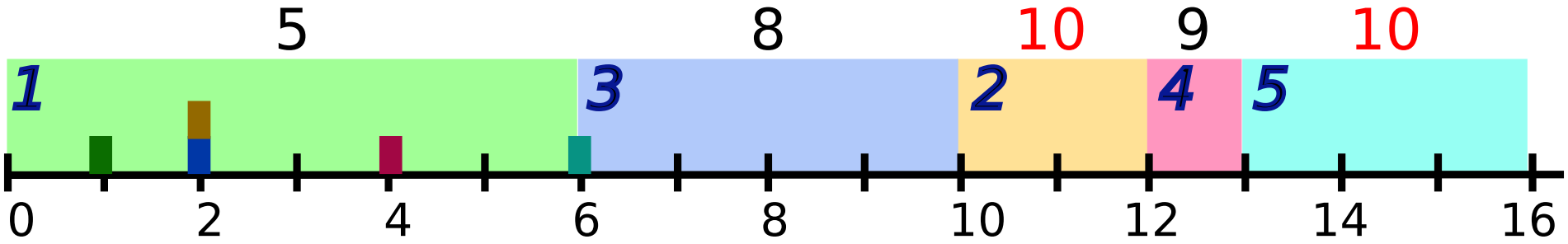

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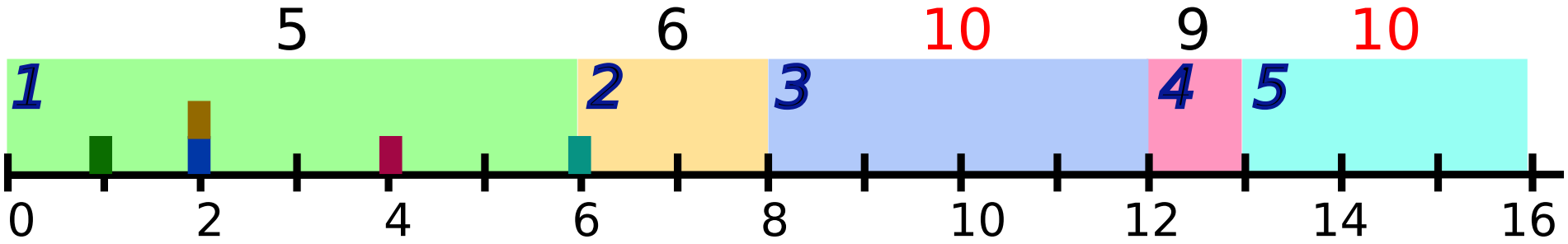

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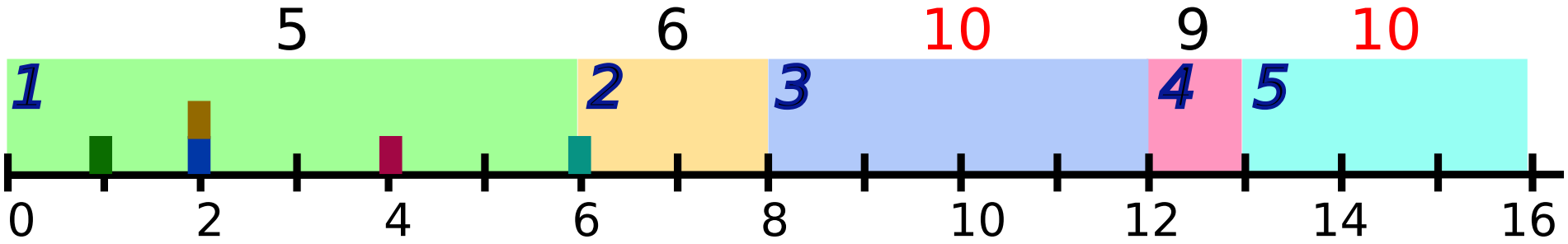

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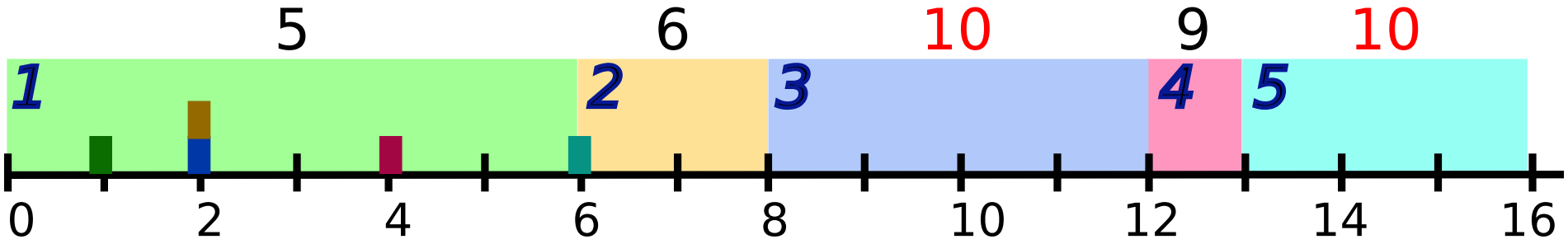

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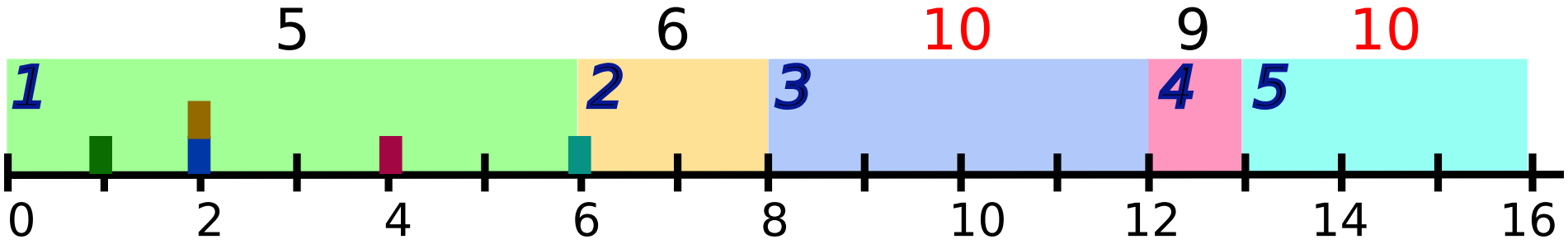

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- Claim 5: The greedy algorithm produces an optimal schedule.

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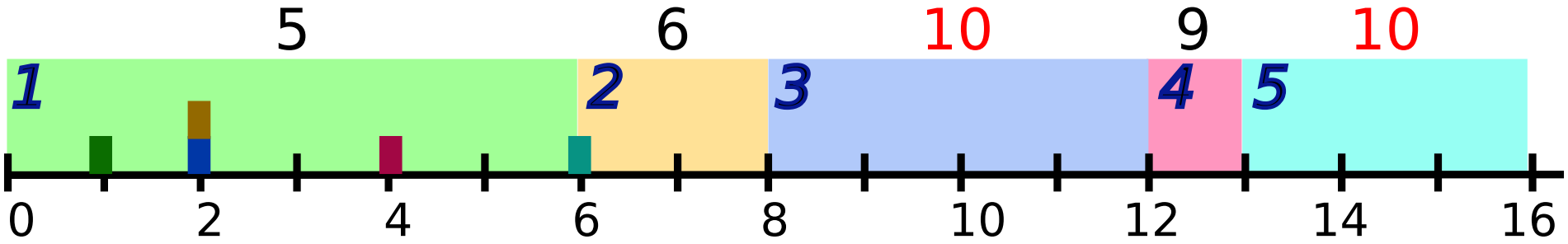

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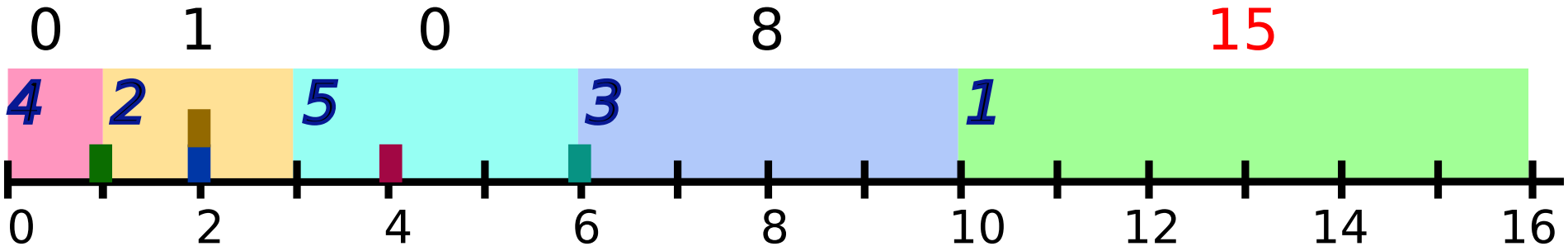
- Claim 5: The greedy algorithm produces an optimal schedule. Follows from Claims 1, 2 and 4.

Proving Claim 4



- Claim 4: There is an optimal schedule with no inversions and no idle time.

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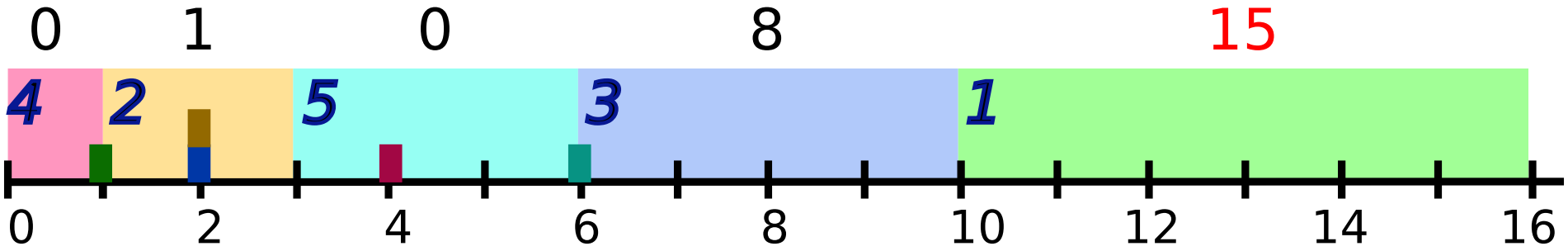

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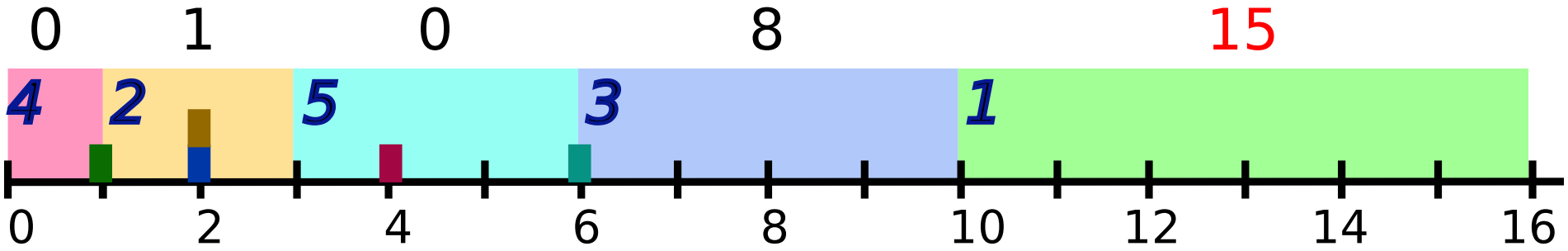

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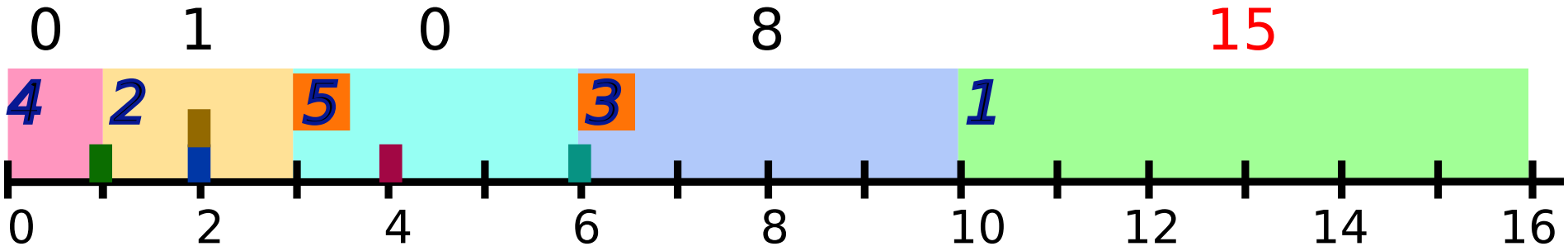

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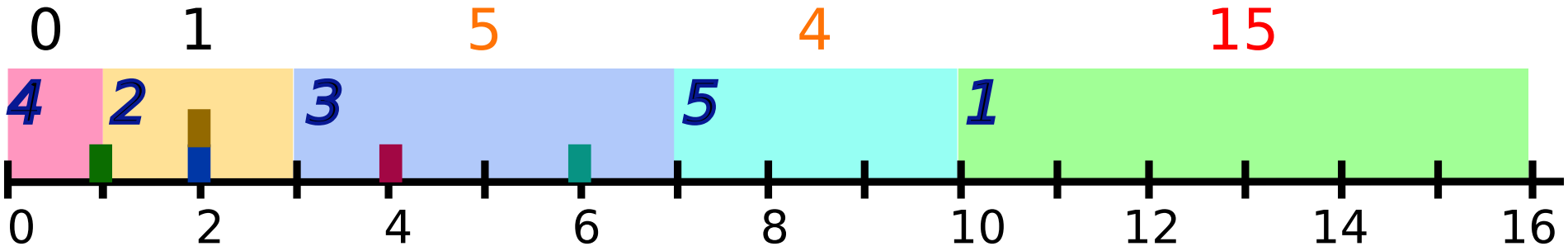
  1. If O has an inversion, let i and j be consecutive inverted jobs in O. After swapping i and j, we get a schedule O' with one less inversion.

Proving Claim 4



- Claim 4: There is an optimal schedule with no inversions and no idle time.
- Proof: Start with an optimal schedule O (that may have inversions) and use an *exchange argument* to convert O into a schedule that satisfies Claim 4 and has lateness not larger than O.
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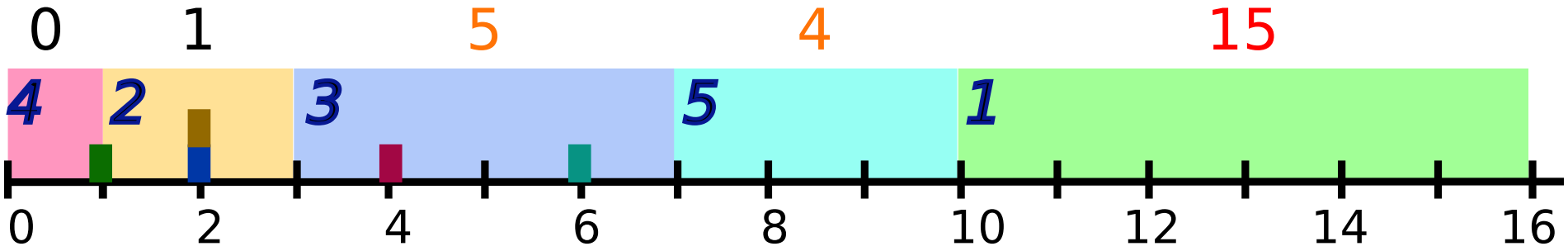

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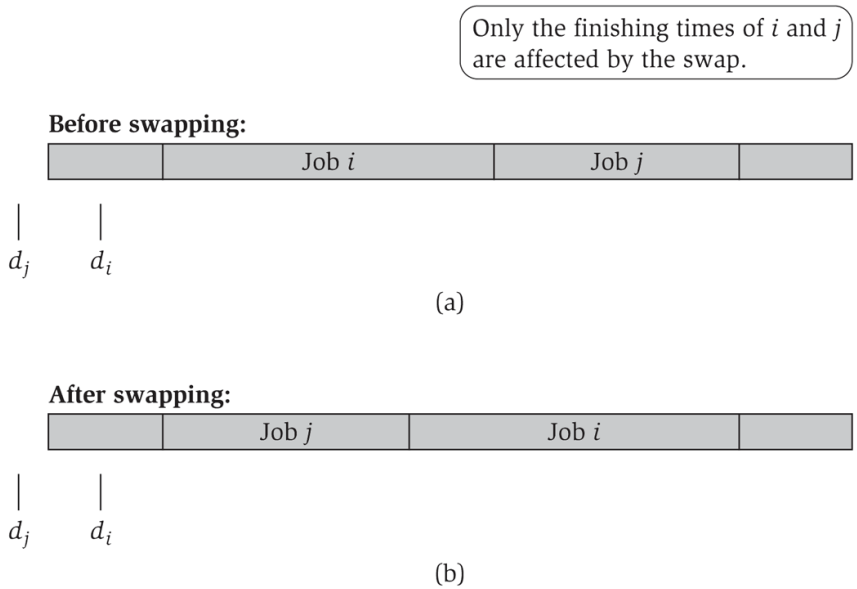
- It is enough to prove the last item, since after $\binom{n}{2}$ swaps, we obtain a schedule with no inversions whose lateness is no larger than that of O.

Swapping Inverted Jobs



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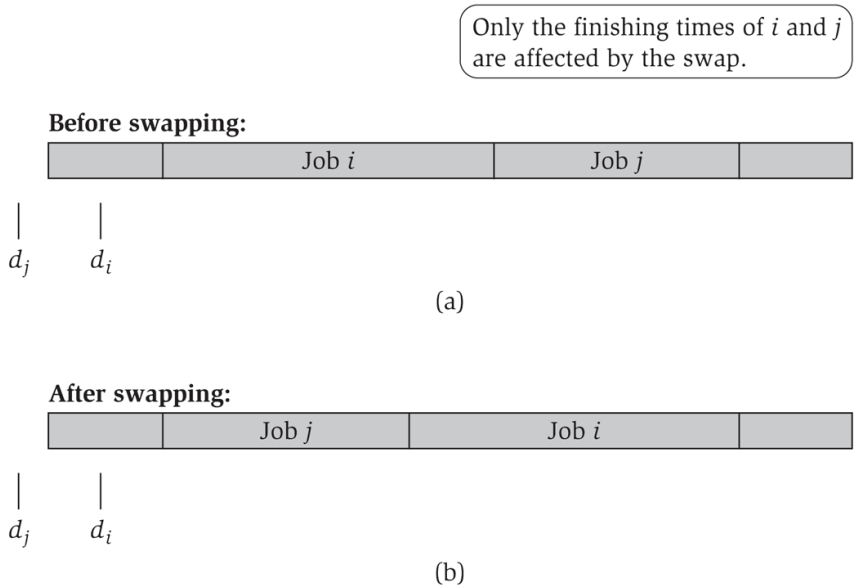

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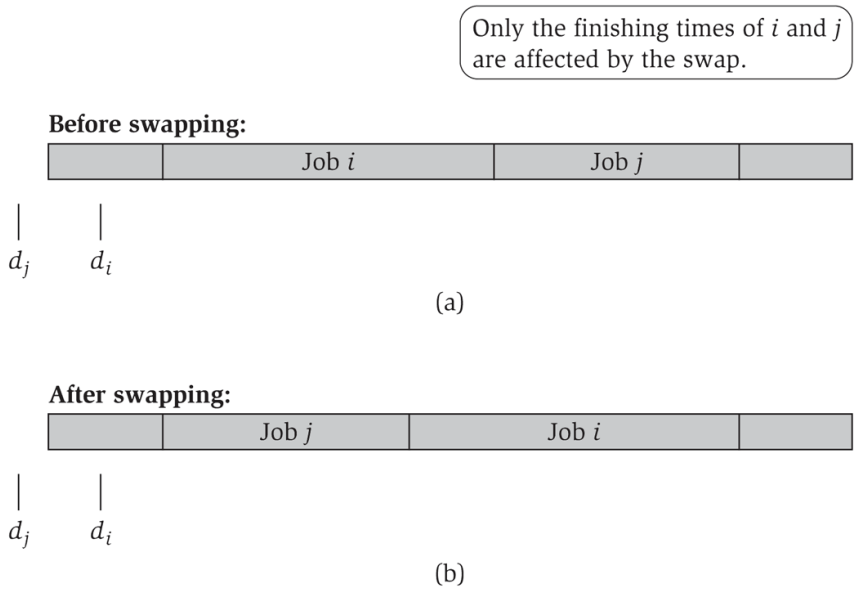

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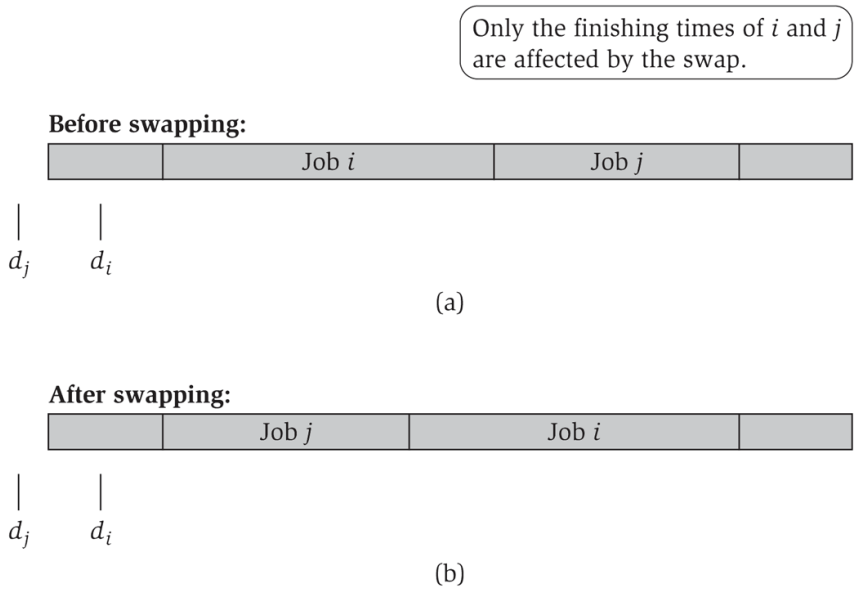

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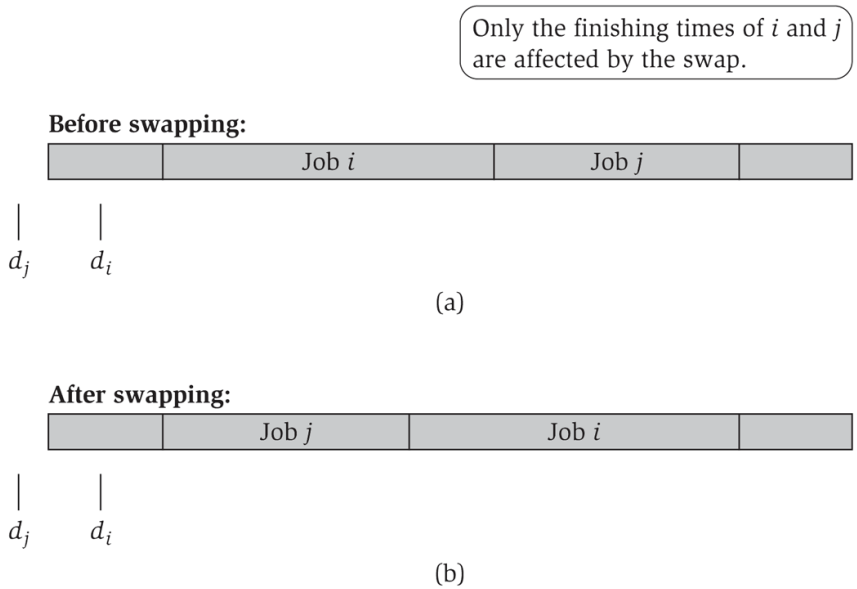

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- Claim: $l'(j) \leq l(j)$.
- Claim: $l'(i) \leq l(j)$ because $l'(i) = f(j) - d_i \leq f(j) - d_j = l(j)$.

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8. Repeat one more step: X_0 has no inversions. What is X_0's location? $(0, l_X)$ or "below" because of #7 and $(0, l_A)$ because of #3.
9. We have a contradiction!
10. Lateness of A cannot be larger than that of O!

Common Mistakes with Exchange Arguments

- Wrong: start with algorithm's schedule A and argue that A cannot be improved by swapping two jobs.
- Correct: Start with an arbitrary schedule O (which can be the optimal one) and argue that O can be converted into the schedule that is essentially the same as the one the algorithm produces without increasing the lateness.

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- Wrong: Swap two jobs that are not neighbouring in O. Pitfall is that the completion times of all intervening jobs changes.
- Correct: Show that an inversion exists between two neighbouring jobs and swap them.

Summary

- Greedy algorithms make local decisions.
- Three analysis strategies:

  Greedy algorithm stays ahead Show that after each step in the greedy algorithm, its solution is at least as good as that produced by any other algorithm.

  Structural bound First, discover a property that must be satisfied by every possible solution. Then show that the (greedy) algorithm produces a solution with this property.

  Exchange argument Transform the optimal solution in steps into the solution by the greedy algorithm without worsening the quality of the optimal solution.