Greedy Graph Algorithms

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Algorithm Design

- Start discussion of different ways of designing algorithms.
- Greedy algorithms, divide and conquer, dynamic programming, flow-based approaches.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.

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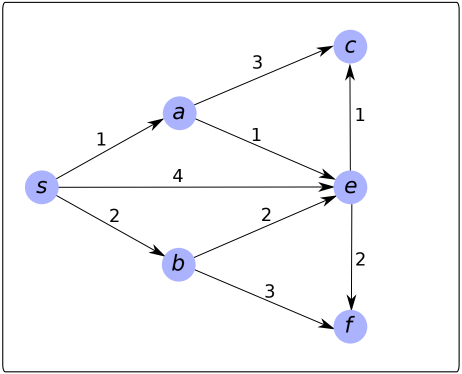
- Greedy algorithms, divide and conquer, dynamic programming, flow-based approaches.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.
- Greedy algorithms: make the current best choice.
  - First discussed greedy algorithms for scheduling (Chapters 4.1 to 4.2).
  - Now we will discuss greedy graph algorithms (Chapters 4.4 to 4.6).

Shortest Paths Problem



- $G(V, E)$ is a connected directed graph. Each edge e has a length $l(e) \geq 0$.
- *Length of a path* P is the sum of the lengths of the edges in P.

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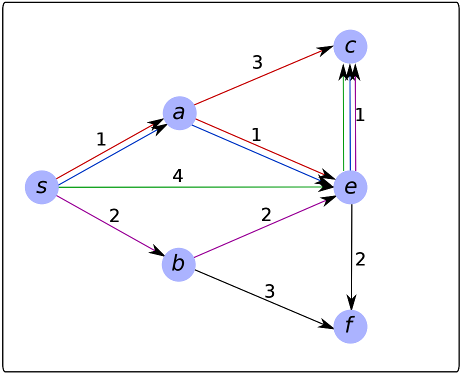

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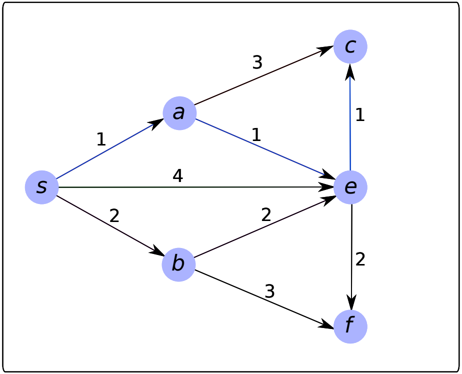

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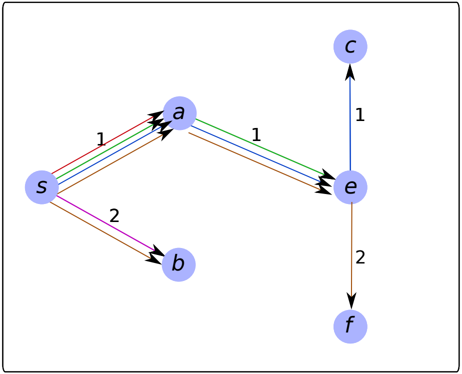

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- *Length of a path* P is the sum of the lengths of the edges in P.
- Goal: compute the shortest path from a specified start node s to each node in V.

Shortest Paths

INSTANCE: A directed graph $G(V, E)$, a function $l : E \to \mathbb{R}^+$, and a node $s \in V$

SOLUTION: A set $\{P_u, u \in V\}$ of paths, where P_u is the shortest path in G from s to u.

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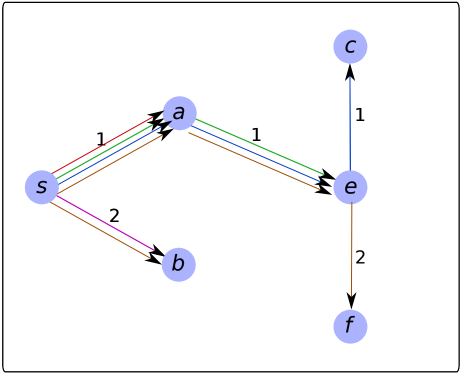

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- *Length of a path* P is the sum of the lengths of the edges in P.
- Goal: compute the shortest path from a specified start node s to each node in V.
- Aside: If G is undirected, convert to a directed graph by replacing each edge in G by two directed edges.

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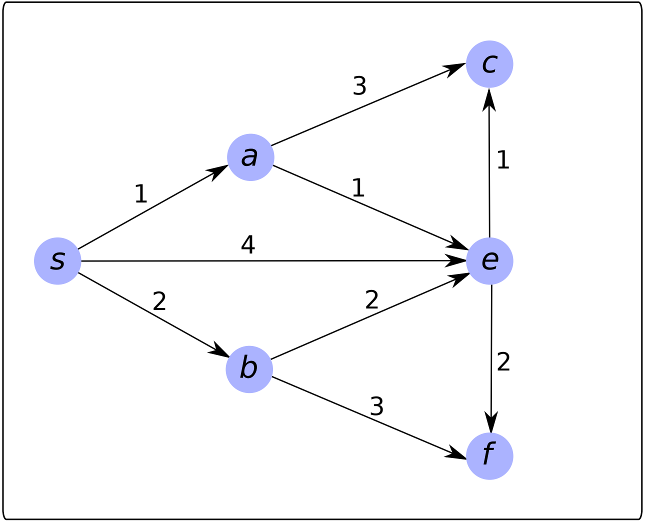
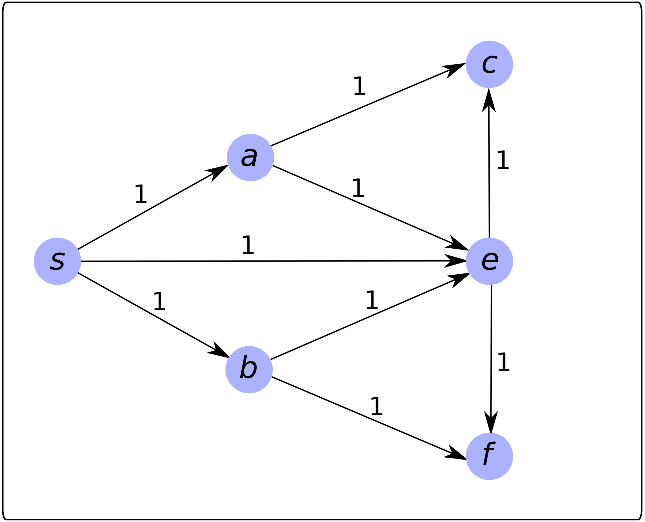
Shortest Paths Problem Instance



Generalizing BFS



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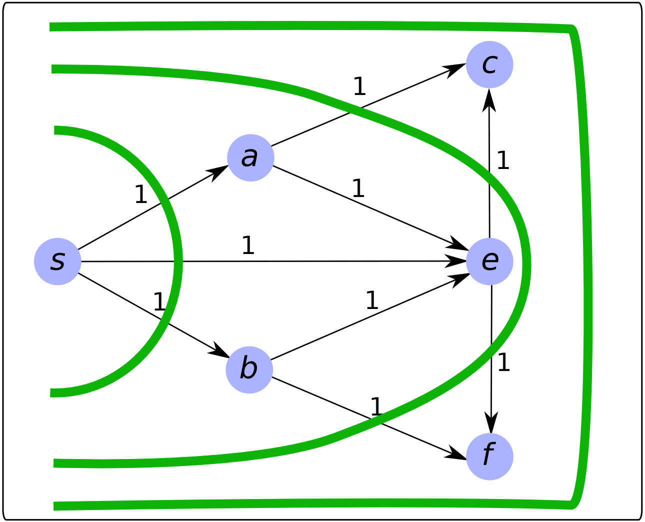
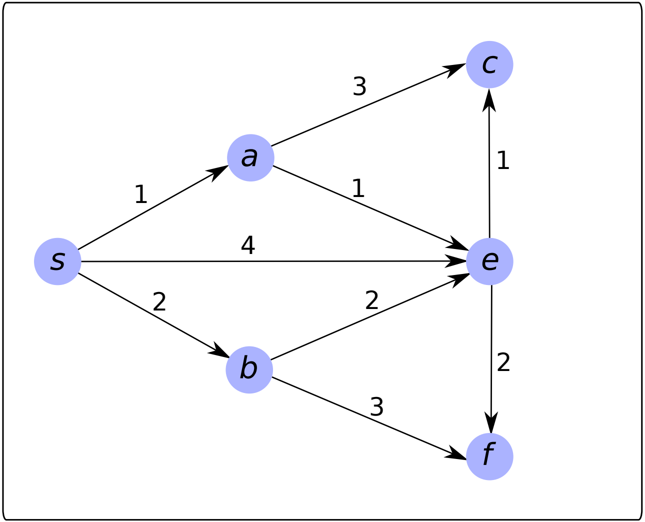


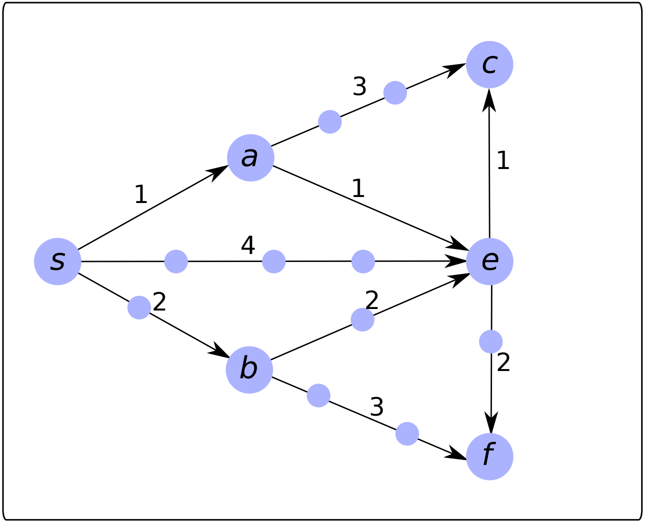
Unweighted graph: Use BFS. Process nodes in non-decreasing order of distance.

Generalizing BFS



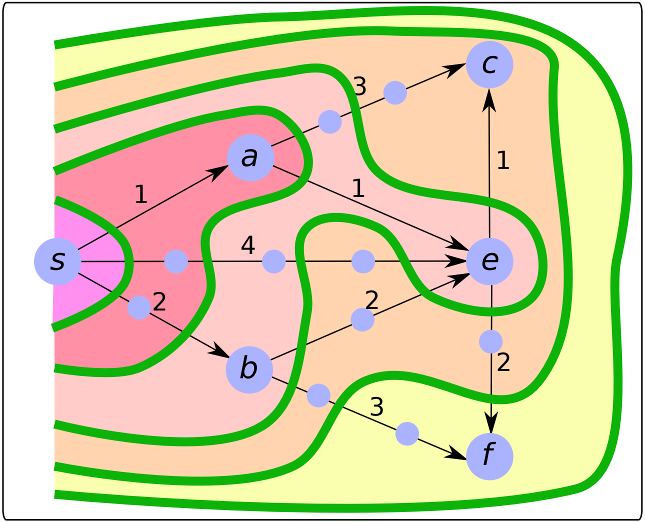
Weighted graph: Edge weights are integers. Can we make the graph unweighted?

Generalizing BFS



Add dummy nodes: Edge of weight w gets $w - 1$ nodes.

Generalizing BFS



Dummy nodes: BFS computes shortest paths correctly. Running time is

▸ Lectures 9-12: Greedy graph algorithms: Running time of BFS with dummy nodes

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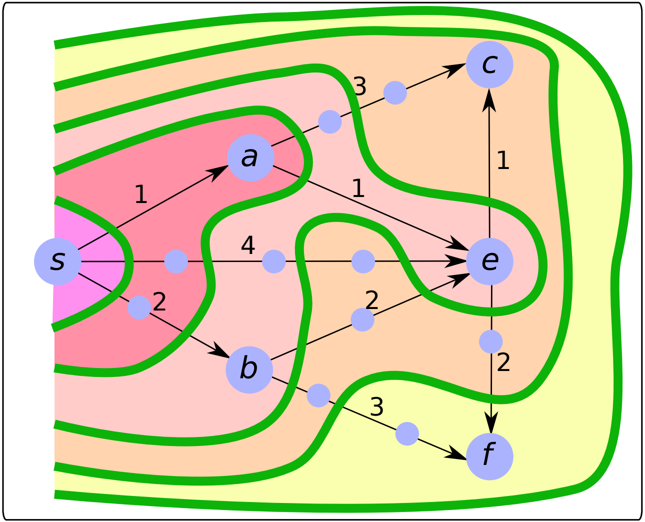
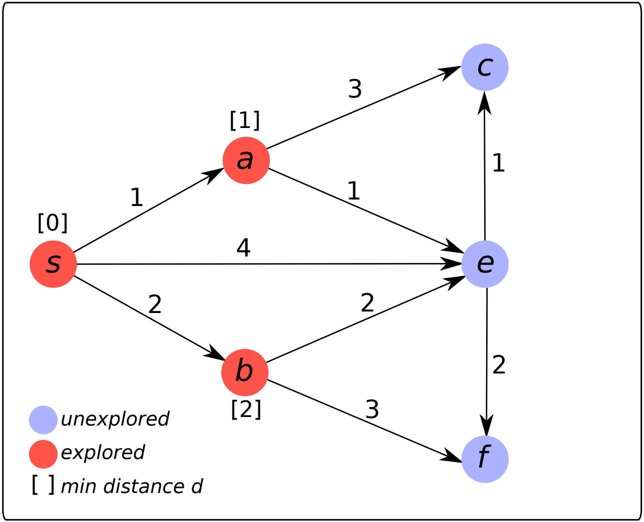


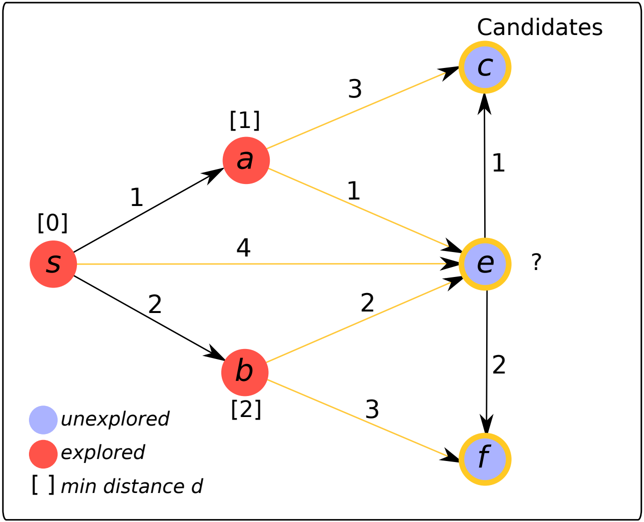
Dummy nodes: BFS computes shortest paths correctly. Running time is $O(m + n + \sum_{e \in E} l(e))$. *Pseudo-polynomial time*: depends on input *values*.

Generalizing BFS to Dijkstra's Algorithm



Like BFS: explore nodes in non-increasing order of distance from s. Once a node is explored, its distance is fixed.

Generalizing BFS to Dijkstra's Algorithm



Unlike BFS: Layers are not uniform. Which node to process next? Candidates are nodes with an edge from a explored node.

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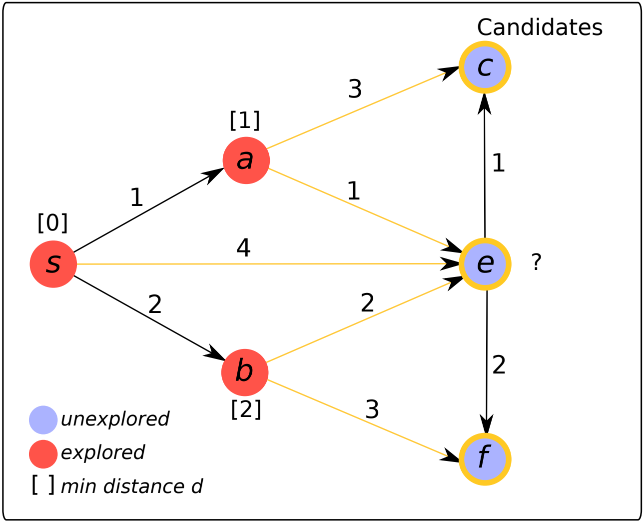

For each unexplored node, determine "best" preceding explored node.

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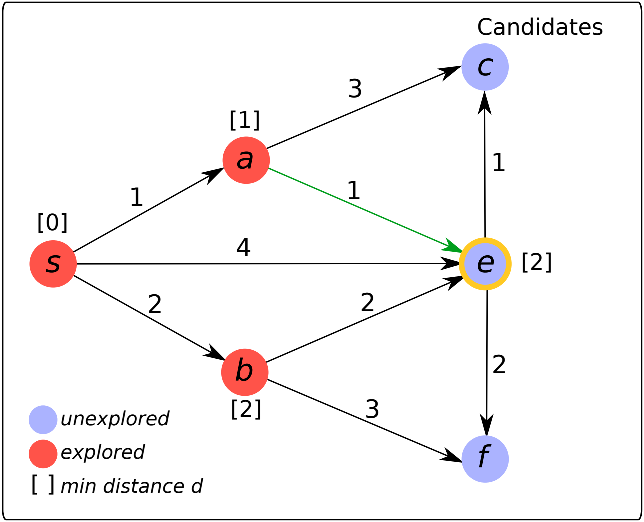

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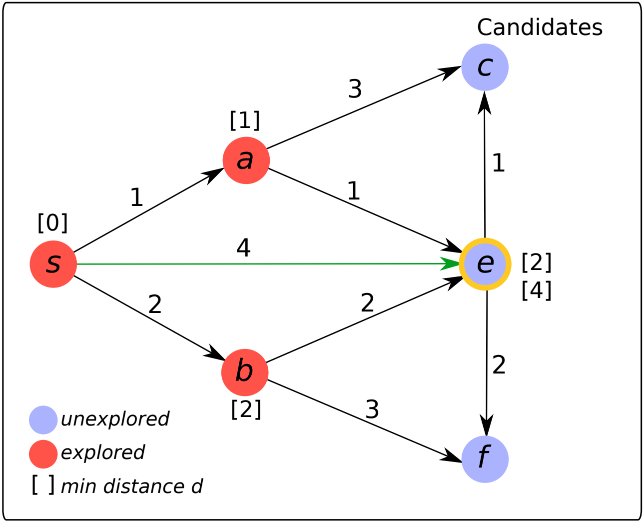

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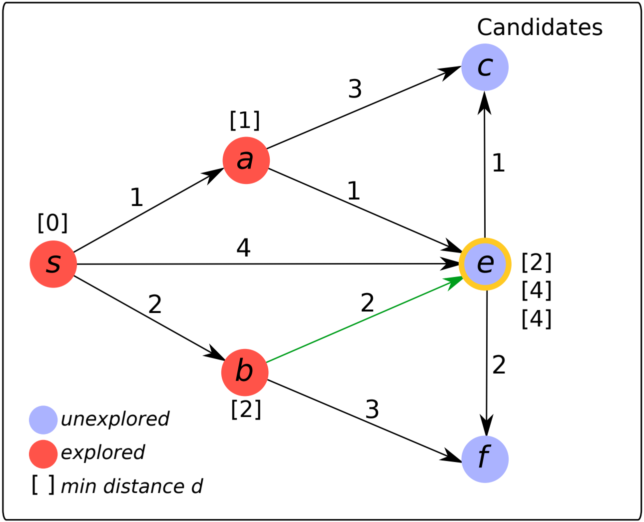
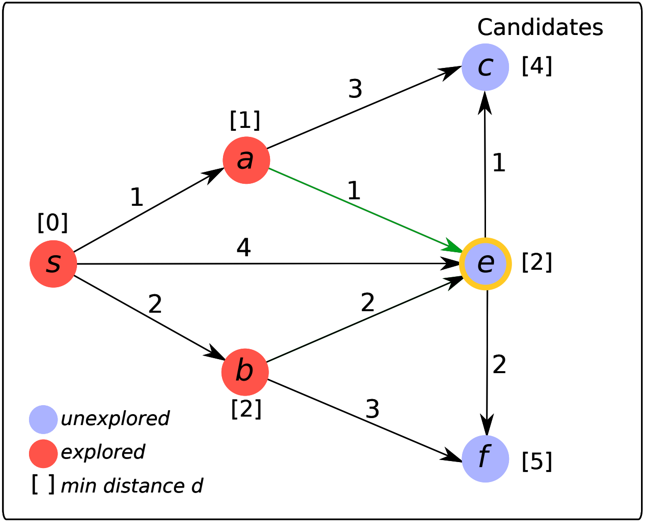


For each unexplored node, determine "best" preceding explored node. Record shortest path length only through explored nodes.

Generalizing BFS to Dijkstra's Algorithm



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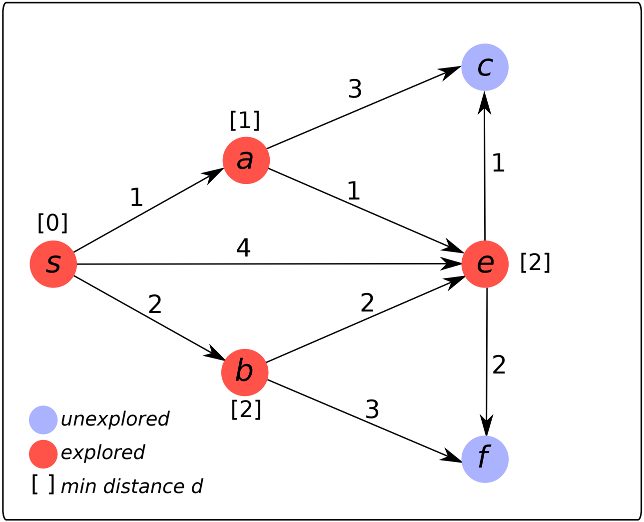
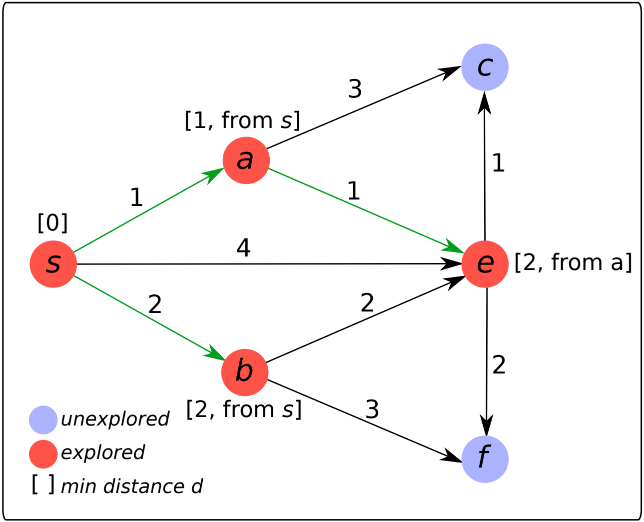


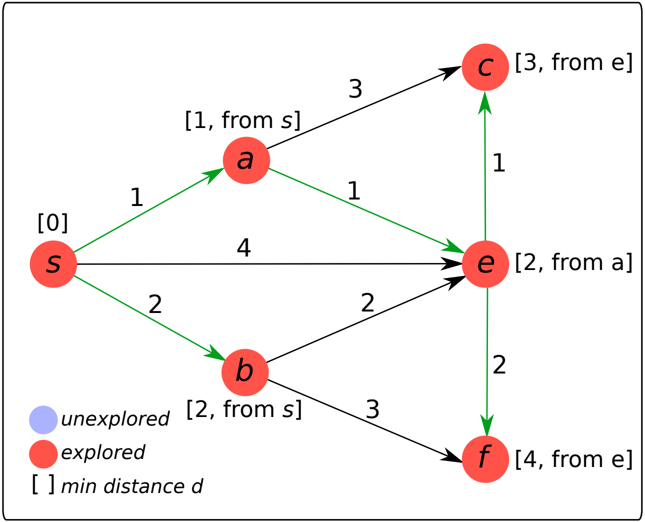
Explore node with smallest path length only through explored nodes.

Generalizing BFS to Dijkstra's Algorithm



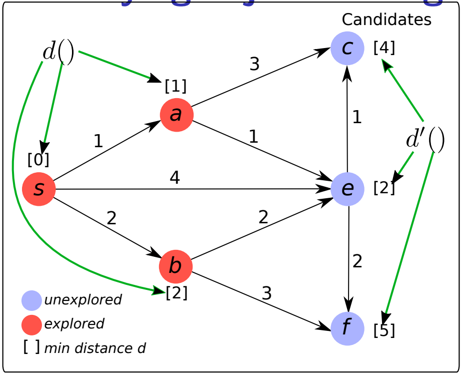
Like BFS: Record previous node in the computed path.

Generalizing BFS to Dijkstra's Algorithm



Follow previous nodes to compute shortest path. Like BFS: these edges form a tree.

Idea Underlying Dijkstra's Algorithm



- Maintain a set S of explored nodes.
  - For each node $u \in S$, compute a value $d(u)$, which (we will prove) is the length of the shortest path from s to u.
  - For each node $x \notin S$, maintain a value $d'(x)$, which is the length of the shortest path from s to x using only the nodes in S (and x, of course).

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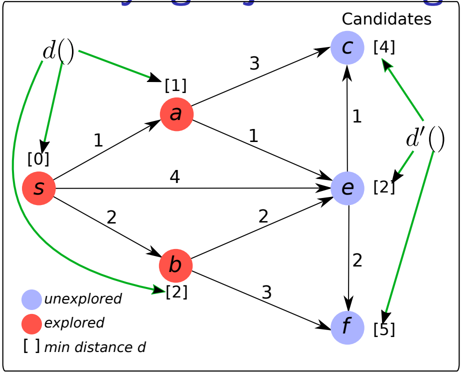

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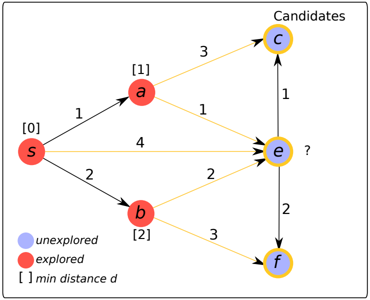
- "Greedily" add a node v to S that has the smallest value of $d'(v)$ (is closest to s using only nodes in S).

Dijkstra's Algorithm

Dijkstra's Algorithm(G, l, s)

1: $S = \{s\}$ and $d(s) = 0$
2: **while** $S \neq V$ **do**
3: **for** every node $x \in V - S$ **do**
4: Set $d'(x) = \min_{(u,x):u \in S}(d(u) + l(u,x))$
5: Set $v = \arg\min_{x \in V-S} d'(x)$
6: Add v to S and set $d(v) = d'(v)$



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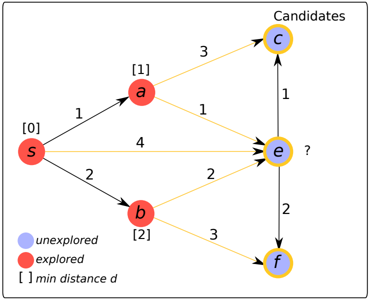

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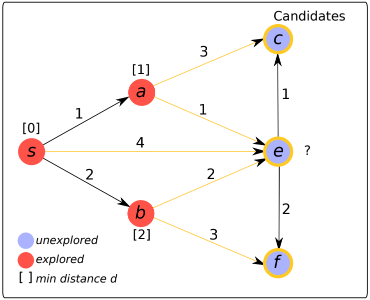

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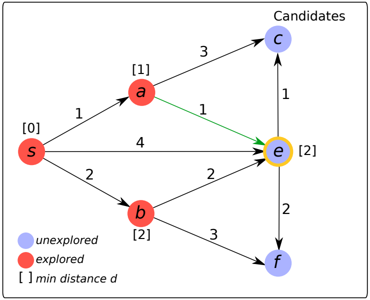

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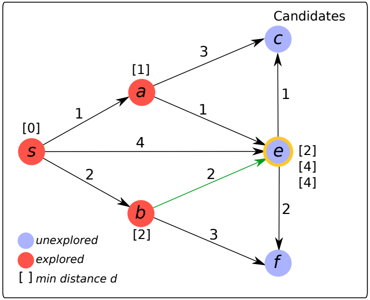

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  - For each such edge, we compute the length of the shortest path from s to x via u, which is $d(u) + l(u, x)$.

Dijkstra's Algorithm

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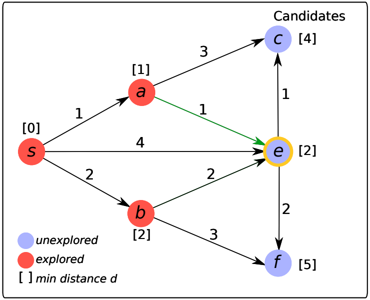
```
S = {s} and d(s) = 0
while S ≠ V do
    for every node x ∈ V − S do
        Set d'(x) = min_{(u,x): u∈S} (d(u) + l(u,x))
    Set v = arg min_{x∈V−S} d'(x)
    Add v to S and set d(v) = d'(v)
```



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  - We store the smallest of these values in $d'(x)$.

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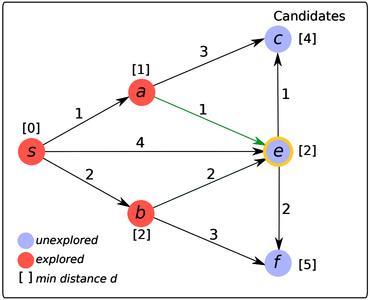

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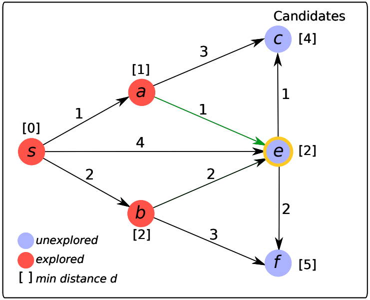

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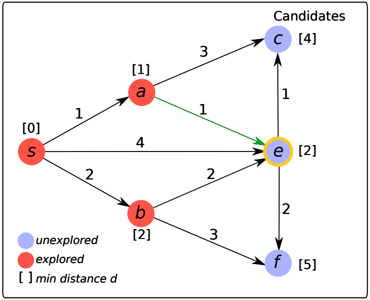

- How do we parse $v = \arg\min_{x \in V-S} d'(x)$?
  - Run over all (unexplored) nodes x in $V - S$.
  - Examine the d' values for these nodes.

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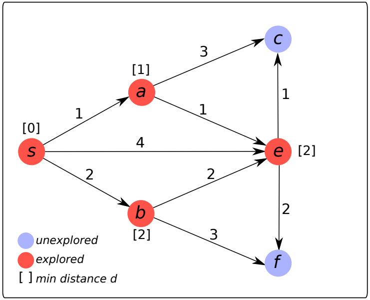

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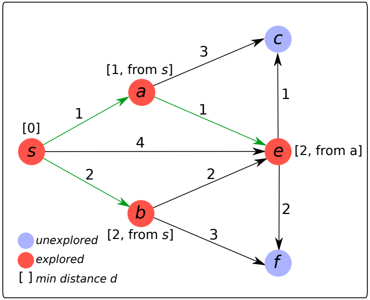

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  - Examine the d' values for these nodes.
  - Return the *argument* (i.e., the node) that has the smallest value of $d'(x)$.
- To compute the shortest paths: when adding a node v to S, store the predecessor u that minimises $d'(v)$.

Proof of Correctness

- Let P_u be the path computed by the algorithm for an arbitrary node u.
- Claim: P_u is the shortest path from s to u.
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  - Inductive step: $|S| = k + 1$ because we add the node v to S. Could there be a shorter path P from s to v? We must prove this cannot be the case.

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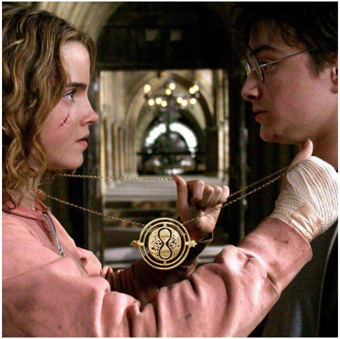
  - Inductive hypothesis: $|S| = k$, for some $k \geq 1$. The algorithm has correctly computed P_u for every node $u \in S$. Strong induction.
  - Inductive step: $|S| = k + 1$ because we add the node v to S. Could there be a shorter path P from s to v? We must prove this cannot be the case.



Proof of Correctness

- Let P_u be the path computed by the algorithm for an arbitrary node u.
- Claim: P_u is the shortest path from s to u.
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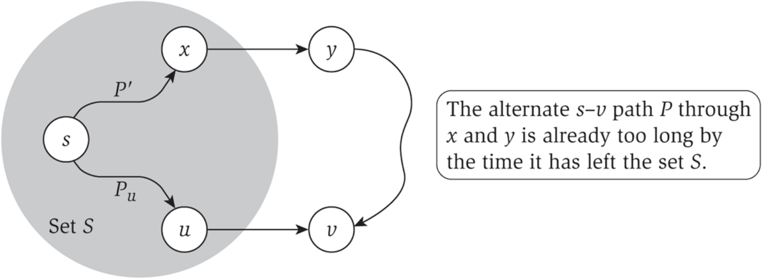

Figure 4.8 The shortest path P_v and an alternate s-v path P through the node y.

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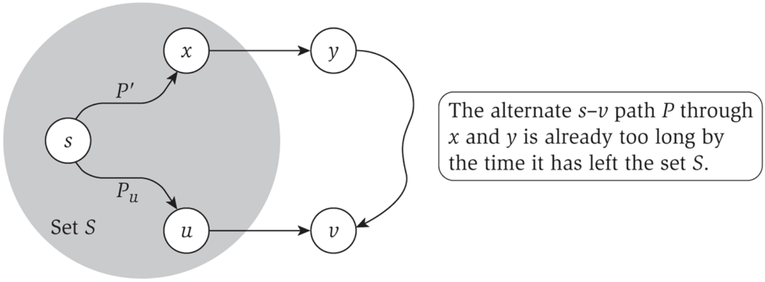

Figure 4.8 The shortest path P_v and an alternate s-v path P through the node y.

Lectures 9–12: Proof of correctness of Dijsktra's algorithm: path P'

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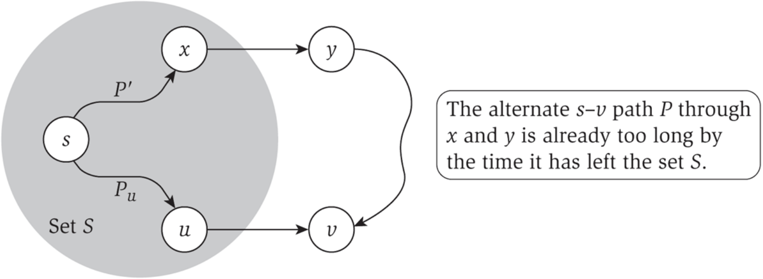

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▸ Lectures 9–12: Proof of correctness of Dijsktra's algorithm: sub-path of P'

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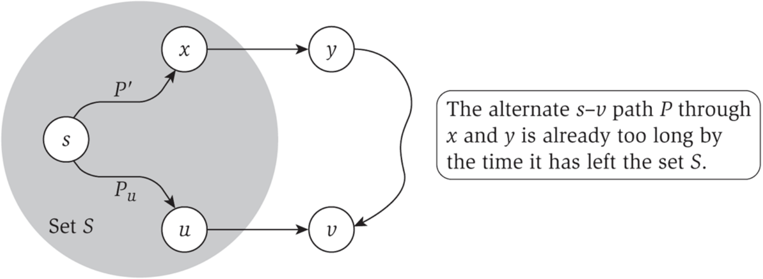

Figure 4.8 The shortest path P_v and an alternate s-v path P through the node y.

Lectures 9–12: Proof of correctness of Dijsktra's algorithm: another sub-path of P'

Comments about Dijkstra's Algorithm

- Algorithm cannot handle negative edge lengths. We will discuss the Bellman-Ford algorithm in a few weeks.
- Union of shortest paths output by Dijkstra's algorithm forms a tree. Why?

Comments about Dijkstra's Algorithm

- Algorithm cannot handle negative edge lengths. We will discuss the Bellman-Ford algorithm in a few weeks.
- Union of shortest paths output by Dijkstra's algorithm forms a tree. Why?
- Union of shortest paths from a fixed source s forms a tree; paths not necessarily computed by Dijkstra's algorithm.

Running time of Dijkstra's Algorithm

DIJKSTRA'S ALGORITHM(G, l, s)

```
S = {s} and d(s) = 0
while S ≠ V do
    for every node x ∈ V − S do
        Set d'(x) = min_{(u,x):u∈S}(d(u) + l(u,x))
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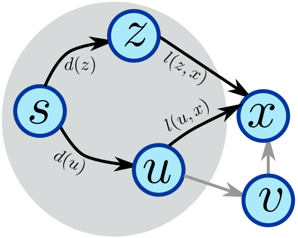

- V has n nodes and E has m edges. ▸ Greedy graph algorithms: Running time of Dijkstra's algorithm: take 1
- How many iterations are there of the while loop?

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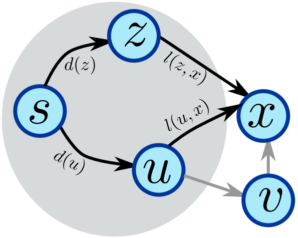

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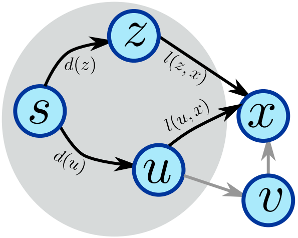

- V has n nodes and E has m edges.

Greedy graph algorithms: Running time of Dijkstra's algorithm: take 1

- How many iterations are there of the while loop? $n - 1$.
- In each iteration, for each node $x \in V - S$, compute

$$d'(x) = \min_{(u,x),u \in S}(d(u) + l(u,x))$$

Running time of Dijkstra's Algorithm

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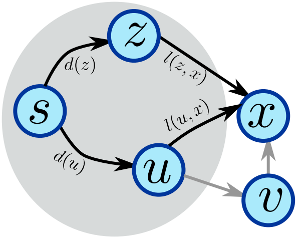

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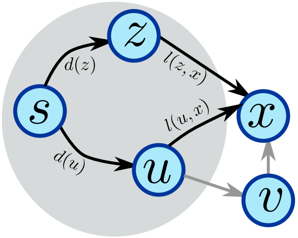

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- In each iteration, for each node $x \in V - S$, compute

$$d'(x) = \min_{(u,x), u \in S}(d(u) + l(u,x))$$

- Running time per iteration is $O(m)$, since the algorithm processes each edge (u, x) in the graph exactly once (when computing $d'(x)$).
- The overall running time is $O(nm)$.

A Faster implementation of Dijkstra's Algorithm

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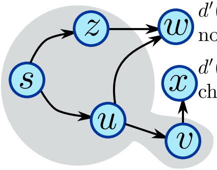

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S = {s} and d(s) = 0
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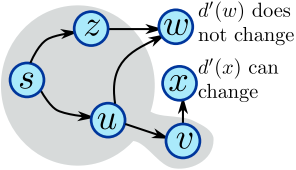

- Observation: If we add v to S, $d'(x)$ changes only

▸ Greedy graph algorithms: Running time of Dijkstra's algorithm: updating d'

A Faster implementation of Dijkstra's Algorithm

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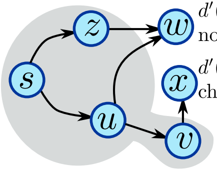

- Observation: If we add v to S, $d'(x)$ changes only if (v, x) is an edge in G and x is not in S.

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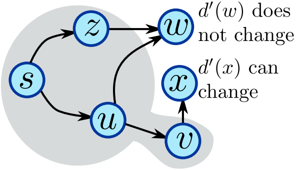

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- Idea: For each node $x \in V - S$, store the current value of $d'(x)$. Upon adding a node v to S, update $d'()$ only for neighbours of v that are not in S.

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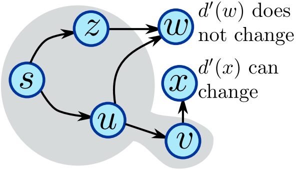

- Observation: If we add v to S, $d'(x)$ changes only if (v, x) is an edge in G and x is not in S.
- Idea: For each node $x \in V - S$, store the current value of $d'(x)$. Upon adding a node v to S, update $d'()$ only for neighbours of v that are not in S.
- How do we efficiently compute $v = \arg\min_{x \in V-S} d'(x)$?

A Faster implementation of Dijkstra's Algorithm

DIJKSTRA'S ALGORITHM(G, l, s)

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S = {s} and d(s) = 0
while S ≠ V do
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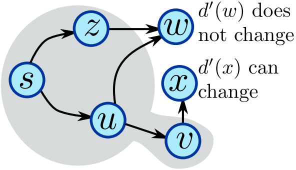

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- How do we efficiently compute $v = \arg\min_{x \in V-S} d'(x)$?
- Use a priority queue!

Faster Dijkstra's Algorithm

```
DIJKSTRA'S ALGORITHM(G, l, s)
1: INSERT(Q, s, 0).
2: while S ≠ V do
3:    (v, d'(v)) = EXTRACTMIN(Q)
4:    Add v to S and set d(v) = d'(v)
5:    for every node x ∈ V − S such that (v, x) is an edge in G do
6:       if d(v) + l(v, x) < d'(x) then
7:          d'(x) = d(v) + l(v, x)
8:          CHANGEKEY(Q, x, d'(x))
```

- For each node $x \in V - S$, store the pair $(x, d'(x))$ in a priority queue Q with $d'(x)$ as the key.
- Determine the next node v to add to S using EXTRACTMIN (line 3).
- After adding v to S, for each node $x \in V - S$ such that there is an edge from v to x, check if $d'(x)$ should be updated, i.e., if there is a shortest path from s to x via v (lines 5–8).
- In line 8, if x is not in Q, simply insert it.

Running Time of Faster Dijkstra's Algorithm

DIJKSTRA'S ALGORITHM(G, l, s)

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```

- How many times does the algorithm invoke EXTRACTMIN?

▸ Greedy graph algorithms: Running time of Dijkstra's algorithm: take 2

Running Time of Faster Dijkstra's Algorithm

DIJKSTRA'S ALGORITHM(G, l, s)

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```

- How many times does the algorithm invoke EXTRACTMIN? $n-1$ times.

Running Time of Faster Dijkstra's Algorithm

Dijkstra's Algorithm(G, l, s)

```
1: Insert(Q, s, 0).
2: while S ≠ V do
3:     (v, d'(v)) = ExtractMin(Q)
4:     Add v to S and set d(v) = d'(v)
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- How many times does the algorithm invoke ExtractMin? $n - 1$ times.
- For every node v, what is the running time of step 5?

Running Time of Faster Dijkstra's Algorithm

DIJKSTRA'S ALGORITHM(G, l, s)

1: INSERT$(Q, s, 0)$.
2: **while** $S \neq V$ **do**
3: $(v, d'(v)) = $ EXTRACTMIN(Q)
4: Add v to S and set $d(v) = d'(v)$
5: **for** every node $x \in V - S$ such that (v, x) is an edge in G **do**
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- How many times does the algorithm invoke EXTRACTMIN? $n - 1$ times.
- For every node v, what is the running time of step 5? $O(d_v)$, the number of *outgoing* neighbours of v.

Running Time of Faster Dijkstra's Algorithm

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Running Time of Faster Dijkstra's Algorithm

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- How many times does the algorithm invoke CHANGEKEY? At most m times.
- What is total running time of the algorithm? $O(m \log n)$.

Running Time of Faster Dijkstra's Algorithm

DIJKSTRA'S ALGORITHM(G, l, s)

1: INSERT$(Q, s, 0)$.
2: **while** $S \neq V$ **do**
3: $\quad (v, d'(v)) = $ EXTRACTMIN(Q)
4: $\quad$ Add v to S and set $d(v) = d'(v)$
5: $\quad$ **for** every node $x \in V - S$ such that (v, x) is an edge in G **do**
6: $\qquad$ **if** $d(v) + l(v, x) < d'(x)$ **then**
7: $\qquad\quad d'(x) = d(v) + l(v, x)$
8: $\qquad\quad$ CHANGEKEY$(Q, x, d'(x))$

- How many times does the algorithm invoke EXTRACTMIN? $n - 1$ times.
- For every node v, what is the running time of step 5? $O(d_v)$, the number of *outgoing* neighbours of v.
- What is the total running time of step 5? $\sum_{v \in V} O(d_v) = O(m)$.
- How many times does the algorithm invoke CHANGEKEY? At most m times.
- What is total running time of the algorithm? $O(m \log n)$.
- State of the art: Fibonacci heaps achieve a running time of $O(m)$ for all CHANGEKEY operations, for a running time of $O(n \log n + m)$.

Network Design

- Connect a set of nodes using a set of edges with certain properties.
- Input is usually a graph and the desired network (the output) should use subset of edges in the graph.
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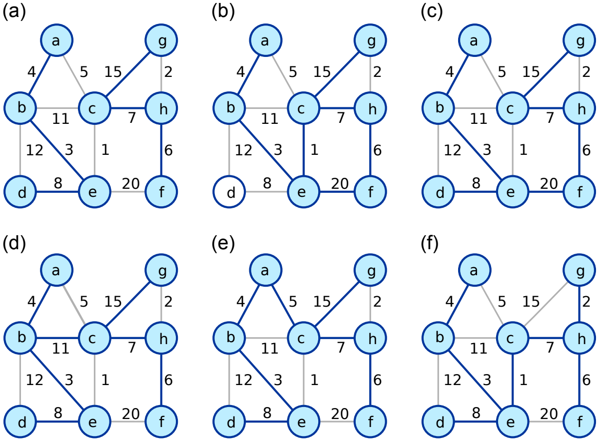
Minimum Spanning Tree (MST)

- Given an undirected graph $G(V, E)$ with a cost $c(e) > 0$ associated with each edge $e \in E$.
- Find a subset T of edges such that the graph (V, T) is connected and the cost $\sum_{e \in T} c(e)$ is as small as possible.



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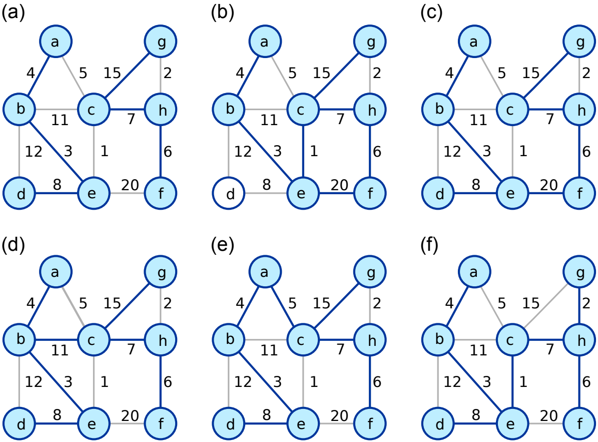
Lectures 9–12: MST: Connected and minimum examples



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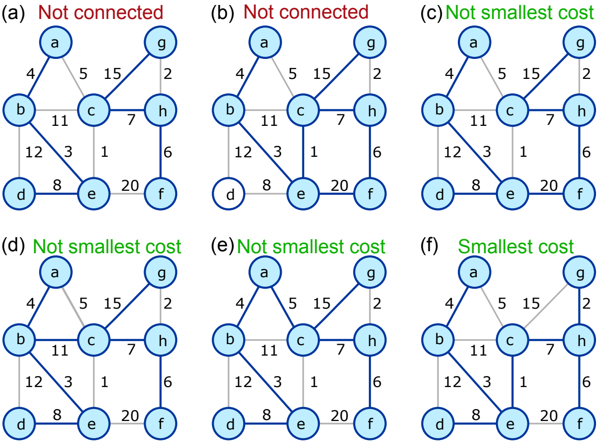
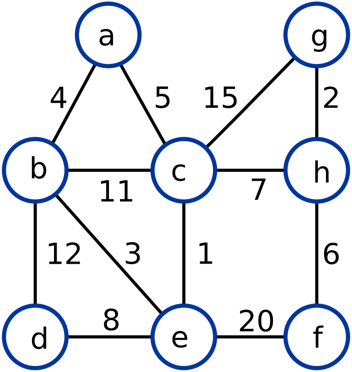


Minimum Spanning Tree (MST)

MINIMUM SPANNING TREE

INSTANCE: An undirected graph $G(V, E)$ and a function $c : E \rightarrow \mathbb{R}^+$

SOLUTION: A set $T \subseteq E$ of edges such that (V, T) is connected and the cost $\sum_{e \in T} c(e)$ is as small as possible.



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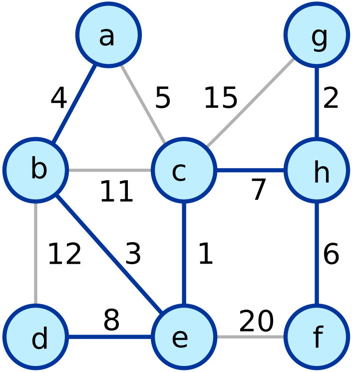

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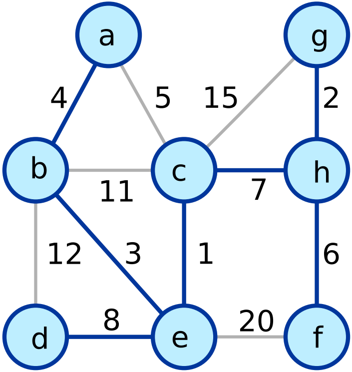

- Claim: If T is a minimum-cost solution to this problem then (V, T) is a tree.
- A subset T of E is a *spanning tree* of G if (V, T) is a tree.

Greedy Algorithm for the MST Problem

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- Simplifying assumption: all edge costs are distinct.

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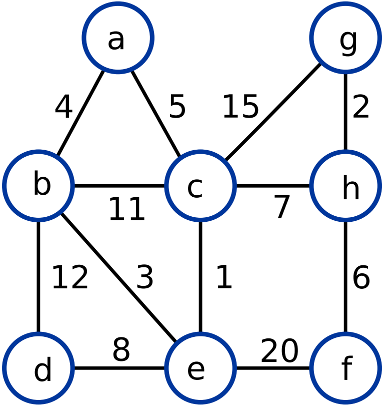
  - What happens when we add an edge to an MST?
  - We obtain a cycle.
  - Which edge in the cycle can we be sure does not belong to an MST?

Graph Cuts

- A *cut* in a graph $G(V, E)$ is a set of edges whose removal disconnects the graph (into two or more connected components).



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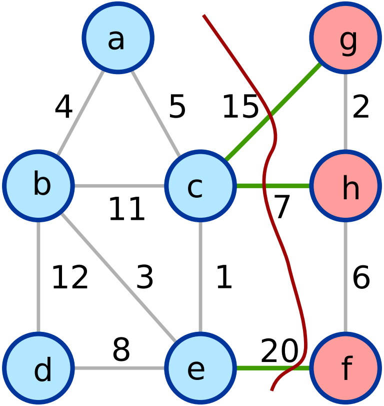

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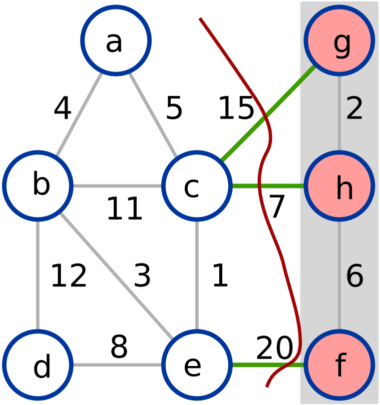

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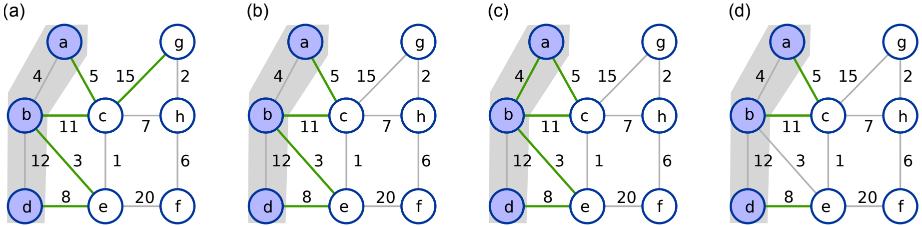

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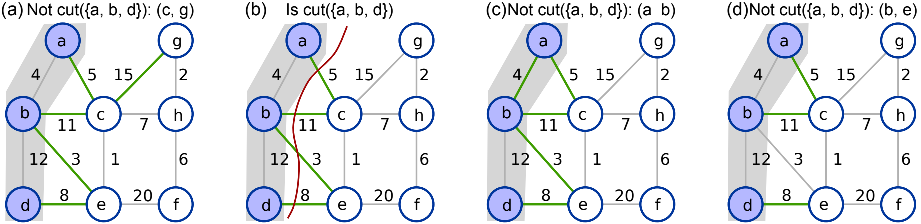

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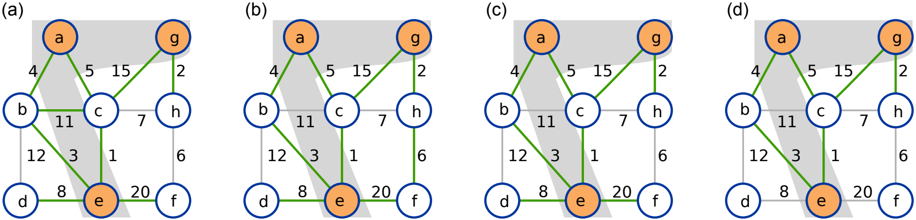

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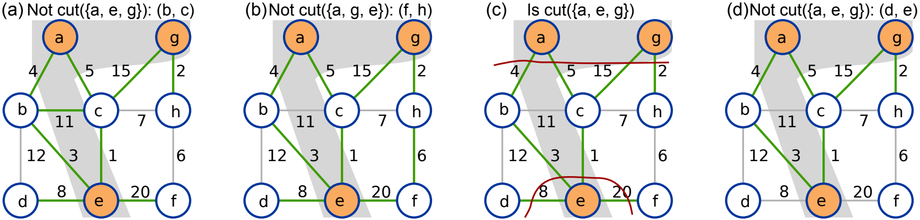

Cut Property

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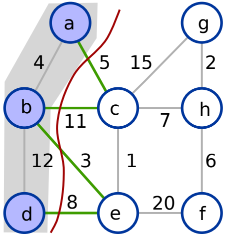

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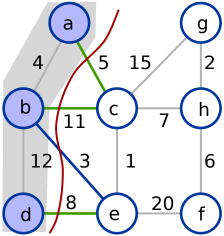

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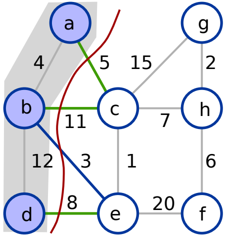

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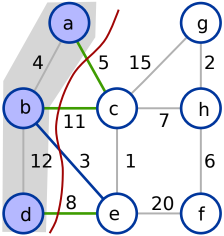

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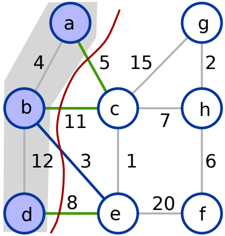

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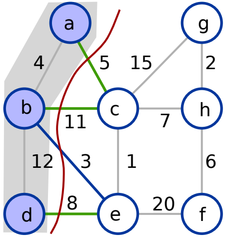

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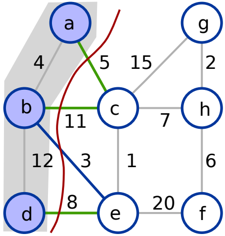

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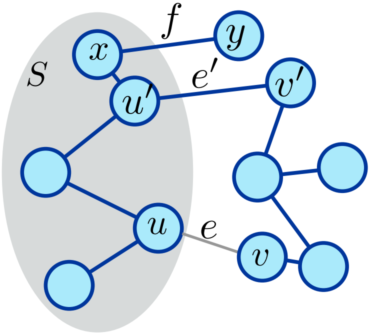

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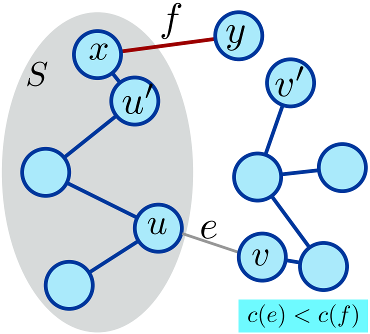

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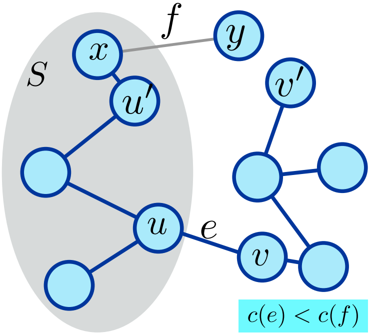

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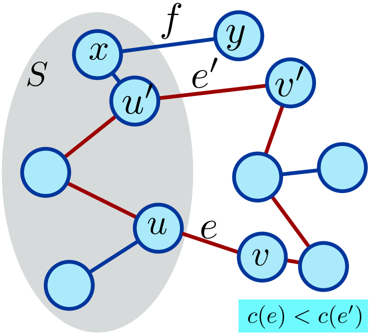

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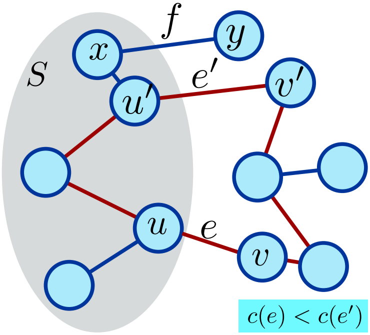

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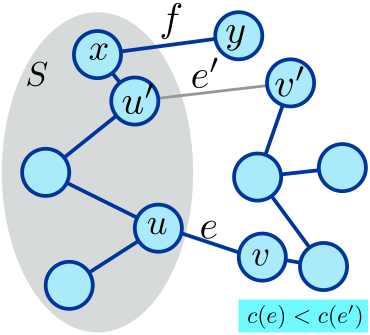

Prim's Algorithm

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Prim's Algorithm(G, c, s)

```
S = {s} and T = ∅
while S ≠ V do
    Compute (u, v) = arg min_{(u,v): u∈S, v∈V−S} c(u, v)
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```

Lectures 9–12: MST: Step 3 of Prim's algorithm

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PRIM'S ALGORITHM(G, c, s)

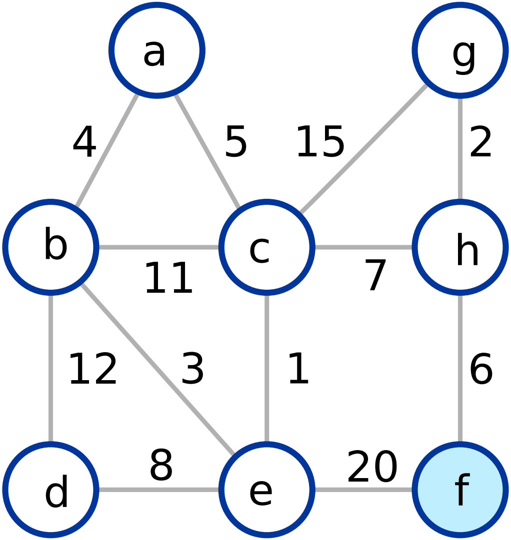
1: $S = \{s\}$ and $T = \emptyset$
2: **while** $S \neq V$ **do**
3: Compute $(u, v) = \arg\min_{(u,v): u \in S, v \in V - S} c(u, v)$ Lectures 9–12: MST: Step 3 of Prim's algorithm
4: Add the node v to S and add the edge (u, v) to T.

- Note that

$$\arg \min_{(u,v), u \in S, v \in V - S} c(u, v) \equiv \arg \min_{(u,v) \in \mathrm{cut}(S)} c(u, v).$$

- In other words, in each step, Prim's algorithm computes and adds the cheapest edge in the current value of cut(S).

Example of Prim's Algorithm



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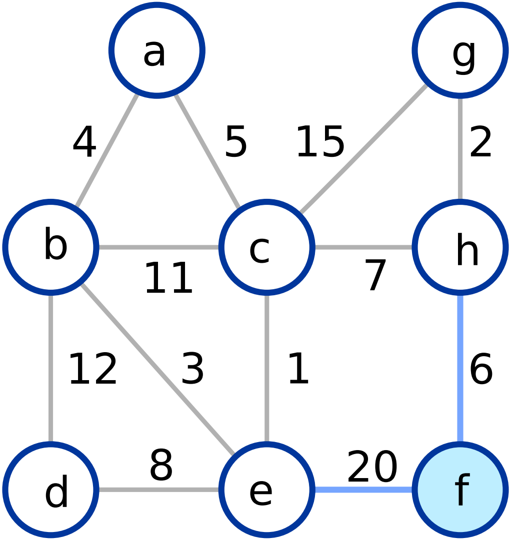

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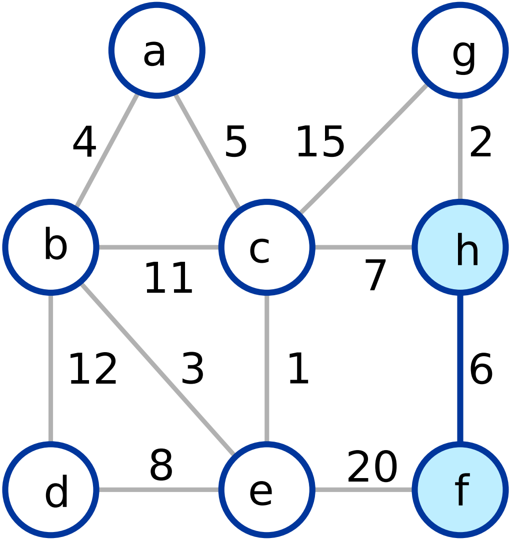

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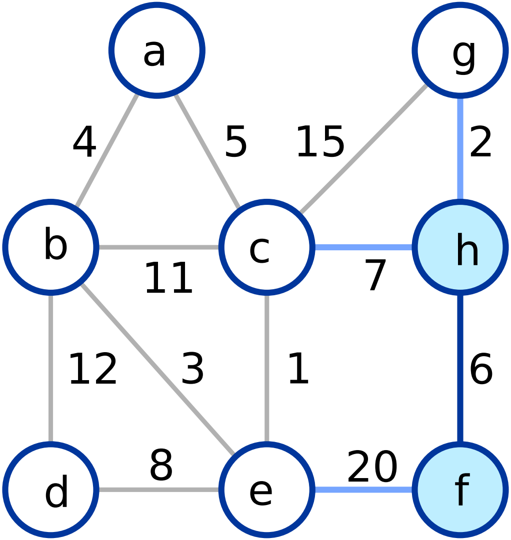

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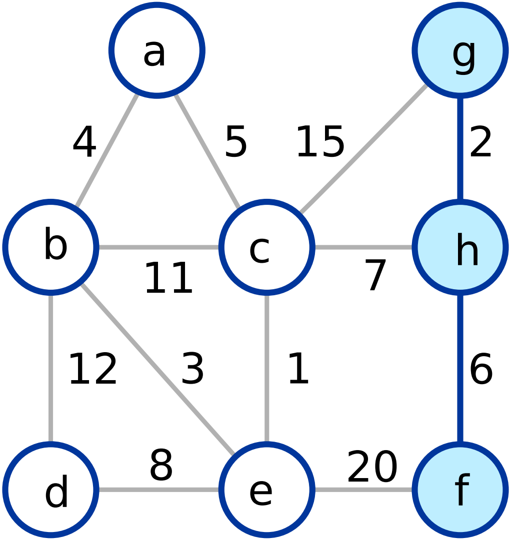

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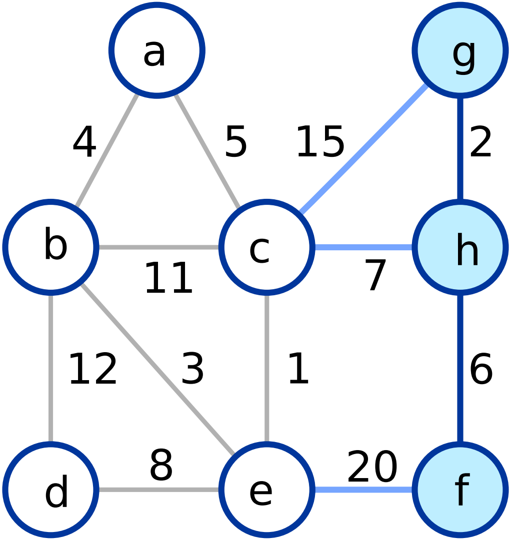

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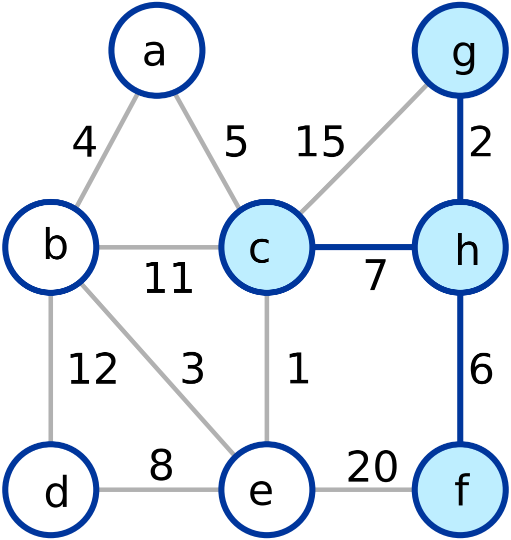

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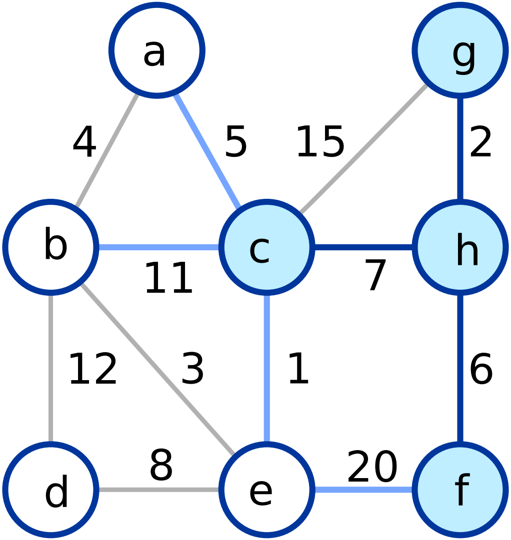

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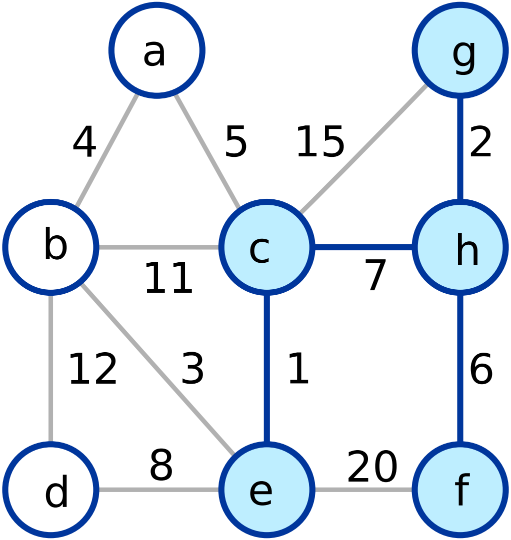

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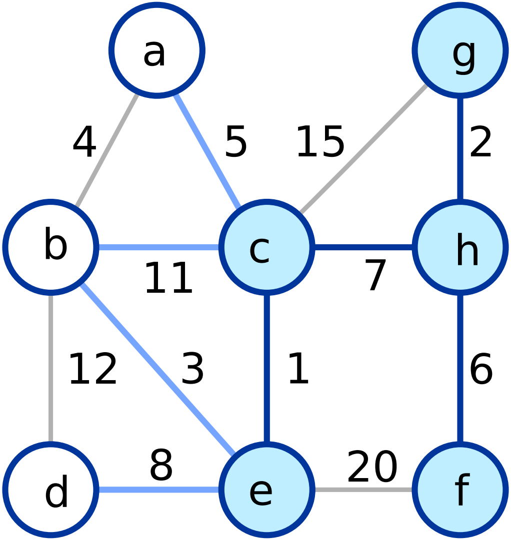

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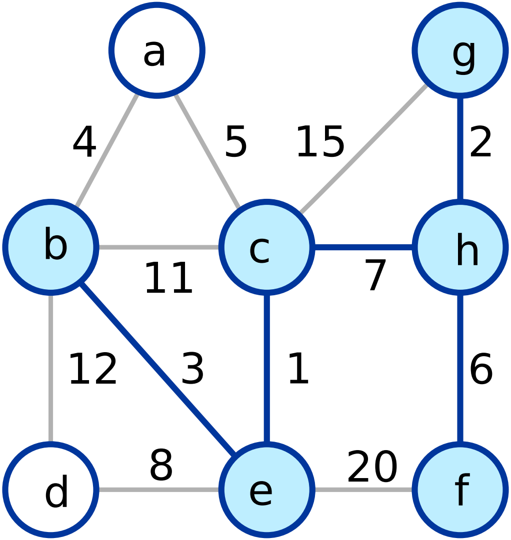

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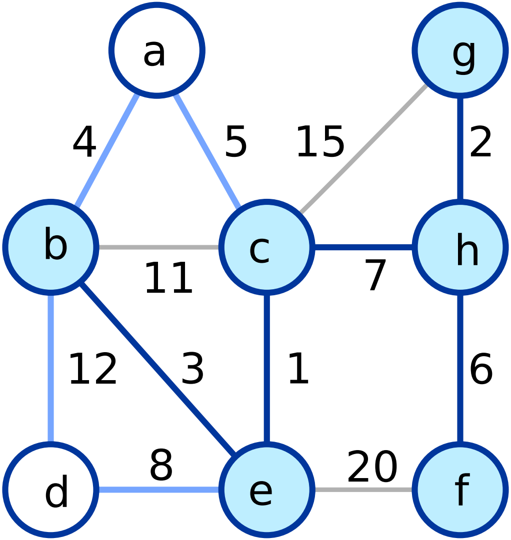

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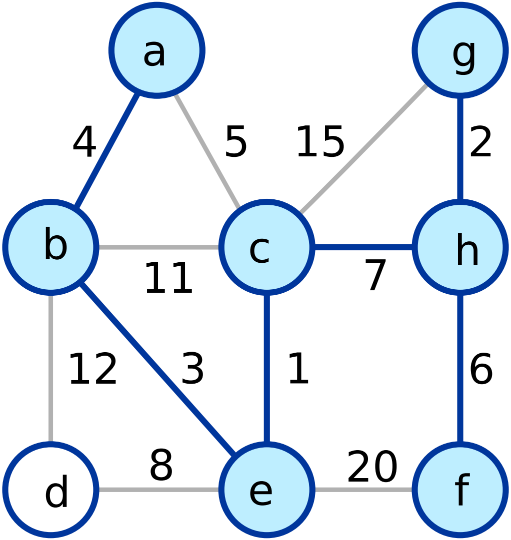

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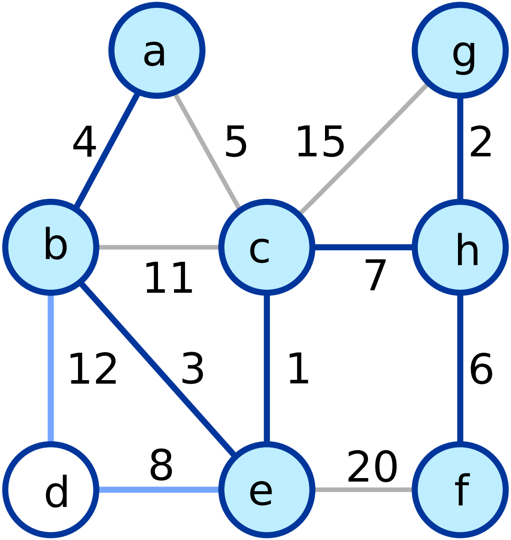

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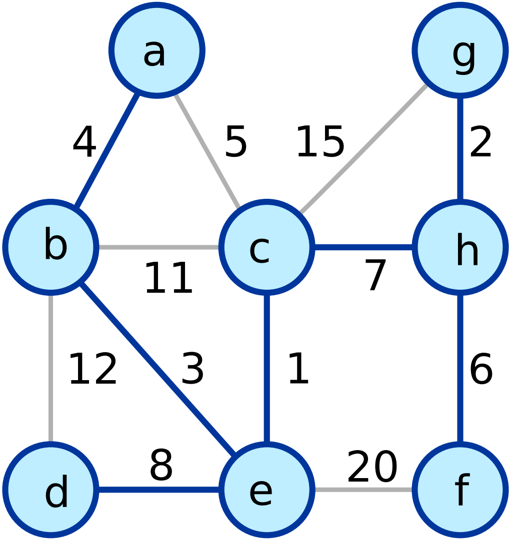

Optimality of Prim's Algorithm

PRIM'S ALGORITHM(G, c, s)

$S = \{s\}$ and and $T = \emptyset$
while $S \neq V$ **do**
    Compute $(u, v) = \arg\min_{(u,v) \in \mathrm{cut}(S)} c(u, v)$
    Add the node v to S and add the edge (u, v) to T.

- Claim: Prim's algorithm outputs an MST.

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- Claim: Prim's algorithm outputs an MST.
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     - By construction, in each iteration (u, v) is the cheapest edge in cut(S) for the current value of S.
  2. Prove that the graph constructed is a spanning tree.
     - Why are there no cycles in (V, T)? ▸ Lectures 9–12: MST: No cycles in output of Prim's algorithm
     - Why is (V, T) a spanning tree (edges in T connect all nodes in V)? ▸ Lectures 9–12: MST: No cycles in output of Prim's algorithml

Kruskal's Algorithm

- Start with an empty set T of edges.
- Process edges in E in increasing order of cost.
- Add the next edge e to T only if adding e does not create a cycle. Discard e if it creates a cycle.

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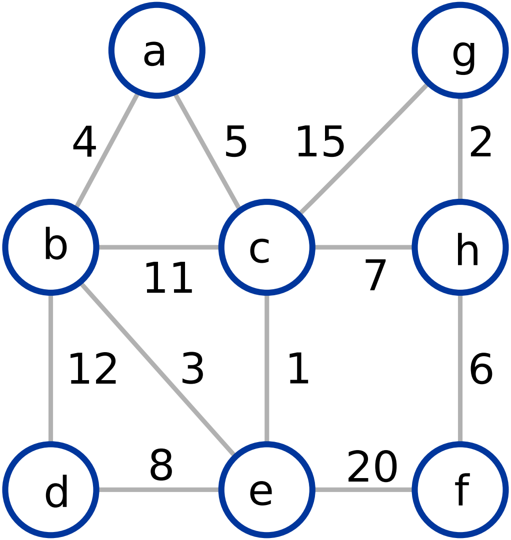

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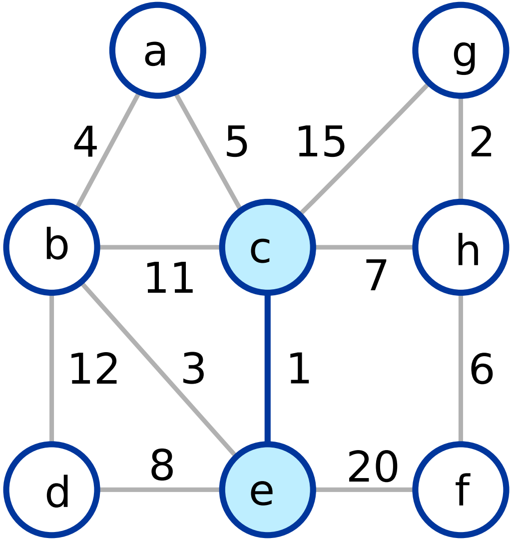

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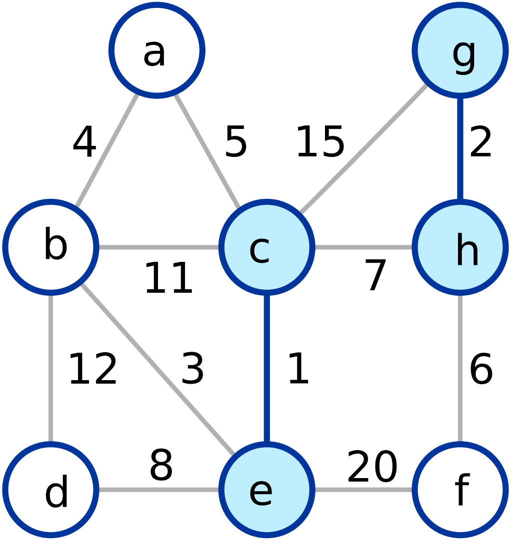

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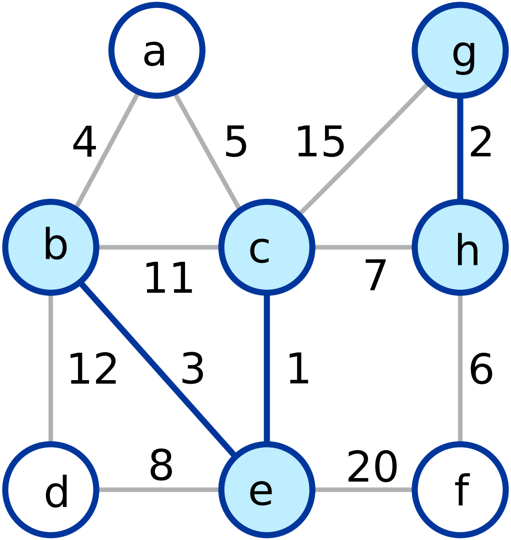

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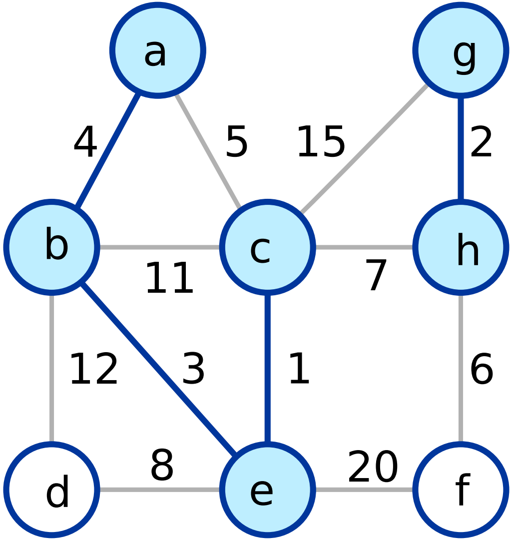

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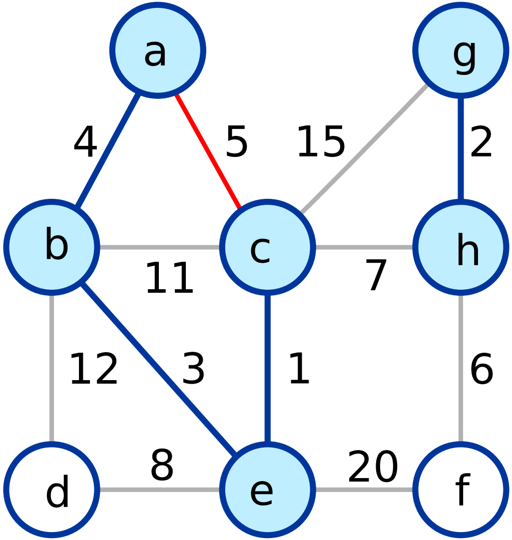

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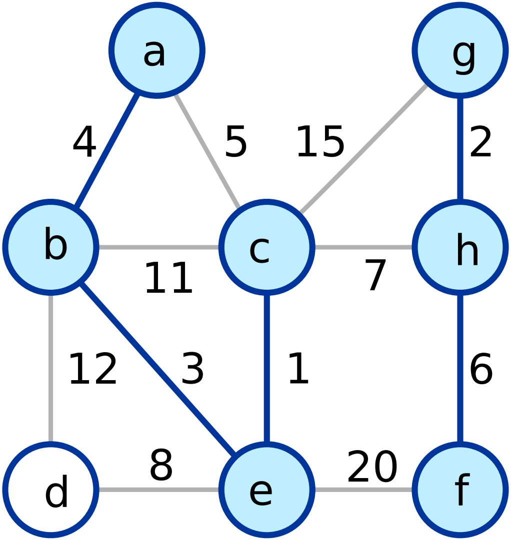

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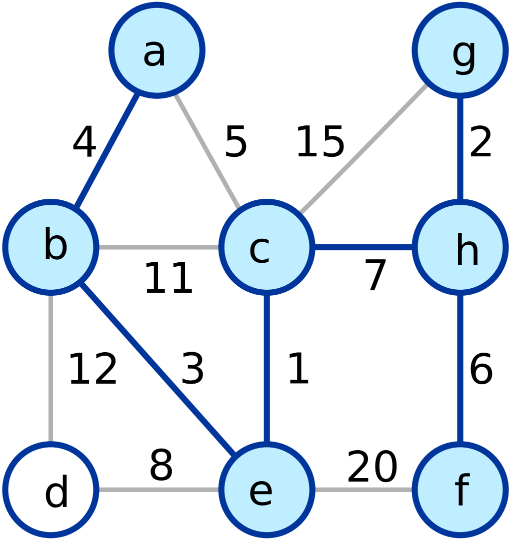

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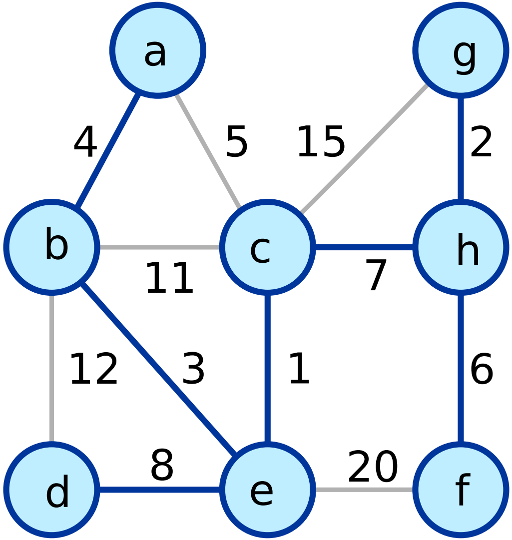

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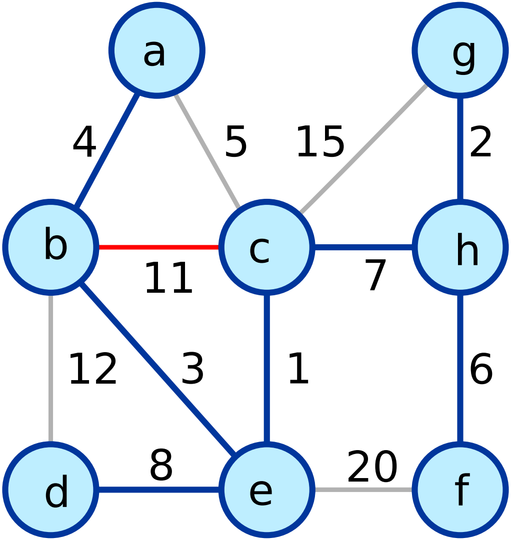

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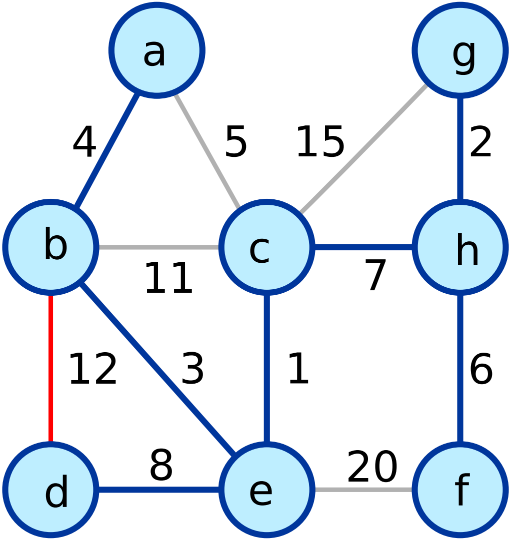

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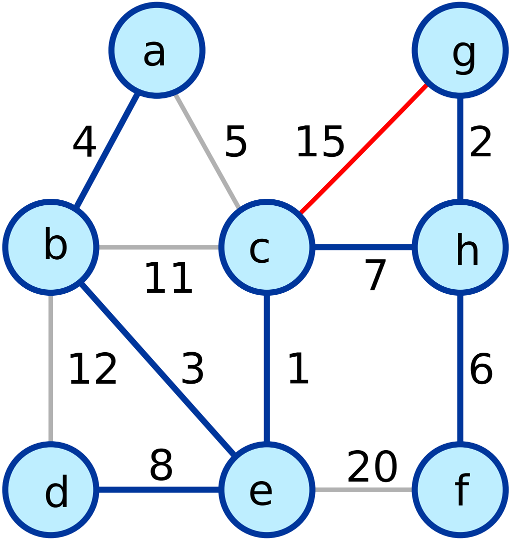

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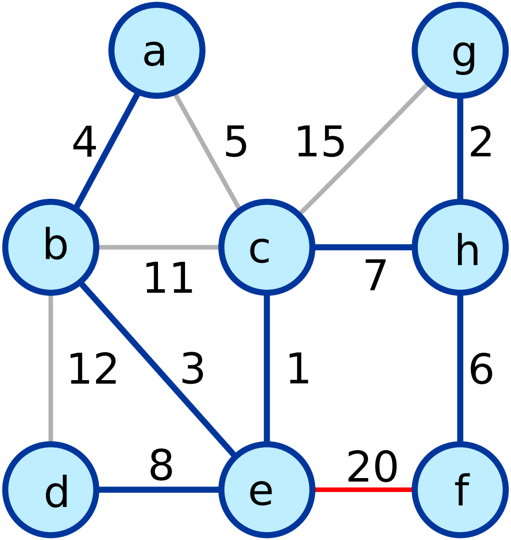

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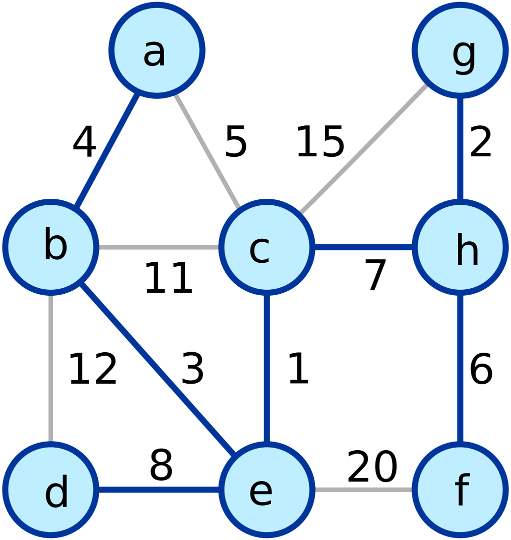

Optimality of Kruskal's Algorithm

- Kruskal's algorithm:
  - Start with an empty set T of edges.
  - Process edges in E in increasing order of cost.
  - Add the next edge e to T only if adding e does not create a cycle. Discard e if it creates a cycle.
- Note: at any iteration, T may contain several connected components and each node in V is in some component.
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  2. Prove that the algorithm computes a spanning tree.

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     - Why is e the cheapest edge in cut(S)?
  2. Prove that the algorithm computes a spanning tree.
     - (V, T) contains no cycles by construction.
     - If (V, T) is not connected, there exists a subset S of nodes not connected to $V - S$. What is the contradiction?

Cycle Property

- When can we be sure that an edge cannot be in *any* MST?

Cycle Property

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- Let C be any cycle in G and let $e = (v, w)$ be the most expensive edge in C.
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Cycle Property

- When can we be sure that an edge cannot be in *any* MST?
- Let C be any cycle in G and let $e = (v, w)$ be the most expensive edge in C.
- Claim: e does not belong to any MST of G.
- Proof: exchange argument. If a supposed MST T contains e, show that there is a tree with smaller cost than T that does not contain e.

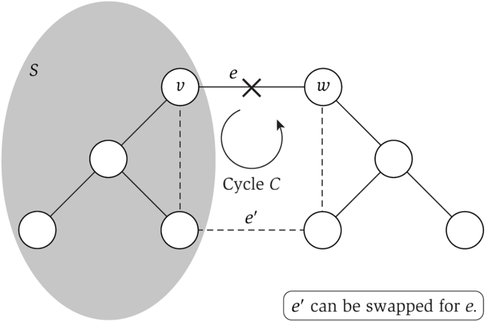

Figure 4.11 Swapping the edge e' for the edge e in the spanning tree T, as described in the proof of (4.20).

Optimality of the Reverse-Delete Algorithm

- Reverse-Delete algorithm: Maintain a set E' of edges.
  - Start with $E' = E$.
  - Process edges in decreasing order of cost.
  - Delete the next edge e from E' only if (V, E') is connected after deletion.
  - Stop after processing all the edges.
- Claim: the Reverse-Delete algorithm outputs an MST.

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     - Since the edge is the first encountered by the algorithm, it is the most expensive edge in C.
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     - Since the edge is the first encountered by the algorithm, it is the most expensive edge in C.
  2. Prove that the graph remaining at the end is a spanning tree.
     - (V, E') is connected at the end, by construction.
     - If (V, E') contains a cycle, consider the costliest edge in that cycle. The algorithm would have deleted that edge.

Implementing Prim's Algorithm

PRIM'S ALGORITHM(G, c, s)

1: $S = \{s\}$ and $T = \emptyset$
2: **while** $S \neq V$ **do**
3: Compute $(u, v) = \arg\min_{(u,v): u \in S, v \in V-S} c(u, v)$
4: Add the node v to S and add the edge (u, v) to T.

- Implementation and analysis are very similar to Dijkstra's algorithm.
- Maintain S and store attachment costs $a(v) = \min_{e \in \text{cut}(S)} c(e)$ for every node $v \in V - S$ in a priority queue. Not the same as Dijsktra's algorithm!
- At each step, extract the node v with the minimum attachment cost from the priority queue and update the attachment costs of the neighbours of v.

Final Version of Prim's Algorithm

```
Prim's Algorithm(G, c, s)
1: Insert(Q, s, 0, ∅)
2: while S ≠ V do
3:    (v, a(v), u) = ExtractMin(Q)
4:    Add node v to S and edge (u, v) to T.
5:    for every node x ∈ V − S such that (v, x) is an edge in G do
6:       if c(v, x) < a(x) then
7:          a(x) = c(v, x)
8:          ChangeKey(Q, x, a(x), v)
```

- Q is a priority queue.
- Each element in Q is a triple: the node, its attachment cost, and its predecessor in the MST.
- In Step 8, if x is not already in Q, simply `Insert` $(x, a(x), v)$ into Q.

▸ Lectures 9–12: MST: Running time of Prim's algorithm

Final Version of Prim's Algorithm

PRIM'S ALGORITHM(G, c, s)

1: INSERT$(Q, s, 0, \emptyset)$
2: **while** $S \neq V$ **do**
3: $(v, a(v), u) =$ EXTRACTMIN(Q)
4: Add node v to S and edge (u, v) to T.
5: **for** every node $x \in V - S$ such that (v, x) is an edge in G **do**
6: **if** $c(v, x) < a(x)$ **then**
7: $a(x) = c(v, x)$
8: CHANGEKEY$(Q, x, a(x), v)$

- Q is a priority queue.
- Each element in Q is a triple: the node, its attachment cost, and its predecessor in the MST.
- In Step 8, if x is not already in Q, simply `Insert` $(x, a(x), v)$ into Q.
- Total of $n - 1$ EXTRACTMIN and m CHANGEKEY/`Insert` operations, yielding a running time of $O(m \log n)$.

Skip implementation of Kruskal's algorithm.

Implementing Kruskal's Algorithm

- Start with an empty set T of edges.
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Implementing Kruskal's Algorithm

- Start with an empty set T of edges.
- Process edges in E in increasing order of cost.
- Add the next edge e to T only if adding e does not create a cycle.
- Sorting edges takes $O(m \log n)$ time.
- Key question: "Does adding $e = (u, v)$ to T create a cycle?"
  - Maintain set of connected components of T.
  - $\textsc{Find}(u)$: return the name of the connected component of T that u belongs to.
  - $\textsc{Union}(A, B)$: merge connected components A and B.

Analysing Kruskal's Algorithm

- How many FIND invocations does Kruskal's algorithm need?

Analysing Kruskal's Algorithm

- How many FIND invocations does Kruskal's algorithm need? $2m$.
- How many UNION invocations does Kruskal's algorithm need?

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- How many FIND invocations does Kruskal's algorithm need? $2m$.
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- Textbook describes two implementations of UNION-FIND: (see appendix to this set of slides)
  - Each FIND takes $O(1)$ time, k invocations of UNION take $O(k \log k)$ time in total.
  - Each FIND takes $O(\log n)$ time and each invocation of UNION takes $O(1)$ time.

Analysing Kruskal's Algorithm

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- How many UNION invocations does Kruskal's algorithm need? $n - 1$.
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  - Each FIND takes $O(1)$ time, k invocations of UNION take $O(k \log k)$ time in total.
  - Each FIND takes $O(\log n)$ time and each invocation of UNION takes $O(1)$ time.
- Total running time of Kruskal's algorithm is $O(m \log n)$.

Comments on Union-Find and MST

- The Union-Find data structure is useful to maintain the connected components of a graph as edges are added to the graph.
- The data structure does not support edge deletion efficiently.

Comments on MST Algorithms

- To handle multiple edges with the same length, perturb each length by a random infinitesimal amount. Read the textbook.
- *Any* algorithm that constructs a spanning tree by including edges that satisfy the cut property and deleting edges that satisfy the cycle property will yield an MST!
- Current best algorithm for MST runs in $O(m\alpha(m, n))$ time (Chazelle 2000) and $O(m)$ randomised time (Karger, Klein, and Tarjan, 1995).
- Holy grail: $O(m)$ deterministic algorithm for MST.

Union-Find Data Structure

- Abstraction of the data structure needed by Kruskal's algorithm.
- Maintain disjoint subsets of elements from a universe U of n elements.
- Each subset has an name. We will set a set's name to be the identity of some element in it.
- Support three operations:
  1. MakeUnionFind(U): initialise the data structure with elements in U.
  2. Find(u): return the identity of the subset that contains u.
  3. Union(A, B): merge the sets named A and B into one set.

Union-Find Data Structure: Implementation 1

- Store all the elements of U in an array Component.
  - Assume identities of elements are integers from 1 to n.
  - Component[s] is the name of the set containing s.
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- $\textsc{Union}$ is very slow because we cannot efficiently find the elements that belong to a set.

Union-Find Data Structure: Implementation 2

- Optimisation 1: Use an array Elements
  - Indices of Elements range from 1 to n.
  - Elements[s] stores the elements in the subset named s in a list.
- Execute Union(A, B) by merging B into A in two steps:
  1. Updating Component for elements of B in $O(|B|)$ time.
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- Union takes $\Omega(n)$ in the worst-case.
- Optimisation 2: Store size of each set in an array (say, Size). If Size[B] $\leq$ Size[A], merge B into A. Otherwise merge A into B. Update Size.

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- Find is fast in the worst case, Union is fast in an amortised sense. Can we make both operations worst-case efficient?

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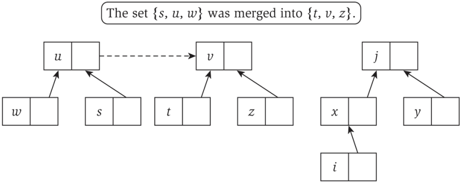

Figure 4.12 A Union-Find data structure using pointers. The data structure has only two sets at the moment, named after nodes v and j. The dashed arrow from u to v is the result of the last Union operation. To answer a Find query, we follow the arrows until we get to a node that has no outgoing arrow. For example, answering the query Find(i) would involve following the arrows i to x, and then x to j.

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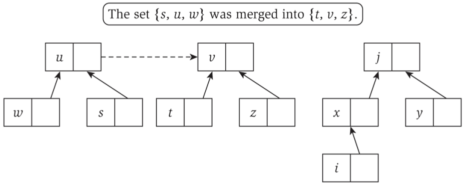

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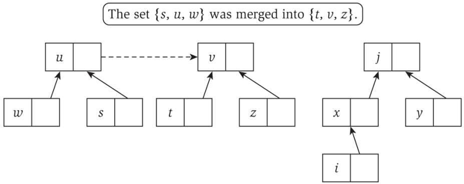
- Implementing Union(A, B): make smaller tree's root a child of the larger tree's root. Takes $O(1)$ time.



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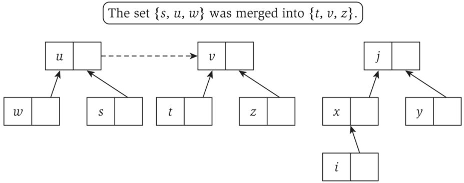

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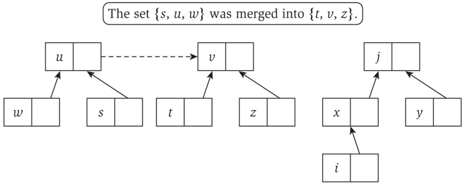

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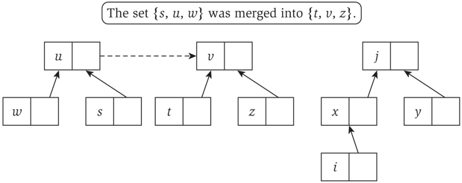
- Why does FIND(u) take $O(\log n)$ time?
- Number of pointers followed equals the number of times the identity of the set containing u changed.
- Every time u's set's identity changes, the set at least doubles in size $\Rightarrow$ there are $O(\log n)$ pointers followed.

Union-Find Data Structure: Improving Implementation



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- Every time we invoke $\text{Find}(u)$, we follow the same set of pointers.

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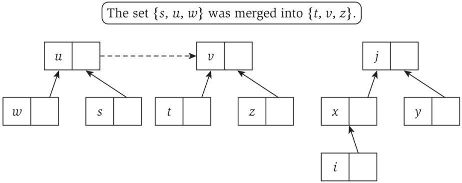

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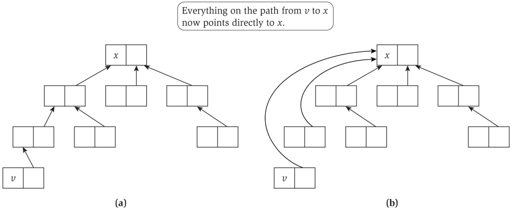

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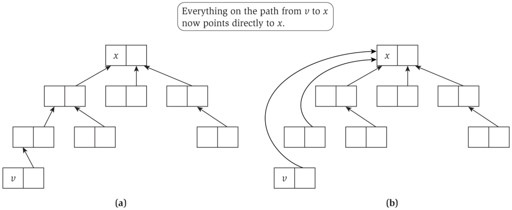

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- Every time we invoke Find(u), we follow the same set of pointers.
- Path compression: make all nodes visited by Find(u) children of the root.
- Can prove that total time taken by n Find operations is $O(n\alpha(n))$, where $\alpha(n)$ is the inverse of the Ackermann function, and grows e-x-t-r-e-m-e-l-y s-l-o-w-l-y with n.